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The defect of weak approximation for homogeneous spaces. II

To Nikolay Vasilievich Kuznetsov
on the occasion of his 70th birthday
with esteem and gratitude

Let X be a right homogeneous space of a connected linear algebraic group G' over a number field k, containing a k-point x. Assume that the stabilizer of x in G' is connected. Using the notion of a quasi-trivial group introduced by Colliot-Thélène, we can represent X in the form $X = H\backslash G$, where G is a quasi-trivial k-group and $H \subset G$ is a connected k-subgroup.

Let S be a finite set of places of k. Applying results of [1], we compute the defect of weak approximation for X with respect to S in terms of the biggest toric quotient H^{tor} of H. In particular, we show that if H^{tor} splits over a metacyclic extension of k, then X has the weak approximation property. We show also that any homogeneous space X with connected stabilizer (without assumptions on H^{tor}) has the real approximation property.

Key words: *linear algebraic groups, homogeneous spaces, weak approximation*

1. Introduction

This note is a sequel for [1], and we use the notation of that paper. Let k be a number field, and let $\bar{k}$ be a fixed algebraic closure of k. We write $\mathscr{V}$ for the set of all places of k, and $\mathscr{V}_\infty$ for the set of its archimedean places. If $v \in \mathscr{V}$, we write k_v for the completion of k at v.

Let X be an algebraic variety over k. We refer to [1] for preliminaries on weak approximation for X. If $S \subset \mathscr{V}$ is a finite set of places, we write (WA_S) for the weak approximation property with respect to S. Thus, "X has (WA_S)" means that $X(k)$ is dense in $\prod_{v\in S} X(k_v)$. We say that X has *the weak approximation property,* if X has (WA_S) for any finite subset $S \subset \mathscr{V}$. We say that X has *the real approximation property,* if X has the weak approximation property (WA_S) with respect to $S = \mathscr{V}_\infty$.

In [1] we considered the case $X = H\backslash G$, where $H \subset G$ is a connected k-subgroup of a connected k-group G, assuming that Ш$(G) = 0$ and $A(G) = 0$ (the assumption $A(G) = 0$ means that G has the weak approximation property). Under these assumptions we constructed a certain abelian group $C_S(H,G)$ which is the defect of weak approximation for X with respect to S: the variety X has (WA_S) if and only if $C_S(H,G) = 0$. We initially constructed $C_S(H,G)$ in terms of H and G, but then we computed it in terms of the Brauer group of X.

In the present note we consider the case of an arbitrary homogeneous space with connected stabilizer $X = H'\backslash G'$, where G' is any connected linear k-group and $H' \subset G'$ is a connected

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k-subgroup. Using the notions of a quasi-trivial k-group and a flasque resolution, introduced by J.-L. Colliot-Thélène [2], we notice that we can represent X in the form $X = H\backslash G$, where G is a *quasi-trivial* group and $H \subset G$ is a connected k-subgroup (Lemma 2.5). We have Ш$(G) = 0$ and $A(G) = 0$, because G is quasi-trivial. Now we can apply [1, Theorem 1.3]. We obtain that X has (WA_S) if and only if $C_S(H, G) = 0$.

Moreover, we have $\mathrm{Pic}(G_K) = 0$ for any field extension K/k, because G is quasi-trivial. Using this fact, we show that the group $C_S(H, G)$ can be computed in terms of H only. Namely, we construct a group $C_S(H)$ in terms of H as in [3] and prove that $C_S(H, G) = C_S(H)$ (Lemma 3.4).

We see that X has (WA_S) if and only if $C_S(H) = 0$. We say that $C_S(H)$ is *the defect of weak approximation for X with respect to S.* Note that the group $C_S(H)$ does not depend on the representation of X in the form $X = H\backslash G$ with quasi-trivial G and connected H, because it can be computed in terms of the Brauer group of X ([1, Theorem 1.11]).

Let H^{tor} denote the biggest quotient torus of H. We show that the canonical homomorphism $C_S(H) \to C_S(H^{\mathrm{tor}})$ is an isomorphism (Proposition 3.7). It follows that X has (WA_S) if and only if $C_S(H^{\mathrm{tor}}) = 0$. We notice that

$$C_S(H^{\mathrm{tor}}) \simeq \operatorname{coker}\left[H^1(k, H^{\mathrm{tor}}) \to \prod_{v\in S} H^1(k_v, H^{\mathrm{tor}})\right].$$

Let L/k be a Galois extension splitting the torus H^{tor}. Let S_0 denote the set of (nonarchimedean, ramified in L) places v of k such that the decomposition group of v in $\mathrm{Gal}(L/k)$ is noncyclic. We prove that $C_S(H) = C_{S\cap S_0}(H)$ (Corollary 3.11).

Assume that $S \cap S_0 = \emptyset$, i.e. all the places in S have cyclic decomposition subgroups in $\mathrm{Gal}(L/k)$. Then $C_S(H) = 0$, hence X has (WA_S) (Theorem 3.12). In particular, $C_{\mathscr{V}_\infty}(H) = 0$ for any H. Thus any homogeneous space X of a connected k-group with connected stabilizer has the real approximation property (Corollary 3.13).

Now assume that H^{tor} splits over a cyclic extension of k (e.g. $H^{\mathrm{tor}} = 1$). Then $S_0 = \emptyset$, hence $C_S(H) = 0$ for any S, and X has the weak approximation property (Corollary 3.14). Moreover, we prove that if H^{tor} splits over a metacyclic extension, then X has the weak approximation property (Theorem 4.2).

These results generalize the results of [3], where we assumed that G is semisimple simply connected. They also generalize results of Sansuc [4] on weak approximation for connected linear groups.

We could state and prove our results thanks to the notion of a quasi-trivial group introduced by Colliot-Thélène [2]. The constructions and proofs are based on results of Kottwitz [5]. Of course, our results are based on the classical results of Kneser, Harder, Chernousov, and Platonov on the Hasse principle and weak approximation for simply connected semisimple groups.

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2. Preliminaries on quasi-trivial groups

The results of this section are actually due to J.-L. Colliot-Thélène [2].

2.1. Let k be a field of characteristic 0, $\overline{k}$ a fixed algebraic closure of k. Let G be a connected linear k-group. We set $\overline{G} = G \times_k \overline{k}$. We use the following notation:
G^{u} is the unipotent radical of G;
$G^{\mathrm{red}} = G/G^{\mathrm{u}}$ (it is reductive);

G^{ss} is the derived group of G^{red} (it is semisimple);
$G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$ (it is a torus);
$G^{\mathrm{ssu}} = \ker[G \to G^{\mathrm{tor}}]$ (it is an extension of G^{ss} by G^{u}).

Definition 2.2 (J.-L. Colliot-Thélène). A connected linear k-group G over a field k of characteristic 0 is called *quasi-trivial*, if G^{tor} is a quasi-trivial torus and G^{ss} is simply connected.

Recall that a k-torus T is called quasi-trivial if its character group $\mathbf{X}(\overline{T})$ is a permutation $\mathrm{Gal}(\overline{k}/k)$-module.

Note that if G is quasi-trivial, then for any field extension K/k the group G_K is quasi-trivial.

Lemma 2.3. *Let G be a quasi-trivial group over a field k of characteristic 0. Then* $\mathrm{Pic}(G) = 0$, *where* Pic *denotes the Picard group.*

P r o o f. If

$$1 \to G' \to G \to G'' \to 1$$

is a short exact sequence of connected linear k-groups, then we have an exact sequence

$$\mathbf{X}(G') \to \mathrm{Pic}(G'') \to \mathrm{Pic}(G) \to \mathrm{Pic}(G'), \tag{1}$$

where $\mathbf{X}(G')$ denotes the group of k-characters of G', see [4, Corollary 6.11].

Since G^{u} is a unipotent k-group, the exponential map $\exp\colon \mathrm{Lie}\, G^{\mathrm{u}} \to G^{\mathrm{u}}$ is a biregular isomorphism of algebraic varieties (because $\mathrm{char}(k) = 0$), hence $\mathrm{Pic}(G^{\mathrm{u}}) = 0$. By [4, Lemme 6.9] $\mathrm{Pic}(G^{\mathrm{ss}}) = 0$ (because G^{ss} is simply connected) and $\mathrm{Pic}(G^{\mathrm{tor}}) = H^1(k, \mathbf{X}(\overline{G}^{\mathrm{tor}}))$. Since $\mathbf{X}(\overline{G}^{\mathrm{tor}})$ is a permutation module, we see that $\mathrm{Pic}(G^{\mathrm{tor}}) = 0$. Using exact sequence (1), we conclude by dévissage that $\mathrm{Pic}(G) = 0$.

Lemma 2.4. *Let G be a quasi-trivial k-group over a number field k. Then* $Ш(G) = 0$ *and* $A(G) = 0$.

P r o o f. By [2, Proposition 9.2] we have $Ш(G^{\mathrm{red}}) = 0$ and $A(G^{\mathrm{red}}) = 0$. By [4, Proposition 4.1] $Ш(G) = Ш(G^{\mathrm{red}})$. By [4, Proposition 3.2] $A(G) = A(G^{\mathrm{red}})$. Thus $Ш(G) = 0$ and $A(G) = 0$.

Lemma 2.5. *Let k be a field of characteristic 0 and X a right homogeneous space with connected stabilizer over k, i.e. $X = H'\backslash G'$, where G' is a connected linear k-group and $H' \subset G'$ is a connected k-subgroup. Then one can represent X as $X = H\backslash G$, where G is a quasi-trivial k-group and $H \subset G$ is a connected k-subgroup.*

P r o o f. By [2, Proposition-Définition 3.1] there exists a flasque resolution of G', i.e. a central extension of connected k-groups

$$1 \to F \to G \to G' \to 1,$$

where G is quasi-trivial and F is a flasque k-torus. Let H be the preimage of H' in G. From the exact sequence

$$1 \to F \to H \to H' \to 1$$

we see that H is connected, because H' and F are connected. We have $X = H\backslash G$.

3. Defect of weak approximation

3.1. Let X be a homogeneous space with connected stabilizer over a number field k, i.e. $X = H'\backslash G'$, where G' is a connected linear k-group and $H' \subset G'$ is a connected k-subgroup. By Lemma 2.5 we may write $X = H\backslash G$, where G is a quasi-trivial k-group and $H \subset G$ is a connected k-subgroup.

By Lemma 2.4 $Ш(G) = 0$ and $A(G) = 0$. Therefore we can apply the results of [1].

3.2. Let X, G, H be as in 3.1. Let $S \subset \mathscr{V}$ be a finite subset. Set

$$B(H) = \mathrm{Hom}(\mathrm{Pic}(H), \mathbb{Q}/\mathbb{Z}) = (\pi_1(H)_\Gamma)_{\mathrm{tors}}$$
$$B_v(H) = B(H_{k_v}) \text{ for } v \in \mathscr{V}$$

with the notation of [1]. Consider the canonical homomorphism

$$\lambda_v \colon B_v(H) \to B(H).$$

Set:

$$B^S(H) = \langle \lambda_v(B_v(H)) \rangle_{v \in \mathscr{V} \smallsetminus S}$$
$$B'(H) = B^{\emptyset}(H) = \langle \lambda_v(B_v(H)) \rangle_{v \in \mathscr{V}}$$
$$C_S(H) = B'(H)/B^S(H),$$

where $\langle \lambda_v(B_v(H)) \rangle_{v \in \mathscr{V} \smallsetminus S}$ denotes the subgroup of $B(H)$ generated by the subgroups $\lambda_v(B_v(H))$ for all $v \in \mathscr{V} \smallsetminus S$.

3.3. For a homogeneous space $X = H\backslash G$ over k, without assuming that G is quasi-trivial, we defined in [1] the following groups:

$$B(H, G) = \ker[B(H) \to B(G)],$$
$$B_v(H, G) = B(H_{k_v}, G_{k_v}) = \ker[B_v(H) \to B_v(G)],$$

and also $B^S(H, G)$, $B'(H, G)$, and $C_S(H, G)$, see [1, Section 1.2].

Lemma 3.4. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial). Then there is a canonical isomorphism $C_S(H, G) \xrightarrow{\sim} C_S(H)$.*

P r o o f. Since G is quasi-trivial, by Lemma 2.3 $\mathrm{Pic}(G) = 0$, hence $B(G) = 0$. Since G_{k_v} is also quasi-trivial, we see that $B_v(G) = 0$. We obtain successively that $B(H, G) = B(H)$, $B_v(H, G) = B_v(H)$, $B^S(H, G) = B^S(H)$, $B'(H, G) = B'(H)$, whence $C_S(H, G) = C_S(H)$.

Theorem 3.5. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial). Let $S \subset \mathscr{V}$ be a finite set of places of k. Then X has* (WA$_S$) *if and only if $C_S(H) = 0$.*

P r o o f. By Lemma 2.4 $\text{Ш}(G) = 0$ and $A(G) = 0$. By [1, Theorem 1.3] X has (WA$_S$) if and only if $C_S(H, G) = 0$. By Lemma 3.4 $C_S(H, G) = C_S(H)$, and the theorem follows.

Lemma 3.6. *Let H be a connected linear k-group over a number field k. Assume that $H^{\mathrm{tor}} = 1$. Then for any place v of k the map $\lambda_v \colon B_v(H) \to B(H)$ is surjective.*

P r o o f. See [6, Proof of Theorem 3.4(b)].

Proposition 3.7 ([3, Theorem 1.4]). *Let H be a connected k-group over a number field k. Let $S \subset \mathscr{V}$ be a finite set of places of k. Then the canonical homomorphism $C_S(H) \to C_S(H^{\mathrm{tor}})$ is an isomorphism.*

P r o o f. Since [3] is not easily accessible, we reproduce the proof here.

First, consider H^{ssu}. Since $(H^{\mathrm{ssu}})^{\mathrm{tor}} = 1$, by Lemma 3.6 for any place v of k we have $\lambda_v(B_v(H^{\mathrm{ssu}})) = B(H^{\mathrm{ssu}})$. We see that $B^S(H^{\mathrm{ssu}}) = B'(H^{\mathrm{ssu}}) = B(H^{\mathrm{ssu}})$.

Consider the canonical short exact sequence

$$1 \to H^{\mathrm{ssu}} \to H \to H^{\mathrm{tor}} \to 1.$$

Exact sequence (1) from the proof of Lemma 2.3 gives us an exact sequence

$$\mathbf{X}(H^{\mathrm{ssu}}) \to \mathrm{Pic}(H^{\mathrm{tor}}) \to \mathrm{Pic}(H) \to \mathrm{Pic}(H^{\mathrm{ssu}}),$$

where clearly $\mathbf{X}(H^{\mathrm{ssu}}) = 0$. We obtain the dual exact sequence

$$B(H^{\mathrm{ssu}}) \to B(H) \to B(H^{\mathrm{tor}}) \to 0$$

and similar exact sequences for the groups B_v. Since $B^S(H^{\mathrm{ssu}}) = B(H^{\mathrm{ssu}})$, we obtain an exact sequence

$$B(H^{\mathrm{ssu}}) \to B^S(H) \to B^S(H^{\mathrm{tor}}) \to 0.$$

Set $\overline{B} = \mathrm{im}[B(H^{\mathrm{ssu}}) \to B(H)]$, then we obtain an exact sequence

$$0 \to \overline{B} \to B^S(H) \to B^S(H^{\mathrm{tor}}) \to 0$$

and a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \overline{B} \arrow[r] \arrow[d, equal] & B^S(H) \arrow[r] \arrow[d, hook] & B^S(H^{\mathrm{tor}}) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \overline{B} \arrow[r] & B'(H) \arrow[r] & B'(H^{\mathrm{tor}}) \arrow[r] & 0 \end{tikzcd}$$

Now the snake lemma gives us an isomorphism $C_S(H) = B'(H)/B^S(H) \xrightarrow{\sim} B'(H^{\mathrm{tor}})/B^S(H^{\mathrm{tor}}) = C_S(H^{\mathrm{tor}})$.

Corollary 3.8. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial and H is connected). Assume that $H^{\mathrm{tor}} = 1$. Then X has the weak approximation property.*

P r o o f. By Proposition 3.7 we have $C_S(H) = C_S(H^{\mathrm{tor}}) = 0$ for any S. By Theorem 3.5, X has (WA_S) for any S.

R e m a r k. In the case when G is semisimple simply connected, this result was proved in [3, Corollary 1.7]. For a simple proof see [6, Theorem 3.4(b)].

The following result relates $C_S(H)$ to the Galois cohomology of H^{tor}.

Proposition 3.9. *Let T be a k-torus over a number field k. Let $S \subset \mathscr{V}$ be a finite set of places of k. Then there is a canonical isomorphism*

$$C_S(T) \xrightarrow{\sim} \mathrm{coker}\left[H^1(k,T) \to \prod_{v\in S} H^1(k_v,T)\right].$$

P r o o f. We have canonical duality isomorphisms

$$\beta_v\colon H^1(k_v,T) \xrightarrow{\sim} \mathrm{Hom}(H^1(k_v,\mathbf{X}(\overline{T})),\mathbb{Q}/\mathbb{Z}) = B_v(T),$$

cf. [7, Chapter I, Corollary 2.3 and Theorem 2.13]. Moreover, we have an exact sequence

$$H^1(k,T) \xrightarrow{\mathrm{loc}} \oplus_{v\in\mathscr{V}} H^1(k_v,T) \xrightarrow{\mu} B(T), \tag{2}$$

where loc is the localization map, $\mu((\xi_v)_{v\in\mathscr{V}}) = \sum \mu_v(\xi_v)$, and μ_v is the composed map

$$\mu_v\colon H^1(k_v,T) \xrightarrow{\beta_v} B_v(T) \xrightarrow{\lambda_v} B(T),$$

cf. [7, Chapter I, Theorem 4.20(b)].

Consider the localization map $\mathrm{loc}_S\colon H^1(k,T) \to \prod_{v\in S} H^1(k_v,T)$. Let $\xi_S = (\xi_v)_{v\in S} \in \prod_{v\in S} H^1(k_v,T) = \oplus_{v\in S} B_v(T)$, where we identify $H^1(k_v,T)$ with $B_v(T)$ using β_v. From exact sequence (2) we see that ξ_S is contained in the image of loc_S if and only if there exists an element $\xi^S \in \oplus_{v\notin S} B_v(T)$ such that $\mu(\xi_S,\xi^S) = 0$. Such an element ξ^S exists if and only if

$$\sum_{v\in S} \mu_v(\xi_v) \in B^S(T) \subset B(T).$$

Set $B_S(T) = \langle \lambda_v(B_v(T)) \rangle_{v\in S}$. Then we see that there is a canonical isomorphism

$$\operatorname{coker}\left[H^1(k,T) \to \prod_{v\in S} H^1(k_v,T)\right] \xrightarrow{\sim} B_S(T)/\left(B_S(T)\cap B^S(T)\right) \simeq$$
$$\simeq \left(B_S(T)+B^S(T)\right)/B^S(T) = B'(T)/B^S(T) = C_S(T).$$

Proposition 3.10. *Let T be a k-torus over a number field k. Let L/k be a Galois extension splitting T. Let S_0 be the set of (nonarchimedean, ramified in L) places v of k whose decomposition groups in* $\mathrm{Gal}(L/k)$ *are noncyclic. Let $S \subset \mathscr{V}$ be any finite set of places of k. Then the canonical homomorphism $C_S(T) \to C_{S\cap S_0}(T)$ is an isomorphism.*

P r o o f. Let $v \in S$. Let w be a place of L lying over v. Let $D_w \subset \mathrm{Gal}(L/k)$ be the decomposition group of w. Then by [4, Lemme 6.9] $\mathrm{Pic}(T_{k_v}) = H^1(D_w, \mathbf{X}(T_L))$. We see that the image $\lambda_v(B_v(T)) \subset B(T)$ depends only on the conjugacy class of $D_w \subset \mathrm{Gal}(L/k)$.

If $v \in S$, $v \notin S_0$, then D_w is cyclic for w lying over v. By Chebotarev's density theorem there exists $v' \notin S$ and w' lying over v' such that $D_{w'} = D_w$. It follows that $\lambda_v(B_v(T)) = \lambda_{v'}(B_{v'}(T))$. But $v' \notin S$, hence $\lambda_{v'}(B_{v'}(T)) \subset B^S(T)$. We see that $\lambda_v(B_v(T)) \subset B^S(T)$. Thus $B^{S\cap S_0}(T) = B^S(T)$. We conclude that $C_{S\cap S_0}(T) = C_S(T)$.

Corollary 3.11. *Let H be a connected linear k-group over a number field k. Let L/k be a Galois extension splitting H^{tor}. Let S_0 be the set of places v of k whose decomposition groups in* $\mathrm{Gal}(L/k)$ *are noncyclic. Let $S \subset \mathscr{V}$ be any finite set of places of k. Then the canonical homomorphism $C_S(H) \to C_{S\cap S_0}(H)$ is an isomorphism.*

P r o o f. We have a commutative diagram of canonical homomorphisms

$$\begin{tikzcd} C_S(H) \arrow[r, "\simeq"] \arrow[d] & C_S(H^{\mathrm{tor}}) \arrow[d, "\simeq"] \\ C_{S\cap S_0}(H) \arrow[r, "\simeq"] & C_{S\cap S_0}(H^{\mathrm{tor}}) \end{tikzcd}$$

By Proposition 3.7 the horizontal arrows are isomorphisms. By Proposition 3.10 the right vertical arrow is an isomorphism. We conclude that the left vertical arrow is also an isomorphism.

Theorem 3.12. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial and H is connected). Let L/k be a Galois extension splitting H^{tor}. Let S_0 be the set of places v of k whose decomposition groups in* $\mathrm{Gal}(L/k)$ *are noncyclic. Let $S \subset \mathscr{V}$ be a finite set of places of k such that $S\cap S_0 = \emptyset$. Then X has* (WA_S).

P r o o f. By Corollary 3.11 $C_S(H) = C_{S\cap S_0}(H) = C_\emptyset(H) = 0$. By Theorem 3.5 X has (WA_S).

Corollary 3.13. *Let X be as in 3.1. Then X has the real approximation property.*

P r o o f. Let L and S_0 be as in Theorem 3.12. Take $S = \mathscr{V}_\infty$. A decomposition group of an archimedean place is either 0 or $\mathbb{Z}/2$, hence cyclic. We see that $\mathscr{V}_\infty \cap S_0 = \emptyset$. By Theorem 3.12 X has $(\mathrm{WA}_{\mathscr{V}_\infty})$, i.e. X has the real approximation property.

A n o t h e r p r o o f. The subgroup H is connected, and we have Ш$(G) = 0$ and $A(G) = 0$. Now by [1, Corollary 1.7] X has the real approximation property.

Corollary 3.14. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial and H is connected). Assume that H^{tor} splits over a* cyclic *extension L of k. Then X has the weak approximation property.*

P r o o f. Let S_0 denote the set of places v of k whose decomposition groups in $\mathrm{Gal}(L/k)$ are noncyclic, then $S_0 = \emptyset$. Thus for any finite $S \subset \mathscr{V}$ we have $S\cap S_0 = \emptyset$. By Theorem 3.12 X has (WA_S) for any S, i.e. X has the weak approximation property.

4. Metacyclic extensions

In this section, inspired by [4, Lemme 1.3], we generalize Corollary 3.14.

4.1. Recall that a finite group is called metacyclic if all its Sylow subgroups are cyclic. For example, the symmetric group S_3 is metacyclic, while the group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is not. Every cyclic group is metacyclic. We say that a Galois extension L/k is metacyclic if $\mathrm{Gal}(L/k)$ is a metacyclic group.

Theorem 4.2. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial and H is connected). Assume that H^{tor} splits over a* metacyclic *extension L of k. Then X has the weak approximation property.*

P r o o f. Set $T = H^{\mathrm{tor}}$, then T is a k-torus splitting over L. By Theorem 3.5 and Proposition 3.7 it suffices to prove that $C_S(T) = 0$. Set

$$Ч^1_S(T) = \mathrm{coker}\left[H^1(k,T) \to \prod_{v\in S} H^1(k_v,T)\right].$$

By Proposition 3.9 $C_S(T) \simeq Ч^1_S(T)$.

We write $\widehat{T}$ for $\mathbf{X}(\overline{T})$. Set

$$\begin{aligned}
Ш^1_S(k,\widehat{T}) &= \ker\left[H^1(k,\widehat{T}) \to \prod_{v\in\mathscr{V}\smallsetminus S} H^1(k_v,\widehat{T})\right]\\
Ш^1_\omega(k,\widehat{T}) &= \bigcup_S Ш^1_S(k,\widehat{T})\\
Ш^1_{S,\emptyset}(k,\widehat{T}) &= Ш^1_S(k,\widehat{T})/\,Ш^1_\emptyset(k,\widehat{T}).
\end{aligned}$$

By [8, Theorem 0.3]

$$Ч^1_S(T) \simeq \mathrm{Hom}(Ш^1_{S,\emptyset}(k,\widehat{T}),\mathbb{Q}/\mathbb{Z}).$$

Now $Ш^1_{S,\emptyset}(k,\widehat{T})$ is by definition a subquotient of $Ш^1_\omega(k,\widehat{T})$. Thus in order to prove the theorem it suffices to show that $Ш^1_\omega(k,\widehat{T}) = 0$.

Denote by $\mathfrak{g}$ the image of $\mathrm{Gal}(L/k)$ in $\mathrm{Aut}(\widehat{T})$. Then $\mathfrak{g}$ is a finite metacyclic group. We may and shall assume that $\mathrm{Gal}(L/k) = \mathfrak{g}$. For a place v of k, let $D_w \subset \mathfrak{g}$ denote the decomposition group of a place w of L extending v. We write $\mathfrak{g}_v$ for D_w, it is defined up to conjugacy in $\mathfrak{g}$. Since $\mathrm{Gal}(\overline{k}/L)$ is a profinite group and $\widehat{T}$ is a free abelian group, we have $H^1(L,\widehat{T}) = 0$. From the inflation-restriction exact sequence

$$0 \to H^1(\mathfrak{g},\widehat{T}) \to H^1(k,\widehat{T}) \to H^1(L,\widehat{T}) = 0$$

it follows that the inflation homomorphism $H^1(\mathfrak{g},\widehat{T}) \to H^1(k,\widehat{T})$ is an isomorphism. Similarly, for each v the homomorphism $H^1(\mathfrak{g}_v,\widehat{T}) \to H^1(k_v,\widehat{T})$ is an isomorphism. Thus we obtain an isomorphism

$$\ker\left[H^1(\mathfrak{g},\widehat{T}) \to \prod_{v\in\mathscr{V}\smallsetminus S} H^1(\mathfrak{g}_v,\widehat{T})\right] \xrightarrow{\sim} Ш^1_S(k,\widehat{T}).$$

It follows from Chebotarev's density theorem that

$$Ш^1_\omega(k,\widehat{T}) \simeq \ker\left[H^1(\mathfrak{g},\widehat{T}) \to \prod_C H^1(C,\widehat{T})\right], \tag{3}$$

where C runs over all cyclic subgroups of $\mathfrak{g}$.

Now let F be any finite group and Y a finitely generated F-module. Let $i \in \mathbb{Z}$. We write $\hat{H}^i(F,Y)$ for the i-th Tate cohomology group. Following an idea of [9, page 734], we set

$$Ш^i_\Omega(F,Y) = \ker\left[\hat{H}^i(F,Y) \to \prod_C \hat{H}^i(C,Y)\right],$$

where C runs over all cyclic subgroups of F. Then by (3) $Ш^1_\omega(k,\widehat{T}) \simeq Ш^1_\Omega(\mathfrak{g},\widehat{T})$. In order to prove Theorem 4.2 it suffices to show that $Ш^1_\Omega(\mathfrak{g},\widehat{T}) = 0$, which follows from the next lemma.

Lemma 4.3. (B. Kunyavskiĭ, private communication). *Let $\mathfrak{g}$ be a metacyclic finite group and Y a finitely generated $\mathfrak{g}$-module. Then $Ш^i_\Omega(\mathfrak{g},Y) = 0$ for all $i \in \mathbb{Z}$.*

P r o o f. Let $y \in Ш^i_\Omega(\mathfrak{g},Y) \subset \hat{H}^i(\mathfrak{g},Y)$. For a subgroup $\mathfrak{h} \subset \mathfrak{g}$ let $\mathrm{Res}_{\mathfrak{h}}(y) \in \hat{H}^i(\mathfrak{h},Y)$ denote the restriction of y to $\mathfrak{h}$. Since $y \in Ш^i_\Omega(\mathfrak{g},Y)$, we have $\mathrm{Res}_C(y) = 0$ for any cyclic subgroup $C \subset \mathfrak{g}$. Since $\mathfrak{g}$ is metacyclic, every Sylow subgroup of $\mathfrak{g}$ is cyclic. We see that $\mathrm{Res}_S(y) = 0$ for for any Sylow subgroup S of $\mathfrak{g}$. By [10, Chapter IV, Section 6, Corollary 4 of Proposition 8] we have $y = 0$. Thus $Ш^i_\Omega(\mathfrak{g},Y) = 0$. This completes the proofs of the lemma and of Theorem 4.2.

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АННОТАЦИЯ

Пусть X – правое однородное пространство связной линейной алгебраической группы G' над полем алгебраических чисел k, содержащее k-точку x. Предположим, что стационарная подгруппа точки x в G' связная. Используя понятие квазитривиальной группы, введенное Кольо – Теленом, мы можем представить X в виде $X = H\backslash G$, где G – некоторая квазитривиальная k-группа и $H \subset G$ – ее связная k-подгруппа. Пусть S – некоторое конечное множество нормирований поля k. Применяя результаты работы [B2], мы вычисляем дефект слабой аппроксимации для X относительно S в терминах наибольшего фактор-тора H^{tor} группы H. В частности, мы показываем, что если тор H^{tor} расщепляется над некоторым метациклическим расширением поля k, то однородное пространство X обладает свойством слабой аппроксимации. Мы показываем также, что любое однородное пространство X со связной стационарной подгруппой (без условий на H^{tor}) обладает свойством вещественной аппроксимации.

Ключевые слова: *линейные алгебраические группы, однородные пространства, слабая аппроксимация*