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Relations between conjectural eigenvalues of Hecke operators on submotives of Siegel varieties

There exist conjectural formulas of relations between L-functions of submotives of Shimura varieties and automorphic representations of the corresponding reductive groups, due to Langlands – Arthur. In the present paper these formulas are used in order to get explicit relations between eigenvalues of p-Hecke operators (generators of the p-Hecke algebra of X) on cohomology spaces of some of these submotives, for the case where X is a Siegel variety. Hence, this result is conjectural as well: the methods related to counting points on reductions of X using the Selberg trace formula are not used.

It turns out that the above relations are linear and their coefficients are polynomials in p which satisfy a simple recurrence formula. The same result can be easily obtained for any Shimura variety.

This result is an intermediate step for the generalization of Kolyvagin's theorem of the finiteness of Tate – Shafarevich group of elliptic curves of analytic rank 0 and 1 over $\mathbb{Q}$, to the case of submotives of other Shimura varieties, particularly of Siegel varieties of genus 3, see [9].

The idea of the proof: on the one hand, the above formulas of Langlands – Arthur give us (conjectural) relations between Weil numbers of a submotive. On the other hand, the Satake map permits us to transform these relations between Weil numbers into relations between eigenvalues of p-Hecke operators on X.

The paper also contains a survey of some related questions, for example explicit finding of the Hecke polynomial for X, and (Appendix) tables for the cases $g = 2, 3$.

Key words: *Siegel varieties, submotives, Hecke correspondences, Weil numbers, Satake map.*

Introduction

The purpose of the present paper is to show that starting from some standard conjectures of Langlands – Arthur, a chain of elementary calculations leads to a simply-formulated and non-expected result on relations between eigenvalues of p-Hecke operators on a Shimura variety (which hence depends on these conjectures).

Namely, let Π be a stable global packet of automorphic representations of a reductive group G (G corresponds to a Shimura variety X). Attached to Π is a parabolic subgroup P of G. Let $\mathcal{M}$ be an irreducible constituent of a submotive conjecturally attached to Π (see (0.1) below). We denote the field of coefficients of $\mathcal{M}$ by E. $H^i(\mathcal{M})$ is a module over the p-Hecke algebra $\mathbb{H}(G) = \mathbb{H}_p(G)$ of X, p is a fixed prime. Any $t \in \mathbb{H}(G)$ acts on $H^i(\mathcal{M})$ by multiplication by an

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element $\mathfrak{m}_{\mathcal{M}}(t)$ of E. If $\{t_i\}$ is a set of generators of $\mathbb{H}(G)$ then numbers $\mathfrak{m}_{\mathcal{M}}(t_i)$ can satisfy some relations.

The main result of the present paper is the Theorem 4.4. We fix the type of a Shimura variety (the level is arbitrary) and a parabolic subgroup P. We find the set of relations between $\mathfrak{m}_{\mathcal{M}}(t_i)$ (depending on P only) such that if the above conjectures are true then these relations are satisfied for all submotives $\mathcal{M}$ corresponding to all stable Π corresponding to P.

Really, the first steps of the calculations of the present paper are made only for Siegel varieties, and the last step only for submotives corresponding to one type of P (the simplest non-trivial). This restriction is not of principle: the reader can easily get analogous results for any Shimura variety and any type of submotives, including the ones that correspond to non-stable packets. The author is interested in the case $g = 3$, because the main result of the present paper can be applied for a generalization of the Kolyvagin's theorem to the case of submotives of Siegel varieties of genus 3. See [9] for applications of the results of the present paper.

The paper also contains some calculations that are not logically necessary for the proof of the main theorem, but can be used for references or for the further development of the subject (Sections 2.2 - 2.4, 3.4 and small parts of other sections). Some tables for genii 2 and 3 are given in the Appendix, which is written for the same reason.

0. Idea of the proof

As it was mentioned above, in order to define a Shimura variety X of a fixed level, we must fix a reductive group G over $\mathbb{Q}$, a map $h : \text{Res}_{\mathbb{C}/\mathbb{R}}(G_m) \to G$ over $\mathbb{R}$, and a level subgroup $\mathcal{K} \subset G(\mathbb{A}_f(\mathbb{Q}))$; these data must satisfy some conditions ([6]). Throughout the paper we consider only the case when p does not divide the level, i.e. $\mathcal{K} \supset G(\mathbb{Z}_p)$. We shall consider only the case of Siegel varieties, i.e. from here we let $G = GSp_{2g}$ over $\mathbb{Q}$. Further, we must choose a compactification of X and a type of cohomology.

Really, all subsequent considerations depend only on G and h and, hence, do not depend on level, compactification and the type of cohomology.

We fix a Borel subgroup B of G and consider all intermediate parabolic subgroups P of G, $B \subset P \subset G$. There is a 1 - 1 correspondence between the set of archimedean cohomological A-parameters of G and the set of such P ([3]; [2]; [4], Section 4.1). We denote by Π_P the packet of automorphic representations of $G(\mathbb{R})$ corresponding to the archimedean cohomological A-parameter corresponding to P ([4], Section 4.2).

Attached to P and X is a set (indexed by k) of stable global packets of automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$. This set clearly depends on the level of X. We denote the k-th packet by $\Pi_P^{glob}(k)$. Let $\pi \in \Pi_P^{glob}(k)$ be a representation, $\pi = \pi_\infty \otimes \pi_f$ its decomposition on archimedean and finite part, and $\pi_f = \otimes_l \pi_l$ ($l =$ prime) the decomposition of π_f. We consider only such π for which π_p is non-ramified (see Step 3 below for a description of π_p). We have $\pi_\infty \in \Pi_P$, and for any other $\pi'_\infty \in \Pi_P$ the representation $\pi' := \pi'_\infty \otimes \pi_f$ also belongs to $\Pi_P^{glob}(k)$.

Conjecturally, $\forall k$ there exists a submotive $\mathcal{M}_P(k)$ (reducible unless $P = B$) such that

$$L_p(\pi, r, s) = L_p(\mathcal{M}_P(k), s) \tag{0.1}$$

where $r : {}^LG \to GL(W)$ is a finite-dimensional representation defined in [4], 5.1; Weil numbers of $\mathcal{M}_P(k)$ and hence its L-function are considered with respect to E, and $L_p(\pi, r, s) = L(\pi_p, r, s)$ is the local L-function (particularly, it does not change if we change π by π').

Now we fix some k and we denote $\mathcal{M}_P(k)$ simply by $\mathcal{M}_P$. It is the main object of the present paper. Since it is conjectural, all subsequent theorems should be understand as follows:

Let $\mathcal{M}_P$ be a motive such that (0.1) is satisfied. Then ... (statement of the theorem).

Some references for properties of $\mathcal{M}_P$: [4], Sections 4.3, 5, 7; [2], Section 9, and the present paper, section 3. Most of these properties are not necessary for the proof of the main theorem. Here we mention only that (0.1) implies that $\sum_i h^i(\mathcal{M}_P) = \dim r$; for $G = GSp_{2g}$ this number is 2^g. See also Remark 0.1 below.

Now let $\mathcal{M} \subset \mathcal{M}_P$, $\mathfrak{m}_{\mathcal{M}}$, E be as in the Introduction. Multiplication by elements of E gives us an inclusion $i_{\mathcal{M}} : E \to \mathbb{C}$. It is clear that the composite map $i_{\mathcal{M}} \circ \mathfrak{m}_{\mathcal{M}} : \mathbb{H}(G) \to \mathbb{C}$ does not depend on $\mathcal{M}$ but only on $\mathcal{M}_P$. So, we denote the map $i_{\mathcal{M}} \circ \mathfrak{m}_{\mathcal{M}}$ by $\mathfrak{m} = \mathfrak{m}_P$. The main result of the present paper is the finding of relations between numbers $\mathfrak{m}(t_i)$, where $t_0 = \tau_p$, $t_i = \tau_{p,i}$, and τ_p, $\tau_{p,i}$ are defined in Sect. 1.1.

Step 1. We use the following notation: if $b_1, \dots, b_g$ is a set of numbers and I is a subset of the set $\{1, \dots, g\}$ then we denote $b_I = \prod_{i \in I} b_i$. In (2.7) we recall a (well-known; see, for example, [4], Sect. 5.1, Example 3) proof of the fact that there exists a set of numbers $a_0, b_1, b_2, \dots, b_g \in \mathbb{C}$ such that eigenvalues of $r(\theta_{\pi_p})$ (where θ_{π_p} is a Langlands element of π_p) have the form $a_0 b_I$ where I runs over the set of all 2^g subsets of $(1, \dots, g)$. Hence, (0.1) means that the Weil numbers of $\mathcal{M}_P$ have the same form.

Step 2. We shall show in Sect. 4 (as a result of calculations of Sect. 3) that for a fixed P numbers $a_0, b_1, b_2, \dots, b_g$ satisfy some relations depending only on P (Prop. 4.3).

Step 3. Finally, using explicit formulas of the Satake map (Sect. 2), we shall show that relations of Prop. 4.3 give us relations between numbers $\mathfrak{m}(t_i)$.

Now we describe steps 2 and 3 in more detail.

Step 2a. Firstly, we recall the description of the set of parabolic subgroups P under consideration: there are 2 types of such subgroups, and subgroups of each type are parametrized by the set of ordered partitions of g, i.e. the set of representations of g as a sum

$$g = \mathfrak{b}_1 + \mathfrak{b}_2 + \dots + \mathfrak{b}_k \tag{0.2}$$

($\mathfrak{b}_i \geq 1$, the order is essential), or, the same, the set of sequences

$$0 = m_1 < m_2 < \dots < m_{k+1} = g$$

where $m_i = \mathfrak{b}_1 + \cdots \mathfrak{b}_{i-1}$.

Step 2b. Using formulas of [4], Sect. 4, we describe explicitly in Sect. 3.3 the set of archimedean cohomological representations belonging to Π_P. Namely, let P be of type 1 given by (0.2). We denote by $\mathfrak{C}$ the set of sequences $\mathfrak{c} = (c_1, \dots, c_k)$ such that $\forall j = 1, \dots, k$ holds $0 \leq c_j \leq \mathfrak{b}_j$. We have: Π_P is isomorphic to $\mathfrak{C}$ factorized by the equivalence relation $(c_1, \dots, c_k) \sim (\mathfrak{b}_1 - c_1, \dots, \mathfrak{b}_k - c_k)$. For P of type 2 the result is the same, but c_1 is omitted. The representation corresponding to $\mathfrak{c} \in \mathfrak{C}$ is denoted by $\pi_{\mathfrak{c}}$.

Step 2c. Now we use formulas of [4], Sect. 4 for the dimensions of $H^{i,j}(\mathfrak{g}, K_c; \pi_{\mathfrak{c}})$, $\mathfrak{g} = \mathfrak{gsp}_{2g}$. We consider for all $i = 1, \dots, k$ the set $\mathfrak{S}(c_i, \mathfrak{b}_i)$ of all subsets of order c_i of the set $\{1, \dots, \mathfrak{b}_i\}$, and we denote

$$\mathfrak{S}(\mathfrak{c}, P) = \prod_{i=1}^{k} \mathfrak{S}(c_i, \mathfrak{b}_i) \text{ or } \mathfrak{S}(\mathfrak{c}, P) = (\mathbb{Z}/2\mathbb{Z})^{\mathfrak{b}_1} \prod_{i=2}^{k} \mathfrak{S}(c_i, \mathfrak{b}_i) \tag{0.3}$$

for P of type 1 and 2 respectively (this is the set of representatives of minimal length for the cosets $\Omega(T, M)/\Omega(T, M \cap wKw^{-1})$ in notations of [4], Sect. 4.3). For $\rho \in \mathfrak{S}(\mathfrak{c}, P)$ an explicit formula for the length $l(\rho)$ is given in (3.5.1).

Further, (3.5.1) gives us relations between Weil numbers of $\mathcal{M}_P$. The exact formula for these relations is given in (4.1). Really, (4.1) is a corollary of a stronger proposition 4.3.

Remark 0.1. A sketch of the description of the structure of $\mathcal{M}_P$ ([2], [4]).

The space generated by $\mathfrak{S}(\mathfrak{c}, P)$ is isomorphic to $\bigoplus_{i,j} H^{i,j}(\mathfrak{g}, K_c; \pi_\mathfrak{c})$, so it is a $\mathfrak{sl}_2$-module with a Hodge structure. We denote this $\mathfrak{sl}_2$-module by $Lie(\mathfrak{S}(\mathfrak{c}, P))$. There are 2 numbers $p_\mathfrak{c}, q_\mathfrak{c}$ associated with each $\mathfrak{c}$ (see [4], Section 4.3 for a formula for them, and (3.4) for explicit values). 2 basis elements of minimal weight in $\mathfrak{S}(\mathfrak{c}, P)$ have Hodge numbers $h^{p_\mathfrak{c}, q_\mathfrak{c}} = h^{q_\mathfrak{c}, p_\mathfrak{c}} = 1$ (case $G = GSp$, $p_\mathfrak{c} \neq q_\mathfrak{c}$; formulas for other Hodge numbers are given for example in [2], Section 9).

There exists a partition of $\mathfrak{C}$:

$$\mathfrak{C} = \bigcup_{i \in \mathfrak{I}} \mathfrak{C}_i \tag{0.1.1}$$

(the union is disjoint) which gives rise to a decomposition of $\mathcal{M}_P$ as a direct sum of submotives. For $\mathfrak{c}_1$, $\mathfrak{c}_2$ a necessary condition to belong to one $\mathfrak{C}_i$ is the following:

(0.1.2) For $j = 1, 2$ numbers $p_{\mathfrak{c}_j} + q_{\mathfrak{c}_j}$ coincide and $\mathfrak{sl}_2$-modules $Lie(\mathfrak{S}(\mathfrak{c}_j, P))$ (but not their Hodge structures!) are isomorphic.

Attached to (0.1.1) is a motive decomposition

$$\mathcal{M}_P = \bigoplus_{i \in \mathfrak{I}} \mathcal{M}_{P,i} \tag{0.1.3}$$

having the following property: $H^*(\mathcal{M}_{P,i})$ has a natural structure of $\mathfrak{sl}_2$-module, and we have an isomorphism of $\mathfrak{sl}_2$-modules with Hodge structures:

$$H^*(\mathcal{M}_{P,i}) = \bigoplus_{\mathfrak{c} \in \mathfrak{C}_i} Lie(\mathfrak{S}(\mathfrak{c}, P)) \tag{0.1.4}$$

and analogously for their components of any fixed weight. It is known that a decomposition of $\mathcal{M}_{P,i}$ in a direct sum indexed by $\mathfrak{c} \in \mathfrak{C}_i$ — like in (0.1.4) — does not exist. Clearly (0.1.3), (0.1.4) give us a description of Hodge numbers of $\mathcal{M}_P$ and primitive elements in the cohomology groups.

See also Appendix, 8 for some explicit properties of $\mathcal{M}_P$, where P is of two simplest types.

Step 2d. To complete Step 2, we must use results of steps 1 and 2c in order to find relations between numbers $a_0, b_1, b_2, \dots, b_g$. These relations are the following (Proposition 4.3):

(0.4) P of the first type: b_{m_i+1} are free variables, $b_{m_i+c} = p^{c-1} b_{m_i+1}$ ($c = 1, \dots, \mathfrak{b}_i$), and a_0 is defined by the equality $a_0^2 \prod b_i = p^{g(g+1)/2}$.

P of the second type: $b_i = p^i$ for $i = 1, \dots, \mathfrak{b}_1$, b_{m_i+1} ($2 \leq i \leq k$) are free variables, b_{m_i+c} and a_0 are like the above.

Step 3. The p-Hecke algebra $\mathbb{H}(G)$ is the ring of polynomials whose generators are denoted by $\tau_{p,*}$, $* = \emptyset, 1, \dots, g$: $\mathbb{H}(G) = \mathbb{Z}[\tau_p, \tau_{p,1}, \dots, \tau_{p,g}]$. Let $\chi : T(\mathbb{Q}_p) \to \mathbb{C}^*$ be a nonramified character such that $\pi_\chi = \pi_p$ where π_p is the p-part of π and $\pi_\chi : G(\mathbb{Q}_p) \to GL(\mathcal{V})$ is the parabolically induced representation. χ does not depend on $\pi \in \Pi_P^{glob}(k)$. π_χ defines an action of $\mathbb{H}(G)$ on a 1-dimensional subspace of $\mathcal{V}^{G(\mathbb{Z}_p)}$ and hence a homomorphism $\alpha_G(\chi) : \mathbb{H}(G) \to \mathbb{C}$. Obviously $\alpha_G(\chi) = \mathfrak{m}$, hence in order to find relations between numbers $\mathfrak{m}(\tau_{p,*})$ we need to represent $\alpha_G(\chi)(\tau_{p,*})$ as polynomials in $a_0, b_1, b_2, \dots, b_g$ and to use (0.4).

To solve this problem we use

(a) the Satake map $S : \mathbb{H}(G) \to \mathbb{H}(T)$ where $\mathbb{H}(T) \overset{i}{\hookrightarrow} \mathbb{Z}[U_*^{\pm 1}, V_*^{\pm 1}]$, ($* = 1, \dots, g$) is the Hecke algebra of a maximal torus T of G;

(b) an explicit expression for a Langlands element $\theta(\pi_\chi) \in \hat{T} \subset {}^L G$ given in (2.7.1), and a decomposition of $r|_{\hat{T}}$ as a sum of characters of $\hat{T}$ (Section 2.6).

Explicit formulas for $S(\tau_{p,*})$ are given in Section 1 ((1.2.1), (1.3.1), (1.5.1)). Further, there exists a map $\alpha_T(\chi) : \mathbb{Z}[U_i^{\pm 1}, V_i^{\pm 1}] \to \mathbb{C}$ such that $\alpha_G(\chi) = \alpha_T(\chi) \circ i \circ S$. (0.1), (2.5.1) and (2.7.1) show us that $\alpha_T(\chi)(V_i) = a_0^{1/g}$, $\alpha_T(\chi)(U_i) = a_0^{1/g} b_i$.

Using explicit formulas for $i \circ S$ (Section 1), we can represent $\alpha_G(\chi)(\tau_{p,*})$ as polynomials in $\alpha_T(\chi)(U_i)$, $\alpha_T(\chi)(V_i)$, i.e. as polynomials in $a_0, b_1, b_2, \dots, b_g$ (2.7.4). The final result follows immediately from (0.4) and (2.7.4).

Structure of the paper. In Section 1.1 we recall the definition of Satake maps S_G, S_T and define generators of Hecke algebras $\mathbb{H}(G)$, $\mathbb{H}(M_s)$. In 1.2, 1.3 we find explicitly S_G of these generators. Remark 1.4 is used only for a proof that the 2 methods of finding of Hecke polynomial give the same result. Remark 1.5 gives a slightly different method of description of the Satake map; some notations of 1.5 will be used later.

Section 2.1 contains a definition of the induced representation and of the corresponding map $\alpha_G(\chi) : \mathbb{H}(G) \to \mathbb{C}$. Sections 2.2 - 2.4 are of survey nature: they contain explicit formulas for $\alpha_G(\chi)$ using the counting of cosets. A formula for $\alpha_G(\chi)$ that will be really used in future is given in 2.5. In 2.6 we recall properties of the map r which is used to define the L-function of $\mathcal{M}$, and in 2.7 we get an expression for Weil numbers of $\mathcal{M}$.

In 3.1 we recall the definition of parabolic subgroups of G and related groups. Contents of other subsections 3.2 — 3.5 correspond to their titles. Finally, Section 4 contains the end of the proof.

1. Explicit description of Satake map

1.1. References: [1], [7]. We let: $T \subset G$ is a torus of diagonal matrices; $M_s = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \right\} \subset G$.

Here we consider elements of p-Hecke algebras $\mathbb{H}(\mathfrak{G})$ ($\mathfrak{G} = G, M_s, T$) as linear combinations of double cosets of $\mathfrak{G}(\mathbb{Z}_p)$. There are inclusions $\mathbb{H}(G) \subset \mathbb{H}(M_s) \subset \mathbb{H}(T)$ defined by Satake maps denoted by S_G, S_T respectively (see [10], [7]).

We need the following matrices:

$T_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, entries are $g \times g$-matrices;

$T_{p,i} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$, diagonal entries are $g-i \times g-i$, $i \times i$, $g-i \times g-i$, $i \times i$-matrices, $i = 0, \dots, g$.

We denote the double cosets $G(\mathbb{Z}_p) T_p G(\mathbb{Z}_p)$, $G(\mathbb{Z}_p) T_{p,i} G(\mathbb{Z}_p)$ (= elements of $\mathbb{H}(G)$) by τ_p, $\tau_{p,i}$ respectively. It is known that $\mathbb{H}(G)$ is the ring of polynomials: $\mathbb{H}(G) = \mathbb{Z}[\tau_p, \tau_{p,1}, \dots, \tau_{p,g}]$. Now we need matrices

$$F_{p,i} = F_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where diagonal entries are $g-i \times g-i$, $i \times i$, $g-i \times g-i$, $i \times i$-matrices, $i = 0, \dots, g$.

We denote the corresponding elements $M_s(\mathbb{Z}_p) F_i M_s(\mathbb{Z}_p)$ of $\mathbb{H}(M_s)$ by Φ_i.

Let us recall the definition of the Satake map S_G. Here we consider for $* = G$ or M_s an element $f \in \mathbb{H}(*)$ as a $*(\mathbb{Z}_p)$-bi-invariant function on $*(\mathbb{Q}_p)$; a function associated to a double coset is its characteristic function. $S_G(f)$ is defined completely by its values on elements $X \in M_s(\mathbb{Q}_p)$ of the form $X = \operatorname{diag}(p^{a_1}, \dots, p^{a_g}, p^{\lambda-a_1}, \dots, p^{\lambda-a_g})$. By definition,

$$S_G(f)(X) = \beta(X) \int_{U(\mathbb{Q}_p)} f(Xu)du \tag{1.1.1}$$

where $\beta(X) = p^{-ga_1-(g-1)a_2-\dots-a_g}$ and $U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$, entries are $g\times g$-matrices (the multiplier $\beta(X)$ differs slightly from the one of [7]).

1.2. Here we apply (1.1.1) to $\tau_p = G(\mathbb{Z}_p)T_pG(\mathbb{Z}_p)$. Let f be its the characteristic function, and $X = F_i$.

For $u = \begin{pmatrix} E_g & A \\ 0 & E_g \end{pmatrix}$ where $A = \begin{pmatrix} u_{11} & u_{12} \\ u_{12}^t & u_{22} \end{pmatrix}$, sizes of diagonal blocks here and below are $g-i$, i, we have $F_iu = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where $B = \operatorname{diag}(1, \dots, 1, p, \dots, p)$, $D = \operatorname{diag}(p, \dots, p, 1, \dots, 1)$, $C = \begin{pmatrix} u_{11} & u_{12} \\ pu_{12}^t & pu_{22} \end{pmatrix}$. Hence, $f(F_iu) = 1 \iff$ entries of $u_{11}, u_{12} \in \mathbb{Z}_p$, entries of $u_{22} \in \frac{1}{p}\mathbb{Z}_p$. This implies that

$$\int_{U(\mathbb{Q}_p)} f(F_iu)du = p^{\frac{i(i+1)}{2}}$$

and $S_G(f)(\Phi_i) = 1$. For other X it is easy to see that $\int f(Xu) = 0$, i.e.

$$S_G(\tau_p) = \Phi_0 + \Phi_1 + \dots + \Phi_g \tag{1.2.1}$$

1.3. Here we apply (1.1.1) to $\tau_{p,i} = G(\mathbb{Z}_p)T_{p,i}G(\mathbb{Z}_p)$, $i \geq 1$. Let f be its characteristic function, and $X = F_jF_k$. We have

$F_jF_k = \operatorname{diag}(1, \dots, 1, p, \dots, p, p^2, \dots, p^2, p^2, \dots, p^2, p, \dots, p, 1, \dots, 1)$, $k > j$, sizes of diagonal blocks here and below are $g-k$, $k-j$, j, $g-k$, $k-j$, j.

For $u = \begin{pmatrix} E_g & A \\ 0 & E_g \end{pmatrix}$ where $A = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12}^t & u_{22} & u_{23} \\ u_{13}^t & u_{23}^t & u_{33} \end{pmatrix}$ we have $F_jF_ku = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where $B = \operatorname{diag}(1, \dots, 1, p, \dots, p, p^2, \dots, p^2)$, $D = \operatorname{diag}(p^2, \dots, p^2, p, \dots, p, 1, \dots, 1)$, $C = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ pu_{12}^t & pu_{22} & pu_{23} \\ p^2u_{13}^t & p^2u_{23}^t & p^2u_{33} \end{pmatrix}$. Hence, $f(F_jF_ku) = 1 \iff$ entries of $u_{11}, u_{12}, u_{13} \in \mathbb{Z}_p$, entries of $u_{22}, u_{23} \in \frac{1}{p}\mathbb{Z}_p$, entries of $u_{33} \in \frac{1}{p^2}\mathbb{Z}_p$, $\operatorname{rank}(\widetilde{pu_{22}}) = k-j-i$, where tilde means the residue map $\mathbb{Z}_p \to \mathbb{F}_p$. (This is because for a symmetric $g \times g$-matrix A such that $\operatorname{rank} \tilde{A} = r$ we have $\begin{pmatrix} p & A \\ 0 & p \end{pmatrix} \in G(\mathbb{Z}_p)T_{p,g-r}G(\mathbb{Z}_p)$).

So, we denote by $R_g(i) = R_g(i,p)$ the quantity of symmetric $g \times g$-matrices with entries in $\mathbb{F}_p$ of corank exactly i (see [1], Chapter 3, Lemma 6.19 for the formula for $R_g(i)$) and we have

$$\int_{U(\mathbb{Q}_p)} f(F_jF_ku)du = R_{k-j}(i) \cdot p^{j(k-j)+j(j+1)}$$

and

$$S_G(\tau_{p,i})(F_jF_k) = \beta(F_jF_k) \int_{U(\mathbb{Q}_p)} f(F_jF_ku)du = R_{k-j}(i) \cdot p^{-\binom{k-j+1}{2}}$$

For other X it is easy to see that $\int f(Xu) = 0$, i.e. we have $(i \geq 1)$:

$$S_G(\tau_{p,i}) = \sum_{j,k\geq 0, j+i\leq k}^{g} R_{k-j}(i) \cdot p^{-\binom{k-j+1}{2}} \Phi_j \Phi_k \tag{1.3.1}$$

Remark 1.4. The above formulas can be used for finding the Hecke polynomial of X. Any element of $\mathbb{H}(G)$ defines a correspondence on X. We denote the algebra of these correspondences by $\mathbb{T}_p$. It is the quotient ring of $\mathbb{H}(G)$ by the only relation $\tau_{p,g} = \mathrm{id}$.

Let us consider the (good) reduction of X at p, denoted by $\tilde{X}$. We denote by $\mathrm{Corr}\ (\tilde{X})$ its algebra of correspondences. Obviously there exists an inclusion $\gamma : \mathbb{T}_p \to \mathrm{Corr}\ (\tilde{X})$. It is known that it can be included in the commutative diagram:

$$\begin{array}{cccc} S_G: & \mathbb{H}(G) & \to & \mathbb{H}(M_s) \\ & \beta_1 \downarrow & & \beta_2 \downarrow \\ \gamma: & \mathbb{T}_p & \to & \mathrm{Corr}\ (\tilde{X}) \end{array}$$

where β_1 is the natural projection, β_2 is an epimorphism with the same kernel $\tau_{p,g} - \mathrm{id}$.

There is the Frobenius map $f : \tilde{X} \to \tilde{X}$, we can consider it as a correspondence, i.e. $f \in \mathrm{Corr}\ (\tilde{X})$. We have: $f = \beta_2(\Phi_0)$ in $\mathrm{Corr}\ (\tilde{X})$, and $\beta_2(\Phi_g)$ is the Verschiebung correspondence. The minimal polynomial satisfied by f over $\mathbb{T}_p$ is called the Hecke polynomial.

An explicit algorithm for finding the Hecke polynomial is a by-product of the calculations of the present paper. There are 2 methods for finding this polynomial: the first one is to eliminate formally $\Phi_1, \dots, \Phi_g$ from (1.2.1), (1.3.1) and to use the relation $\tau_{p,g} = 1$. The second one is to use a description of Langlands parameters of unramified representations — this gives us formula (2.7.2). See Appendix, Table 4 for the explicit formulas for the cases $g = 2, 3$.

Remark 1.5. There is a slightly different method of finding the Hecke polynomial. We denote by $\Omega(G)$ the Weyl group of G. It enters in the exact sequence

$$0 \to (\mathbb{Z}/2\mathbb{Z})^g \to \Omega(G) \to S(g) \to 0$$

and there exists a section $i : S(g) \to \Omega(G)$. Let U_i, V_i $(i = 1, \dots, g)$ be independent variables. We have: (see [7], Ch. 7 for example) $\mathbb{H}(T)$ is a subring of $\mathbb{Q}[U_i^{\pm 1}, V_i^{\pm 1}]$ generated by $(U_i V_i^{-1})^{\pm 1}$ and $\prod_{i=1}^{g} U_i$. $\Omega(G)$ acts on $\mathbb{H}(T)$ in the obvious manner ($S(g)$ permutes indices in U_i, V_i, and $(\mathbb{Z}/2\mathbb{Z})^g$ interchanges U, V). Then $\mathbb{H}(G)$, $\mathbb{H}(M_s)$ are subrings of $\mathbb{H}(T)$ stable with respect to $\Omega(G)$, $i(S(g))$ respectively, and Satake maps S_G, S_T are identical inclusions.

For a subset I of $1, \dots, g$ we denote $U_I = \prod_{i \in I} U_i \prod_{i \notin I} V_i \in \mathbb{H}(T)$. Then we have:

$$S_T(\Phi_i) = \sum_{\#(I)=i} U_I \tag{1.5.1}$$

(particularly, $\prod_{i=1}^{g} V_i$ is the Frobenius element and $\prod_{i=1}^{g} U_i$ is the Verschiebung). Using (1.2.1), (1.3.1) and (1.5.1) it is easy to find images of τ_p, $\tau_{p,i}$ in $\mathbb{Q}[U_i^{\pm 1}, V_i^{\pm 1}]$ (for example, $\tau_p = \sum_I U_I = \prod_{i=1}^{g}(U_i + V_i)$).

Roots of Hecke polynomial are $(\mathbb{Z}/2\mathbb{Z})^g$-conjugates of $\prod_{i=1}^{g} V_i$, i.e. elements U_I. We denote the i-th coefficient of the Hecke polynomial by $\mathfrak{h}_i \in \mathbb{H}(G)$. Hence, $\mathfrak{h}_i = (-1)^i \sigma_i(U_I)$, $i = 0, \dots, 2^g$, where σ_i is the i-th symmetric polynomial. $\mathfrak{h}_i$ can be found explicitly using (1.2.1), (1.3.1).

2. Description of Weil numbers of $\mathcal{M}_P$

2.1. Let $T \subset B \subset G$ be the standard Borel pair, i.e. T is as above and $B = \left\{ \begin{pmatrix} (D^t)^{-1} & * \\ 0 & D \end{pmatrix} \in G \;\middle|\; D \text{ is an upper-triangular } g \times g\text{-matrix} \right\}$.

Let $\chi : T(\mathbb{Q}_p) \to \mathbb{C}^*$ be a nonramified character. χ is defined uniquely by the numbers

$$a_0 = \chi\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right), a_i = \chi\left(\begin{pmatrix} \rho_i & 0 \\ 0 & \rho_i^{-1} \end{pmatrix} \right)$$

where $\rho_i =$ diag $(1, \dots, p, \dots, 1)$, p being at the i-th place, $i = 1, \dots, g$. It is convenient to denote $b_i = p^i a_i$.

From here and until (2.7) we shall assume that χ is arbitrary, i.e. b_i are arbitrary numbers. From (2.7) we shall treat only one χ defined in Introduction, Step 3.

We can expand χ on $B(\mathbb{Q}_p)$ using the projection $B \to T$, and let $\pi_\chi : G(\mathbb{Q}_p) \to GL(\mathcal{V})$ be the parabolically induced representation. Recall its definition: $\mathcal{V}$ is a space of functions $f : G(\mathbb{Q}_p) \to \mathbb{C}$ which satisfy

$$\forall b \in B(\mathbb{Q}_p) \quad f(bg) = \chi(b) \cdot f(g)$$

and the action is right translation:

$$[\pi_\chi(t)(f)](g) = f(gt)$$

There exist a 1-dimensional subspace $\mathcal{V}^{G(\mathbb{Z}_p)} \subset \mathcal{V}$ of $G(\mathbb{Z}_p)$-invariant functions, an action of $\mathbb{H}(G)$ on $\mathcal{V}^{G(\mathbb{Z}_p)}$ and hence a homomorphism $\alpha_G(\chi) : \mathbb{H}(G) \to \mathbb{C}$.

There are 2 methods of description of $\alpha_G(\chi)$: the first one is based on consideration of decomposition of a double coset $G(\mathbb{Z}_p)TG(\mathbb{Z}_p)$, $T \in G$, as a union of ordinary cosets. Really, if $G(\mathbb{Z}_p)TG(\mathbb{Z}_p) = \cup_i \gamma_i G(\mathbb{Z}_p)$ then $\alpha_G(\chi)(G(\mathbb{Z}_p)TG(\mathbb{Z}_p)) = \sum_i \chi(\gamma_i)$. We treat this decomposition in Sections 2.2 - 2.4.

The second method (which is much more convenient) is treated in 2.5. So, Sections 2.2 -2.4 are entirely of survey nature.

2.2. Here we consider for simplicity the case of $G = GL_n$ and a double coset $G(\mathbb{Z}_p)T_{p,i}G(\mathbb{Z}_p)$ for $T_{p,i} =$ diag $(1, \dots, 1, p, \dots, p)$, p occurs i times. This coset decomposition is the following:

$$G(\mathbb{Z}_p)T_{p,i}G(\mathbb{Z}_p) = \bigcup_I \bigcup_{\{c_{jk}\}} \gamma_{I,\{c_{jk}\}} G(\mathbb{Z}_p)$$

where I runs through the set of all subsets of $\{1, \dots, n\}$ containing i elements, c_{jk} belongs to a fixed set of representatives of $\mathbb{F}_p$ in $\mathbb{Z}$, $c_{jk} = 0$ unless $j \notin I$, $k \in I$, $j < k$, and

$$\gamma_{I,\{c_{jk}\}} = \sum_{j \in I} p \cdot e_{jj} + \sum_{j \notin I} e_{jj} + \sum_{j,k} c_{jk} \cdot e_{kj}$$

($j \notin I$, $k \in I$, $j < k$), where e_{lm} are elementary matrices.

We can transform the above decomposition as follows:

$$G(\mathbb{Z}_p)T_{p,i}^{-1}G(\mathbb{Z}_p) = \bigcup_* G(\mathbb{Z}_p)\gamma_*^{-1} = \bigcup_* {\gamma_*^{-1}}^t G(\mathbb{Z}_p);$$

$$G(\mathbb{Z}_p)pT_{p,i}^{-1}G(\mathbb{Z}_p) = G(\mathbb{Z}_p)T_{p,n-i}G(\mathbb{Z}_p) = \bigcup_* p{\gamma_*^{-1}}^t G(\mathbb{Z}_p)$$

So, we have:

$$\gamma^{-1}_{I,\{c_{jk}\}}=\sum_{j\in I}p^{-1}\cdot e_{jj}+\sum_{j\notin I}e_{jj}+\sum_{j,k}-p^{-1}c_{jk}\cdot e_{kj}$$

$(j\notin I,\ k\in I,\ j<k)$ and hence

$$p\gamma_*^{-1^t}=\sum_{j\in I}e_{jj}+\sum_{j\notin I}pe_{jj}+\sum_{j,k}-c_{jk}\cdot e_{jk}$$

$(j\notin I,\ k\in I,\ j<k)$. These elements are in B. Further, for a fixed I we have

$$\chi(p\gamma^{-1}_{I,\{c_{jk}\}}{}^t)=\prod_{i\notin I}a_i$$

and hence

$$\alpha_G(\chi)(T_{p,n-i})=\sum_{I,\{c_{jk}\}}\chi(p\gamma^{-1}_{I,\{c_{jk}\}}{}^t)=\sum_{I,\#I=i}\prod_{i\notin I}a_i\cdot p^{\#\{(j,k)|j\notin I,k\in I,j<k\}}$$

which gives us

$$\alpha_G(\chi)(T_{p,n-i})=p^{-\frac{i(i+1)}{2}}\sigma_i(b_*)$$

2.3. Here we consider the case $G=GSp_{2g}(\mathbb{Q})$, $T=T_p$. We have the following decomposition: $G(\mathbb{Z}_p)T_pG(\mathbb{Z}_p)=\cup_iG(\mathbb{Z}_p)\gamma_i$ where the set $\{\gamma_i\}$ is described as follows:

1. We consider all subsets $I\subset\{1,\dots,g\}$ (there are 2^g of them);
2. If such I is fixed then we consider the set of $\gamma=\begin{pmatrix}A&B\\0&D\end{pmatrix}$ such that

$$D=\sum_{j\in I}p\cdot e_{jj}+\sum_{j\notin I}e_{jj}+\sum_{j,k}c_{jk}\cdot e_{jk},$$

$$A=pD^{t^{-1}}=\sum_{j\in I}e_{jj}+p\cdot\sum_{j\notin I}e_{jj}+\sum_{j,k}-c_{jk}\cdot e_{kj},$$

$(j\notin I,\ k\in I,\ j<k)$

$$B=\sum_{j,k\in I}b_{jk}e_{jk},$$

b_{jk}, c_{jk} belong to a fixed set of representatives of $\mathbb{F}_p$ in $\mathbb{Z}$, and $b_{jk}=b_{kj}$.

Now we use the same transformations as above. We have: $\chi(p\gamma_i^{-1})=a_0\prod_{i\in I}a_i$ and it is easy to see that

$$\alpha_G(\chi)(T_p)=a_0\prod_{i=1}^{g}(1+b_i)\tag{2.3.1}$$

2.4. Here we consider the case $G=GSp_{2g}(\mathbb{Q})$, $T=T_{p,i}$. Firstly we describe a set J such that

$$\bigcup_{i=0}^{g}G(\mathbb{Z}_p)T_{p,i}G(\mathbb{Z}_p)=\cup_{j\in J}G(\mathbb{Z}_p)\gamma_j$$

and then for each $j\in J$ we find the corresponding $i\in 0,\dots,g$.

We have: $\gamma_j=\begin{pmatrix}A&B\\0&D\end{pmatrix}\in GSp_{2g}$ with $\lambda(\gamma_j)=p^2$. D is an upper-triangilar matrix whose diagonal entries D_{ii} are p^{d_i}, $d_i=0,1,2$, i.e. we have 3^g possibilities for the choice of d_i. To choose a set d_i is the same as to choose a partition $\{1,\dots,g\} = I_0 \cup I_1 \cup I_2$, $i \in I_k \iff d_i = k$. Non-diagonal entries of D are described as follows:

(1) If $i \in I_0, j \in I_1, i < j$ then D_{ij} runs through a system of representatives in $\mathbb{Z}$ of $\mathbb{Z}/p$;

(2) If $i \in I_0, j \in I_2, i < j$ then D_{ij} runs through a system of representatives in $\mathbb{Z}$ of $\mathbb{Z}/p^2$;

(3) If $i \in I_1, j \in I_1, i < j$ then D_{ij} runs through a system of representatives in $\mathbb{Z}$ of $\mathbb{Z}/p$, and the Jordan normal form of this part of D has blocks of size 1 or 2 (i.e. its square is 0);

(4) If $i \in I_1, j \in I_2, i < j$ then $D_{ij} = pD'_{ij}$, where D'_{ij} runs through a system of representatives in $\mathbb{Z}$ of $\mathbb{Z}/p$;

Other D_{ij} are 0. We denote submatrices of D described in (1) - (4) above by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $p\mathfrak{D}$ respectively. Further, we have $A = p^2 D^{-1t}$, and the description of $B = \{B_{ij}\}$ is the following. Firstly, $B_{ij} = 0$ if $i \in I_0$ or $j \in I_0$. Further, we denote submatrices of B formed by elements B_{ij} with $i \in I_r$, $j \in I_s$ $(r, s = 1, 2)$ by $\mathfrak{B}_{rs}$. Entries of $\mathfrak{B}_{11}$, $\mathfrak{B}_{21}$ (resp. $\mathfrak{B}_{12}$, $\mathfrak{B}_{22}$) run through a system of representatives in $\mathbb{Z}$ of $\mathbb{Z}/p$, (resp. of $\mathbb{Z}/p^2$).

Finally, the above matrices satisfy the following relations (which are equivalent to a condition $\gamma_j \in GSp_{2g}$):

(1) $\mathfrak{B}_{11}^t(pI + \mathfrak{C}) = (pI + \mathfrak{C}^t)\mathfrak{B}_{11}$

(2) $(pI + \mathfrak{C}^t)\mathfrak{B}_{12} = p\mathfrak{B}_{11}^t\mathfrak{D} + p^2\mathfrak{B}_{21}^t$

(3) $\mathfrak{D}^t\mathfrak{B}_{12} + p\mathfrak{B}_{22} = \mathfrak{B}_{12}^t\mathfrak{D} + p\mathfrak{B}_{22}^t$

For a given γ_j it is possible to find i such that $\gamma_j \in G(\mathbb{Z}_p)T_{p,i}G(\mathbb{Z}_p)$. It is obvious that $i \le \#I_1$.

For each set $\mathfrak{d} = \{d_i\}$ we denote by $C(\mathfrak{d}, k)$ the quantity of matrices γ_j described above such that $\gamma_j \in G(\mathbb{Z}_p)T_{p,k}G(\mathbb{Z}_p)$. In these notations we have the following formula:

$$\alpha_G(\chi)(T_{p,k}) = \sum_{\mathfrak{d}} C(\mathfrak{d}, k) \prod_{i=1}^{g} a_i^{d_i}$$

Really, it is more convenient to denote $\tilde{C}(\mathfrak{d}, k) = C(\mathfrak{d}, k)p^{-\sum_{i=1}^{g} id_i}$, so

$$\alpha_G(\chi)(T_{p,k}) = \sum_{\mathfrak{d}} \tilde{C}(\mathfrak{d}, k) \prod_{i=1}^{g} b_i^{d_i} \tag{2.4.1}$$

Formulas for $\tilde{C}(\mathfrak{d}, k)$ and $\alpha_G(\chi)(T_{p,k})$ for $g = 2, 3$ are given in the appendix, tables 5, 6.

2.5. It is well-known that there exists a map $\alpha_T(\chi) : \mathbb{H}(T) \to \mathbb{C}$ given by the formula

$$\alpha_T(\chi)(V_i) = a_0^{1/g}, \quad \alpha_T(\chi)(U_i) = a_0^{1/g} b_i \tag{2.5.1}$$

such that

$$\alpha_G(\chi) = \alpha_T(\chi) \circ S_T \circ S_G \tag{2.5.2}$$

Combining (2.5.1), (2.5.2) with (1.2.1), (1.3.1), (1.5.1), we get

$$\alpha_G(\chi)(\tau_{p,i}) = a_0^2 \sum_{j,k \ge 0, j+i \le k}^{g} R_{k-j}(i) \cdot p^{-\binom{k-j+1}{2}} \sum_{\#(J)=j} b_J \sum_{\#(K)=k} b_K \tag{2.5.3}$$

Comparing (2.4.1) and (2.5.3) we get immediately that for $i = 1, \dots, g$

$$\tilde{C}(\mathfrak{d}, i) = \sum_{j=0}^{[(q_1-1)/2]} R_{q_1-2j}(i) p^{-(q_1-2j+1)(q_1-2j)/2} \binom{q_1}{j} \tag{2.5.4}$$

where $q_1 = \#I_1$ is the quantity of ones in $\mathfrak{d}$ and $[x]$ is the integer part of x.

2.6. Here we recall a description of the finite-dimensional representation $r : {}^LG \to GL(W)$ ([4], 5.1), and its restriction to $\hat{T} \subset {}^LG$ for our case $G = GSp_{2g}(\mathbb{Q})$ (this is well-known, see for example [4], 5.1, Example C). So, firstly we describe the spin representation and its restriction to Cartan subalgebra. The following facts can be found in many sources; we use [8].

The dual of GSp_{2g} is the spinor group $GSpin_{2g+1}$. Since $G = GSp_{2g}$ is over $\mathbb{Q}$, we have: ${}^LG = W_{\mathbb{Q}} \times GSpin_{2g+1}$, and $r : {}^LG \to GL(W)$ (see, for example, [4], (5.1)) is trivial on $W_{\mathbb{Q}}$. It is known that $r : GSpin_{2g+1} \to GL(W)$ is the spin representation. There exists a 2-fold covering $\eta : GSpin_{2g+1} \to GO_{2g+1}$. Recall the definition of the corresponding representation of Lie algebras $\mathfrak{r} : \mathfrak{GO}_{2g+1} \to GL(W)$.

Let V be a vector space of dimension $2g+1$, $u_1, \dots, u_{2g+1}$ its basis and B a quadratic form whose matrix in this basis is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, the size of diagonal entries is $1, g, g$. We consider the corresponding orthogonal Lie algebras $\mathfrak{GO}(B)$, $\mathfrak{O}(B)$. Their Cartan subalgebras of diagonal matrices $\mathfrak{TGO}(B)$, resp. $\mathfrak{TO}(B)$ have bases $\theta_0, \theta_1, \dots, \theta_g$, resp. $\theta_1, \dots, \theta_g$, where θ_0 is the $2g+1 \times 2g+1$ unit matrix and $\theta_i = e_{i+1,i+1} - e_{i+g+1,i+g+1}$ for $i > 0$, e_{ij} is an elementary matrix ([8], p. 139, (63)).

The Clifford algebra $C = C(V, B)$ is the quotient of $\sum_{n=0}^{\infty} V^{\otimes n}$ (the tensor algebra of V) by relations $v_1 \otimes v_2 + v_2 \otimes v_1 = 2B(v_1, v_2)$. Let $L(C)$ be the corresponding Lie algebra, $M_1 \subset C$ the natural projection of $V = V^{\otimes 1} \subset \sum_{n=0}^{\infty} V^{\otimes n}$ to C, and $M_2 = [M_1, M_1]$. It is known ([8], p. 231, Th. 7) that M_2 is a Lie subalgebra of $L(C)$, and it is isomorphic to $\mathfrak{O}(B)$. Further, M_1 is isomorphic to V as a vector space, and the Lie action of M_2 on M_1 defined by the formula $x(y) = xy - yx$ (here $x \in M_2$, $y \in M_1$, multiplication is in C), coincides with the action (of a matrix on a vector) of $\mathfrak{O}(B)$ on V.

This formula permits us to get an explicit identification of $\mathfrak{O}(B)$ and M_2. Namely, we denote $v_i = u_1 u_{i+1}$, $w_i = u_1 u_{i+1+g}$, multiplication is in C, v_i, $w_i \in M_2$. We have:

$$\text{for } i > 0 \quad \theta_i \in \mathfrak{O}(B) \text{ corresponds to } \frac{1}{2} + \frac{1}{2} v_i w_i \in M_2 \tag{2.6.1}$$

(calculations are similar to [8], p. 233, (34) or can be deduced easily from these formulas; it is necessary to take into consideration that h_i of page 139 are θ_i and h_i of page 233 are $\theta_i - \theta_{i+1}$).

For $I = (\alpha_1, \dots, \alpha_k) \subset (1, \dots, g)$ we set $x_I = v_1 \cdot \ldots \cdot v_g \cdot w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_k}$. The space of spin representation W is a subspace of C spanned on all vectors x_I. The action of $\mathfrak{O}(B)$ is the right multiplication by the corresponding elements of M_2. This is exactly $\mathfrak{r}$ restricted on $\mathfrak{O}(B)$. (2.6.1) shows that

$$\theta_i(x_I) = \epsilon x_I \tag{2.6.2}$$

where $\epsilon = \frac{1}{2}$ if $i \in I$ and $\epsilon = -\frac{1}{2}$ if $i \notin I$.

Finally, it is known that θ_0 acts on W by multiplication by $\frac{1}{2}$.

Let $\hat{T} \subset GSpin_{2g+1}$ be the dual torus of $T \subset GSp_{2g}$, $\hat{\mathfrak{T}}$ its Lie algebra and $\eta_{Lie} : \hat{\mathfrak{T}} \to \mathfrak{TGO}(B)$ the restriction of η. For $t \in T$ we set $t = \operatorname{diag}(x_1, \dots, x_g, \lambda x_1^{-1}, \dots, \lambda x_g^{-1})$, so we can consider $\lambda, x_1, \dots, x_g$ as a basis of $X^*(T)$. We denote the dual basis of $X^*(\hat{T})$ by $\lambda', x_1', \dots, x_g'$ and we consider $\lambda', x_1', \dots, x_g'$ as coordinates of an element $t \in \hat{T}$. Further, we denote by $\nu_0, \nu_1, \dots, \nu_g$ the basis of $\hat{\mathfrak{T}}$ dual to $\lambda', x_1', \dots, x_g'$. Formulas for η_{Lie} in bases ν_*, θ_* are the following:

$$\eta_{Lie}(\nu_0) = 2\theta_0, \quad \eta_{Lie}(\nu_i) = \theta_i + \theta_0 \tag{2.6.3}$$

(2.6.2), (2.6.3) imply formulas for the action of ν_i on x_I: $\mathfrak{r}(\nu_i)(x_I) = x_I$ if $i = 0$ or $i \in I$, $\mathfrak{r}(\nu_i)(x_I) = 0$ if $i \notin I$. In its turn, these formulas imply formulas for $r|_{\hat{T}}$:

$$r(\lambda', x_1', \dots, x_g')(x_I) = (\lambda' \prod_{i \in I} x_i') x_I \tag{2.6.4}$$

(see also [4], end of (5.1)).

2.7. Here we apply r to a Langlands element $\theta_{\pi_\chi} \in {}^L G$ of π_χ in order to find Weil numbers of $\mathcal{M}_P$.

We can choose θ_{π_χ} in $\hat{T}$; namely, it is known that $\lambda', x_1', \dots, x_g'$-coordinates of θ_{π_χ} are

$$(a_0, b_1, \dots, b_g) \tag{2.7.1}$$

(2.6.4) and (2.7.1) imply that

(2.7.2) $\forall I \subset \{1, \dots, g\}$ the element x_I is an eigenelement of $r(\theta_{\pi_\chi})$ with eigenvalue $a_0 b_I = \alpha_T(\chi)(U_I)$.

From here we fix χ such that $\pi_\chi = \pi_p$ — the p-part of π of Introduction. (2.7.2) and (0.1) give us immediately

(2.7.3) The 2^g Weil numbers of $\mathcal{M}_P$ have the form $a_0 b_I$.

Moreover, the existence of pairing in cohomology of X shows that numbers b_i satisfy a relation $a_0^2 \prod_{i=1}^g b_i = p^{g(g+1)/2}$ ($\iff$ $a_0^2 \prod_{i=1}^g a_i = 1$).

Since $\alpha_G(\chi) = \mathfrak{m}$ ($\mathfrak{m}$ of the Introduction), (2.3.1), (2.5.3) and (2.7.3) give us expressions of $\mathfrak{m}(\tau_{p,*})$ in terms of Weil numbers of $\mathcal{M}_P$:

$$\mathfrak{m}(\tau_p) = \sum_{I \in 2^g} a_0 b_I \tag{2.7.4a}$$

$$\mathfrak{m}(\tau_{p,i}) = a_0^2 \sum_{j,k \geq 0, j+i \leq k}^{g} R_{k-j}(i) \cdot p^{-\binom{k-j+1}{2}} \sum_{\#(J)=j} b_J \sum_{\#(K)=k} b_K \tag{2.7.4b}$$

where b_i should be interpreted as numbers entering in the formula (2.7.3) for Weil numbers of $\mathcal{M}_P$.

Remark. The above formulas give us a simple proof that $\alpha_G(\chi)(\sum \mathfrak{h}_i T^i)$ is the characteristic polynomial of $r(\theta_{\pi_\chi})$ (this is well-known; see, for example, [5] for less explicit proof in a more general situation). Really, roots of the Hecke polynomial are U_I (I runs over 2^g), and $\alpha_T(\chi)(U_I) = r_I(\theta_{\pi_\chi})$.

3. Some explicit formulas for archimedean cohomological representations of G

3.1. Description of parabolic subgroups of G.

The set of simple positive roots that corresponds to a Borel pair (T, B) of G is:

$$\omega_0 = x_1^2 \lambda^{-1}, \omega_i = x_{i+1} x_i^{-1}, \quad i = 1, \dots, g-1,$$

λ, x_i of 2.6. We denote this set by Δ.

Parabolic subgroups that contain B are in one-to-one correspondence to the set of subsets of Δ. We shall tell that such a subgroup is of the first type if the corresponding subset of Δ does not contain ω_0, and of the second type, if it contains ω_0. The set of subgroups of both types is isomorphic to the set of ordered partitions of g, i.e. the set of representations of g as a sum

$$g = \mathfrak{b}_1 + \mathfrak{b}_2 + \cdots + \mathfrak{b}_k \tag{3.1.1}$$

where $\mathfrak{b}_i \geq 1$, the order is essential. We denote $m_i = \mathfrak{b}_1 + \cdots \mathfrak{b}_{i-1}$ $(i = 1, \ldots, k)$. The subset of Δ that corresponds to (3.1.1) is $\Delta - \{\omega_0, \omega_{m_2}, \omega_{m_3}, \ldots, \omega_{m_k}\}$ for the first type, $\Delta - \{\omega_{m_2}, \omega_{m_3}, \ldots, \omega_{m_k}\}$ for the second type. We denote the corresponding parabolic subgroup by P and its Levi decomposition by $P = MN$. Their description is the following:

First type:

$$M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \tag{1M}$$

where A, D are block diagonal matrices with sizes of blocks $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_k$. We denote block entries by A_i, D_i respectively $(i = 1, \ldots, k)$;

$$P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \tag{1P}$$

where A (resp. D) is a lower (resp. upper) block triangular matrix (with the same size of blocks clearly), and

(1N) $N \subset P$ is its subset of matrices whose block entries are unit matrices.

For the second type we have

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2M}$$

where A, D are like in (1M), and B, C contain only the upper left corner of size $\mathfrak{b}_1$ of non-zero elements. These matrices are denoted by B_1, C_1 respectively; clearly the $2\mathfrak{b}_1 \times 2\mathfrak{b}_1$-matrix $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in GSp_{2\mathfrak{b}_1}$;

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2P}$$

where A, D are like in (1P), C is like in (2M);

$$N = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \tag{2N}$$

where A, D are like in (1N), and the upper left corner of size $\mathfrak{b}_1$ of B is the 0-matrix.

To apply formulas of [4] we need to describe a Borel pair (T_c, B_c) such that $T_c(\mathbb{R})$ is a compact modulo $Z(\mathbb{R})$, where Z is the center of G. Namely,

T_c is the set of matrices $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$ where X, Y are diagonal $g \times g$-matrices such that $X^2 + Y^2 = \lambda E_g$.

Let $\alpha = \begin{pmatrix} A & iD \\ iA & D \end{pmatrix}$ $(i = \sqrt{-1})$ where A, D are any scalar $g \times g$-matrices such that $AD = \frac{1}{2}$; let $A = D = \frac{1}{\sqrt{2}} E_g$. We have

$$T_c = \alpha T \alpha^{-1} \tag{3.1.2}$$

We denote $M_c = \alpha M \alpha^{-1}$ and analogically for other objects (N, K etc.).

3.2. Finding of $\Omega_{\mathbb{R}}(G)$.

Here we recall an explicit description of $\Omega_{\mathbb{R}}(G)$ which is necessary for finding Π_P, see 3.3 below. It is possible to use the fact that it contains a subgroup $\Omega(K_c)$ of index 2, but we give a direct calculation. We denote the normalizer by $\mathfrak{N}$. There is an isomorphism $\mathfrak{N}(T_c)/T_c = \Omega(G)$ and a section of sets $\gamma : \Omega(G) \to \mathfrak{N}(T_c)$. (3.1.2) implies that $\mathfrak{N}(T_c) = \alpha\mathfrak{N}(T)\alpha^{-1}$. Let $j = 1, \dots, g$, $e_j = (1, \dots, 1, -1, 1, \dots, 1) \in (\mathbb{Z}/2\mathbb{Z})^g \subset \Omega(G)$ (-1 is at the j-th place). A representative of e_j in $\mathfrak{N}(T)$ is $\mathfrak{w}_j = \begin{pmatrix} E_g - e_{jj} & ie_{jj} \\ ie_{jj} & E_g - e_{jj} \end{pmatrix}$. It commutes with α, i.e. we can set $\gamma(e_j) = \mathfrak{w}_j$. Equality $i\prod_{j=1}^g \mathfrak{w}_j = \begin{pmatrix} 0 & -E_g \\ -E_g & 0 \end{pmatrix} \in G(\mathbb{R})$ shows that a representative of $(-1, \dots, -1) \in (\mathbb{Z}/2\mathbb{Z})^g \subset \Omega(G)$ belongs to $\Omega_{\mathbb{R}}(G)$.

Further, for $\mathfrak{w} \in S(g) \subset \Omega(G)$ we denote by $M_{\mathfrak{w}}$ the $g \times g$-matrix whose (j,k)-th entry is $\delta_j^{\mathfrak{w}(k)}$ (the matrix of permutation). Then we have $\gamma(\mathfrak{w}) = \begin{pmatrix} M_{\mathfrak{w}} & 0 \\ 0 & M_{\mathfrak{w}} \end{pmatrix}$. It belongs to $\mathfrak{N}(T)$, $\mathfrak{N}_c(T)$ and commutes with α.

This means that $\Omega_{\mathbb{R}}(G)$ contains a subgroup $X \subset \Omega(G)$ given by an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to X \to S(g) \to 0$$

where $\mathbb{Z}/2\mathbb{Z} \subset (\mathbb{Z}/2\mathbb{Z})^g$ is the diagonal embedding. Really, it is possible to show that $X = \Omega_{\mathbb{R}}(G)$, i.e. elements of $(\mathbb{Z}/2\mathbb{Z})^g$, except the diagonal element, cannot be lifted to $G(\mathbb{R})$.

Finally, for a subset I of $\{1, \dots, g\}$ — or, the same, an element $I \in (\mathbb{Z}/2\mathbb{Z})^g \subset \Omega(G)$ — we set $\gamma(I) = \prod_{j \in I} \mathfrak{w}_j$, and we denote this element by $\mathfrak{w}_I$.

3.3. Finding of Π_P.

The members of Π_P are parametrized by the double coset space

$$\Omega(M_c)\backslash\Omega(G)/\Omega_{\mathbb{R}}(G)$$

([4], 4.2). We have: $\Omega(M_c) = S(\mathfrak{b}_1) \times \dots \times S(\mathfrak{b}_k)$ for P of the first type and $\Omega(M_c) = \Omega(GSp_{2\mathfrak{b}_1}) \times S(\mathfrak{b}_2) \times \dots \times S(\mathfrak{b}_k)$ for P of the second type. The set of representatives of $\Omega(G)/\Omega_{\mathbb{R}}(G)$ can be chosen as half of $(\mathbb{Z}/2\mathbb{Z})^g$ (we choose one element in each pair of elements $(a, (-1, \dots, -1)a)$, $a \in (\mathbb{Z}/2\mathbb{Z})^g$). The above groups $\Omega(M_c)$ act on this set of representatives from the left, hence the invariant of their action is the quantity of $1, -1$ in the segments of length $\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_k$ (first type); $\mathfrak{b}_2, \dots, \mathfrak{b}_k$ (second type) in the whole segment of length g. This means that the set Π_P coincides with

First type: the set of sequences of numbers $c_1, \dots, c_k$ where $0 \le c_j \le \mathfrak{b}_j$ factorized by the equivalence relation $c_1, \dots, c_k \sim \mathfrak{b}_1 - c_1, \dots, \mathfrak{b}_k - c_k$; representatives w of the corresponding double cosets are

$$w = (\underbrace{1, \dots, 1}_{c_1 \text{ times}}, \underbrace{-1, \dots, -1}_{\mathfrak{b}_1 - c_1 \text{ times}}, \dots, \underbrace{1, \dots, 1}_{c_k \text{ times}}, \underbrace{-1, \dots, -1}_{\mathfrak{b}_k - c_k \text{ times}}) \in (\mathbb{Z}/2\mathbb{Z})^g \subset \Omega(G) \tag{3.3.1}$$

Second type: the same, but the sequences are $c_2, \dots, c_k$, and

$$w = (\underbrace{1, \dots, 1}_{\mathfrak{b}_1 \text{ times}}, \underbrace{1, \dots, 1}_{c_2 \text{ times}}, \underbrace{-1, \dots, -1}_{\mathfrak{b}_2 - c_2 \text{ times}}, \dots, \underbrace{1, \dots, 1}_{c_k \text{ times}}, \underbrace{-1, \dots, -1}_{\mathfrak{b}_k - c_k \text{ times}}) \in (\mathbb{Z}/2\mathbb{Z})^g \subset \Omega(G) \tag{3.3.2}$$

Notation: such a sequence $c_1, \dots, c_k$ or $c_2, \dots, c_k$ is denoted by $\mathfrak{c}$ and the set of all there sequences by $\mathfrak{C}$. We denote the set of $w \in \Omega(G)$ of the form (3.3.1), (3.3.2) by $\mathfrak{W}$, i.e. there is a $1 - 1$ correspondence between $\mathfrak{C}$ and $\mathfrak{W}$: $w = w(\mathfrak{c})$, $\mathfrak{c} = \mathfrak{c}(w)$. The representation $\pi \in \Pi_P$ that corresponds to $\mathfrak{c}$ is denoted by $\pi_{\mathfrak{c}}$ or (like in [4]) by π_w.

3.4. Finding of p_w, q_w.[1]

Numbers p_w, q_w are defined in [4], 4.3; here we use notations of this paper. Firstly we recall the definition of $\mathcal{P}_c^{\pm}$ and find them explicitly. Let $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} G_m \to G$ be a Deligne map for the Siegel

[1]This section is not logically necessary for the proof of the theorem.

variety. We use the following h: for $z=x+iy\quad h(z,\bar z)=\begin{pmatrix}x & y\\ -y & x\end{pmatrix}$. Let $i_1: G_m\to \text{Res}_{\mathbb{C}/\mathbb{R}}G_m$ be the map $z\to(z,1)$ and $\mu=h\circ i_1$. $\mathcal{P}_c^{\pm}$ are the subspaces of $\mathfrak{GSp}_{2g}$ on which ad $\mu(t)$ acts by t^{-1} and t respectively (see for example [6] or [4], 4.3). An element of $\mathfrak{GSp}_{2g}$ is a matrix $\begin{pmatrix}A & B\\ C & -A^t+(\lambda-1)E_g\end{pmatrix}$ where B, C are symmetric. A calculation gives us:

$$\mathcal{P}_c^+=\begin{pmatrix}C & iC\\ iC & -C\end{pmatrix},\mathcal{P}_c^-=\begin{pmatrix}C & -iC\\ -iC & -C\end{pmatrix}$$

where C is a symmetric $g\times g$-matrix.

For $w\in\mathfrak{W}\subset\Omega(G)$ we have $\gamma(w)=\mathfrak{w}_I$ for $I=$ the set of $-1'$s in (3.3.1), (3.3.2); we denote it simply by $\mathfrak{w}$. Further, we denote by $\mathcal{N}$, $\mathcal{N}_{\mathfrak{w}}$, $\mathcal{N}_c$, $\mathcal{N}_{c\mathfrak{w}}$ the Lie algebras of N, $\mathfrak{w}^{-1}N\mathfrak{w}$, N_c, $\mathfrak{w}^{-1}Nc\mathfrak{w}$ respectively. Numbers $p_w=\dim\,(\mathcal{N}_{c\mathfrak{w}}\cap\mathcal{P}_c^+)$, $q_w=\dim\,(\mathcal{N}_{c\mathfrak{w}}\cap\mathcal{P}_c^-)$ are defined in [4], 4.3. It is more convenient to conjugate with α: we set $\mathcal{P}^{\pm}=\alpha^{-1}\mathcal{P}_c^{\pm}\alpha$. A calculation gives: $\mathcal{P}^+=\begin{pmatrix}0 & C\\ 0 & 0\end{pmatrix}$, $\mathcal{P}^-=\begin{pmatrix}0 & 0\\ C & 0\end{pmatrix}$ where C is a symmetric $g\times g$-matrix. So, $p_w=\dim\,(\mathcal{N}\cap\mathfrak{w}\mathcal{P}^+\mathfrak{w}^{-1})$, $q_w=\dim\,(\mathcal{N}\cap\mathfrak{w}\mathcal{P}^-\mathfrak{w}^{-1})$.

Further, $\mathcal{N}$ has the same description like in (1N), (2N), but the diagonal blocks are 0-matrices.

Let $e_{i,j}$ be the elementary (i,j)-matrix. Matrices $\mathfrak{w}e_{i,g+j}\mathfrak{w}^{-1}$ are given by the following table (here and below we indicate in the third column of the table whether $\mathfrak{w}e_{i,g+j}\mathfrak{w}^{-1}\in\mathcal{N}$ or not).

First type:

Subtype	$\mathfrak{w}e_{i,g+j}\mathfrak{w}^{-1}$	
1. $i\notin I, j\notin I$	$e_{i,g+j}$	always $\in\mathcal{N}$
2. $i\in I, j\notin I$	$e_{g+i,g+j}$	$\in\mathcal{N}\iff j>i$ and$(*)$
3. $i\notin I, j\in I$	$e_{i,j}$	$\in\mathcal{N}\iff i>j$ and$(*)$
4. $i\in I, j\in I$	$e_{g+i,j}$	never $\in\mathcal{N}$

where (*) means: i, j do not belong to the same segment of partition $g=\mathfrak{b}_1+\dots+\mathfrak{b}_k$.

Since C is a symmetric matrix, we can take always $j\ge i$, and hence the quantity of pairs (i,j) such that $\mathfrak{w}e_{i,g+j}\mathfrak{w}^{-1}\in\mathcal{N}$ is:

Subtype 1. $\frac{(g-\sum_{l=1}^k c_l)(g+1-\sum_{l=1}^k c_l)}{2}$;

Subtype 2. $c_1(\mathfrak{b}_2-c_2+\mathfrak{b}_3-c_3+\dots+\mathfrak{b}_k-c_k)+c_2(\mathfrak{b}_3-c_3+\dots+\mathfrak{b}_k-c_k)+\dots+c_{k-1}(\mathfrak{b}_k-c_k)$, hence

$$p_w=\frac{(g-\sum_{l=1}^k c_l)(g+1-\sum_{l=1}^k c_l)}{2}+\sum_{1\le i<j\le k}c_i\mathfrak{b}_j-\sigma_2(c_*)$$

Analogously, in order to find q_w, we have:

Subtype	$\mathfrak{w}e_{g+i,j}\mathfrak{w}^{-1}$	
1. $i\notin I, j\notin I$	$e_{g+i,j}$	never $\in\mathcal{N}$
2. $i\in I, j\notin I$	$e_{i,j}$	$\in\mathcal{N}\iff i>j$ and$(*)$
3. $i\notin I, j\in I$	$e_{g+i,g+j}$	$\in\mathcal{N}\iff j>i$ and$(*)$
4. $i\in I, j\in I$	$e_{i,g+j}$	always $\in\mathcal{N}$

with the same notations and assumptions, hence

$$q_w=\frac{(\sum_{l=1}^k c_l)(1+\sum_{l=1}^k c_l)}{2}+\sum_{1\le j<i\le k}c_i\mathfrak{b}_j-\sigma_2(c_*)$$

Type 2 is analogous to the type 1. We set $c_1 = 0$, the above tables are the same with the following exception: for subtype 1 (i.e. $i \notin I$, $j \notin I$) we have: $\mathfrak{w} e_{i,g+j} \mathfrak{w}^{-1} = e_{i,g+j} \in \mathcal{N}$ always except $i, j \in [1, \mathfrak{b}_1]$. This changes the value of p_w:

$$p_w \text{ of the second type } = p_w \text{ of the first type } - \frac{\mathfrak{b}_1(\mathfrak{b}_1+1)}{2}$$

$$= \frac{(g - \sum_{l=2}^k c_l)(g+1-\sum_{l=2}^k c_l)}{2} + \sum_{2 \le i < j \le k} c_i \mathfrak{b}_j - \sigma_2(c_*) - \frac{\mathfrak{b}_1(\mathfrak{b}_1+1)}{2}$$

and for q_w we have the same formula like in the first type:

$$q_w = \frac{(\sum_{l=2}^k c_l)(1+\sum_{l=2}^k c_l)}{2} + \sum_{1 \le j < i \le k} c_i \mathfrak{b}_j - \sigma_2(c_*)$$

Remarks. 1. Change of $(c_1, \dots, c_k)$ to $(\mathfrak{b}_1 - c_1, \dots, \mathfrak{b}_k - c_k)$ leads to interchange of p_w, q_w.

2. We have: $p_w + q_w = \frac{g(g+1)}{2} - \sum_{l=1}^k c_l(\mathfrak{b}_l - c_l)$ (type 1),
$p_w + q_w = \frac{g(g+1)}{2} - \frac{\mathfrak{b}_1(\mathfrak{b}_1+1)}{2} - \sum_{l=2}^k c_l(\mathfrak{b}_l - c_l)$ (type 2).

3.5. Finding the length of representatives of $\Omega(M_c)/\Omega(M_c \cap \mathfrak{w} K_c \mathfrak{w}^{-1})$.

We continue to work with the same $w \in \mathfrak{W}$, $\mathfrak{w} \in \mathfrak{N}(T_c)$ from 3.4. To prove proposition 4.3 below, we must find representatives of the minimal length of $\Omega(M_c)/\Omega(M_c \cap \mathfrak{w} K_c \mathfrak{w}^{-1})$, and find their length (see [4], 4.3 or [2], proof of (9.1)). Firstly we find K_c — the centralizer of μ in $G(\mathbb{R})$. It is clear that K_c is the centralizer of $h(\mathrm{Res}_{\,\mathbb{C}/\mathbb{R}} G_m)$ as well. Replacing h by $\alpha^{-1} h \alpha$ we see that im $\alpha^{-1} h \alpha = \left\{ \begin{pmatrix} Z & 0 \\ 0 & \lambda Z^{-1} \end{pmatrix} \right\}$ where Z is a scalar matrix.

We define K to be the centralizer of im $\alpha^{-1} h \alpha$ in G; we have:

1) $K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \lambda A^{t-1} \end{pmatrix} \right\}$ where $A \in GL_g$;

2) $K_c = \alpha K \alpha^{-1}$;

3) $\Omega(T_c, K_c) = \Omega(T, K) = S(g)$.

Now we see that conjugating with α we get $\Omega(M_c)/\Omega(M_c \cap \mathfrak{w} K_c \mathfrak{w}^{-1}) = \Omega(M)/\Omega(M \cap \mathfrak{w} K \mathfrak{w}^{-1})$. Like in (3.4), we have a table of $\mathfrak{w}$-conjugates of elementary matrix $e_{i,j}$ $(1 \le i, j \le g)$:

Subtype	$\mathfrak{w} e_{i,j} \mathfrak{w}^{-1}$	
1. $i \notin I, j \notin I$	$e_{i,j}$	$\in M \iff (*)$
2. $i \in I, j \notin I$	$e_{g+i,j}$	$\notin M$ (Type 1); $\in M \iff i, j \in (1, \dots, \mathfrak{b}_1)$ (Type 2)
3. $i \notin I, j \in I$	$e_{i,g+j}$	$\notin M$ (Type 1); $\in M \iff i, j \in (1, \dots, \mathfrak{b}_1)$ (Type 2)
4. $i \in I, j \in I$	$e_{g+i,g+j}$	$\in M \iff (*)$

where (*) here means: i, j belong to the same segment of the partition $g = \mathfrak{b}_1 + \dots + \mathfrak{b}_k$.

This means that

$$M \cap \mathfrak{w} K \mathfrak{w}^{-1} = \prod_{l=1}^{g} GL(c_l) \times GL(\mathfrak{b}_l - c_l) \text{ (Type 1)};$$

$$= GL(\mathfrak{b}_1) \times \prod_{l=2}^{g} GL(c_l) \times GL(\mathfrak{b}_l - c_l) \text{ (Type 2)}$$

is the set of block diagonal simplectic matrices with block sizes

$$c_1, \mathfrak{b}_1 - c_1, \dots, c_k, \mathfrak{b}_k - c_k, c_1, \mathfrak{b}_1 - c_1, \dots, c_k, \mathfrak{b}_k - c_k \text{(Type 1)};$$

$$\mathfrak{b}_1, c_2, \mathfrak{b}_2 - c_2, \dots, c_k, \mathfrak{b}_k - c_k, \mathfrak{b}_1, c_2, \mathfrak{b}_2 - c_2, \dots, c_k, \mathfrak{b}_k - c_k \text{ (Type 2)},$$

and

$$\Omega(M_c)/\Omega(M_c \cap \mathfrak{w}K_c\mathfrak{w}^{-1}) = \prod_{l=1}^{g} S(\mathfrak{b}_l)/(S(c_l) \times S(\mathfrak{b}_l - c_l))\text{(Type 1)};$$

$$= (\mathbb{Z}/2\mathbb{Z})^{\mathfrak{b}_1} \times \prod_{l=2}^{g} S(\mathfrak{b}_l)/(S(c_l) \times S(\mathfrak{b}_l - c_l))\text{(Type 2)}.$$

The set $S(b)/(S(c) \times S(b-c))$ is isomorphic to $\mathfrak{S}(c,b)$ — the set of all subsets of order c of the set $(1, \dots, b)$ (see Introduction, Step 2c). Let $D \in \mathfrak{S}(c,b)$, $D = (d_1, \dots, d_c)$, where $1 \le d_1 < \dots < d_c \le b$. The equivalence class that corresponds to this D is the set of permutations of $(1, \dots, b)$ that send $(1, \dots, c)$ to $(d_1, \dots, d_c)$. Since the length of a permutation (considered as an element of $S(b) = \Omega(GL(b+1))$) is the quantity of inversions of elements, it is easy to see that the permutation with the minimal length in the equivalence class corresponding to D is the permutation that sends j to d_j for $j = 1, \dots, c$, and analogously (in increasing order) for $j = c+1, \dots, b$. We denote this permutation by $m_D \in S(b)$; we have $l(m_D) = \sum_{j=1}^{c} d_j - \frac{c(c+1)}{2}$.

Further, let $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_{\mathfrak{b}_1}) \in (\mathbb{Z}/2\mathbb{Z})^{\mathfrak{b}_1} \subset \Omega(G)$, where $\mathfrak{a}_i = 0, 1$. It is known that $l(\mathfrak{a}) = \sum_{i=1}^{\mathfrak{b}_1} i\mathfrak{a}_i$.

Finally, the set of representatives of $\Omega(M_c)/\Omega(M_c \cap \mathfrak{w}K_c\mathfrak{w}^{-1})$ of minimal length is $\mathfrak{S}(\mathfrak{c}, P)$ of (0.3). Really, let $\rho \in \mathfrak{S}(\mathfrak{c}, P)$, $\rho = (D_1, \dots, D_k)$ for P of the first type, $\rho = (\mathfrak{a}, D_2, \dots, D_k)$ for P of the second type, where D_i is a subset of order c_i of the i-th segment of the partition $g = \mathfrak{b}_1 + \dots + \mathfrak{b}_k$ of $(1, \dots, g)$ and $\mathfrak{a}$ is as above. We have $m_{D_i} \in S(\mathfrak{b}_i)$. For P of type 1 the representative of minimal length is $m_\rho = m_{D_1} \times \dots \times m_{D_k} \in S(\mathfrak{b}_1) \times \dots \times S(\mathfrak{b}_k) \subset S(g) \subset \Omega(G)$, and we have

$$l(m_\rho) = \sum_{i=1}^{k} l(m_{D_i}) \tag{3.5.1}$$

For P of type 2 we let $m'_\rho = m_{D_2} \times \dots \times m_{D_k} \in S(\mathfrak{b}_2) \times \dots \times S(\mathfrak{b}_k) \subset S(g) \subset \Omega(G)$ and $m_\rho = (\mathfrak{a}$ multiplied semidirectly by $m'_\rho) \in \Omega(G)$. We have

$$l(m_\rho) = l(\mathfrak{a}) + \sum_{i=2}^{k} l(m_{D_i}) \tag{3.5.2}$$

Remark. It is convenient to treat numbers $f_i = d_i - i$ instead of d_i, so $f_1 \le f_2 \le \dots \le f_c \le b - c$. For the case $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ (i.e. P of type 1, $k = 1$), $\mathfrak{c} = \{c\}$, $w = w(\mathfrak{c})$ we have $h^{p_w + r, q_w + r}(\mathfrak{g}, K_c; \pi_w) =$ the quantity of Young diagrams of weight r in the rectangle with sides $c, g - c$. Analogous formulas exist for other P.

4. Relations between Hecke eigenvalues

Formulas (3.5.1), (3.5.2) can be used in order to find dimensions of

$$H^{i,j}(\mathfrak{g}, K_c; \pi_w)$$

([4], 4.3). They give us also relations between Weil numbers of $\mathcal{M}_P$. The preliminary form of these relations is the following:

Proposition 4.1. For any $\mathfrak{c} \in \mathfrak{C}$ there is a number $\mathfrak{x}_\mathfrak{c}$ such that the set of all Weil numbers of $\mathcal{M}_P$ is the following:

$$p^{l(m_\rho)} \mathfrak{x}_\mathfrak{c}$$

where $\mathfrak{c}$ runs over $\mathfrak{C}$, for a fixed $\mathfrak{c}$ ρ runs over $\mathfrak{S}(\mathfrak{c},P)$.

We shall not give a proof of (4.1), because we need a more general proposition 4.3, see below.

Comparing (4.1) with (2.7.2), we get the following problem:

(4.2). Find relations between $b_i=p^i a_i$ such that both (2.7.2), (4.1) are satisfied.

The solution to (4.2) — and even a more exact result — is given by the following proposition. Recall that $m_j=\mathfrak{b}_1+\cdots\mathfrak{b}_{j-1}$ $(j=1,\dots,k)$.

Proposition 4.3. First type: b_{m_j+1} are free variables, $b_{m_j+i}=p^{i-1}b_{m_j+1}$ $(i=1,\dots,\mathfrak{b}_j)$, and a_0 is defined by the equality $a_0^2\prod_{i=1}^g b_i=p^{g(g+1)/2}$.

Second type: $b_j=p^j$ for $j=1,\dots,\mathfrak{b}_1$, b_{m_j+1} $(2\le j\le k)$ are free variables, b_{m_j+i} and a_0 are like in the first type.

Proof. It follows immediately from [2], proof of Proposition 9.1. Let us recall some definitions of loc. cit., page 62 (here $\pi\in\Pi_P$, $\mathfrak{g}=\mathfrak{G}\mathfrak{S}\mathfrak{p}_{2g}$):

$$V_\pi=\bigoplus_i H^i(\mathfrak{g},K_c;\pi),\quad V_P=V_{\Psi_P}=\bigoplus_{\pi\in\Pi_P}V_\pi$$

(V_P is denoted in [2] by V_ψ and is defined on the page 59, two lines below (9.2)). Spaces V_π, W are $\mathfrak{sl}_2(\mathbb{C})$-modules (see loc. cit. for the definition of the action of $\mathfrak{sl}_2(\mathbb{C})$), and all V_π and hence V_P have the Hodge decomposition.

There exist bases $B(W)$, $B(V_\pi)$, $B(V_P)$ of W, V_π, V_P respectively and an isomorphism $\mathfrak{d}$: $B(V_P)\to B(W)$ (see [2], line below (9.6)) which gives an isomorphism of $\mathfrak{sl}_2(\mathbb{C})$-modules $V_P\to W$.

Arthur uses a slightly different description of $B(W)$ than the one used in (2.6). Namely, the set of elements of $B(W)$ is isomorphic to the set of cosets $\Omega(G)/\Omega(K_c)$, where $\Omega(K_c)\overset{i}{\hookrightarrow}\Omega(G)$ is equal to $S(g)\overset{i}{\hookrightarrow}\Omega(G)$ of Remark 1.5. It is clear that $\Omega(G)/\Omega(K_c)=(\mathbb{Z}/2\mathbb{Z})^g$. Let $I\subset\{1,\dots,g\}$; we can treat I as an element of $(\mathbb{Z}/2\mathbb{Z})^g$ as usually. The element of $B(W)$ that correspond to I according loc.cit. is exactly x_I of (2.6).

Now let $\pi=\pi_w$, $w=w(\mathfrak{c})$. The set $B(V_\pi)$ is isomorphic to $\Omega(M_c)/\Omega(M_c\cap\mathfrak{w}K_c\mathfrak{w}^{-1})=\mathfrak{S}(\mathfrak{c},P)$. For any finite group A and its subgroups B, C we have

$$A/B=\bigcup_{a\in C\backslash A/B}C/(C\cap aBa^{-1})\tag{4.3.1}$$

here and below all unions are disjoint.

Now we apply (4.3.1) to the case $A=\Omega(G)$, $B=K_c$, $C=\Omega(M_c)$ in order to get an inclusion:

$$B(V_\pi)=\Omega(M_c)/\Omega(M_c\cap\mathfrak{w}K_c\mathfrak{w}^{-1})=\mathfrak{S}(\mathfrak{c},P)\overset{\mathfrak{d}_w}{\hookrightarrow}\Omega(G)/\Omega(K_c)=(\mathbb{Z}/2\mathbb{Z})^g=B(W)$$

It follows from loc. cit. that for $\rho\in\mathfrak{S}(\mathfrak{c},P)$ as in the end of (3.5) we have: $\mathfrak{d}_w(\rho)=x_I$ where for Type 1:

$$I=D_1\cup\cdots\cup D_k\tag{4.3.2}$$

for Type 2:

$$I=I_{\mathfrak{a}}\cup D_2\cup\cdots\cup D_k\tag{4.3.3}$$

where $I_{\mathfrak{a}}\subset\{1,\dots,\mathfrak{b}_1\}$ is the set of ones (additive writing of $(\mathbb{Z}/2\mathbb{Z})^g$) in $\mathfrak{a}$, the union is in $\{1,\dots,g\}$. The Hodge type of $\mathfrak{d}_w(\rho)$ is

$$p_w+l(m_\rho),q_w+l(m_\rho)\tag{4.3.4}$$

Finally, we have

$$B(V_P)=\bigcup_{\pi\in\Pi_P}B(V_\pi)$$

and $\mathfrak{d}:B(V_P)\to B(W)$ is the union of $\mathfrak{d}_w$ in the obvious sense.

Let X, Y, H be the standard basis of $\mathfrak{sl}_2(\mathbb{C})$. We have the following properties of the action of ad X on V_π and W:

(4.3.5) If $v\in V_\pi$ is of the pure Hodge type (p,q) then ad $X(v)$ is of the pure Hodge type $(p+1,q+1)$.

(4.3.6) If $w\in W$ is a $r(\theta_{\pi_\chi})$-eigenelement of eigenvalue λ, then ad $X(w)$ is a $r(\theta_{\pi_\chi})$-eigenelement of eigenvalue $p\lambda$.

Type 1. We use notations $\mathfrak{e}_j=(0,\dots,0,1,0,\dots,0)\in\mathfrak{C}$ (1 is at the j-th place), $I(n)$ is a subset of $\{1,\dots,g\}$ consisting of the single element n, and we denote $x_{I(n)}$ simply by x_n. We fix some j and we set $\mathfrak{c}=\mathfrak{e}_j$. (4.3.2) shows that $\mathfrak{d}(V_{\pi_\mathfrak{c}})$ is generated by $x_{m_j+1},\dots,x_{m_j+\mathfrak{b}_j}$. According (4.3.4), $\forall i=1,\dots,\mathfrak{b}_j$ the Hodge type of x_{m_j+i} is $p_\mathfrak{c}+i$, $q_\mathfrak{c}+i$. (4.3.5) implies that

$$\text{ad } X(x_{m_j+i})=c_{j,i}x_{m_j+i+1}\tag{4.3.7}$$

where $c_{j,i}$ is some non-0 coefficient. Now, (2.7.2), (4.3.6) and (4.3.7) imply immediately that $b_{m_j+i+1}=pb_{m_j+i}$ which is 4.3 for Type 1.

Type 2. The idea of the proof is the same. Firstly we consider $\mathfrak{c}=(0,\dots,0)$. $B(V_{\pi_\mathfrak{c}})$ is the set of subsets of $\{1,\dots,\mathfrak{b}_1\}$. (4.3.3) shows that $\mathfrak{d}(V_{\pi_\mathfrak{c}})$ is generated by x_I, where $I\subset\{1,\dots,\mathfrak{b}_1\}$. (4.3.5) implies that for $\forall i=1,\dots,\mathfrak{b}_1$

$$(\text{ad } X)^i(x_\emptyset)=\sum_{I\subset\{1,\dots,\mathfrak{b}_1\}\text{ such that } l(I)=i}c_Ix_I\tag{4.3.8}$$

where coefficients c_I can be easily found using methods of [11]. For us it is sufficient to use the fact that $c_{I(i)}\neq 0$. (2.7.2), (4.3.6) and (4.3.8) imply by induction by i that $b_i=p^i$.

Finally, we consider $\mathfrak{c}=\mathfrak{e}_j$ like in Type 1, but with the first zero omitted. $B(V_{\pi_\mathfrak{c}})=(\mathbb{Z}/2\mathbb{Z})^{\mathfrak{b}_1}\times\{1,\dots,\mathfrak{b}_j\}$. (4.3.3) shows that $\mathfrak{d}(V_{\pi_\mathfrak{c}})$ is generated by $x_{J\cup I(m_j+i)}$, where $J\subset\{1,\dots,\mathfrak{b}_1\}$ and $i\in\{1,\dots,\mathfrak{b}_j\}$. The Hodge type of $x_{J\cup I(m_j+i)}$ is $p_\mathfrak{c}+l(J)+i,q_\mathfrak{c}+l(J)+i$. (4.3.5) implies that for $\forall i=1,\dots,\mathfrak{b}_j$

$$(\text{ad } X)^{i-1}(x_{\emptyset\cup I(m_j+1)})=\sum c_{J,j,n}x_{J\cup I(m_j+n)}\tag{4.3.9}$$

the sum is over the pairs (J,n), $J\subset\{1,\dots,\mathfrak{b}_1\}$, $n\in\{1,\dots,\mathfrak{b}_j\}$ such that $l(J)+n=i$.

Again it is sufficient to use the fact that $c_{\emptyset,j,i}\neq 0$. As earlier (2.7.2), (4.3.6) and (4.3.9) imply by induction by i that $b_{m_j+i}=p^{i-1}b_{m_j+1}$. □

Remark 1. There are $g-k$ (first type); $g-k+1$ (second type) relations between eigenvalues of τ_p, $\tau_{p,i}$ on $\mathcal{M}_P$.

Remark 2. (4.1) is obviously a corollary of (4.3); numbers $\mathfrak{x}_\mathfrak{c}$ are products of some b_i and powers of p.

Remark 3. Formulas of (4.3) are not direct corollaries of (4.1), (2.7.2): it is easy to construct an example of numbers a_i having another form as in (4.3) but such that both (4.1), (2.7.2) are satisfied.

We denote $\mathfrak{m}(\tau_p)$, $\mathfrak{m}(\tau_{p,i})$ by $\mathfrak{m}_p$, $\mathfrak{m}_{p,i}$ respectively.

Theorem 4.4. Relations between $\mathfrak{m}_p^2$, $\mathfrak{m}_{p,i}$ are linear. Particularly, for the parabolic subgroup P of the second type such that $k=g$, all $\mathfrak{b}_i$ are 1 (see Appendix, 8b) the only relation between $\mathfrak{m}_p^2$, $\mathfrak{m}_{p,i}$ is the following:

$$\frac{\mathfrak{m}_p^2}{(p+1)^2}+\sum_{i=1}^{g} Y_i\mathfrak{m}_{p,i}=0 \tag{4.5}$$

where $\mathfrak{m}_{p,g}=1$ and Y_i are polynomials in p (particularly, they do not depend on g) defined as follows: $Y_1=-1$ and Y_n is defined by the recurrence relation

$$[\sum_{i=1}^{n-1} Y_iR_{n-1}(i)](1+p^2)p^{-\frac{(n-1)n}{2}}+[\sum_{i=1}^{n} Y_iR_n(i)]p^{-\frac{n(n+1)}{2}+1}$$

$$+[\sum_{i=1}^{n-2} Y_iR_{n-2}(i)]p^{-\frac{(n-2)(n-1)}{2}+1}+2=0 \tag{4.6}$$

Proof. Follows immediately from (2.7.4) and (4.3). □

Y_2, Y_3 are given in the Appendix, Table 7.

Appendix

1. Some relations satisfied by $\tau_{p,i}$.

We set $W_n(p)=\prod_{i=1}^n(p^i+1)$. Let $\deg:\mathbb{T}_p\to\mathbb{Z}$ be a map of the degree of a double coset (= the quantity of ordinary cosets in it). We have equalities:

$$(\tau_p)^2=\sum_{i=0}^{g}\tau_{p,i}W_i(p);\qquad \deg\tau_p=W_g(p)$$

$$\deg\tau_{p,k}=\sum_{\mathfrak{d}\in 3^g}C(\mathfrak{d},k)$$

$$=p^{(g-k)(g-k+1)/2}\frac{W_g(p)}{W_k(p)}\frac{(p^g-1)(p^{g-1}-1)\cdot\ldots\cdot(p^{g-k+1}-1)}{(p^k-1)(p^{k-1}-1)\cdot\ldots\cdot(p-1)}.$$

Particularly, $\deg\tau_{p,0}=p^{(g)(g+1)/2}W_g(p)$, for $g=2$: $\deg\tau_{p,0}=p^6+p^5+p^4+p^3$,

$\deg\tau_{p,1}=p^4+p^3+p^2+p$,

for $g=3$: $\deg\tau_{p,0}=p^{12}+p^{11}+p^{10}+2p^9+p^8+p^7+p^6$,
$\deg\tau_{p,1}=p^{10}+p^9+2p^8+2p^7+2p^6+2p^5+p^4+p^3=p^3(p^2+p+1)(p^2+1)(p^3+1)$,
$\deg\tau_{p,2}=p^6+p^5+p^4+p^3+p^2+p$

Table 2. Numbers $R_g(k)$.

Source: [1], Chapter 3, Lemma 6.19. We have: $R_g(g)=1$, $R_g(g-1)=p^g-1$.

Numbers $R_g(k)$:

k \ g	2	3
0	p^3-p^2	$p^6-p^5-p^3+p^2$
1	p^2-1	p^5-p^2
2	1	p^3-1
3		1

Table 3. Explicit formulas for Satake map S_G.

$g = 2$:

$$\tilde{\tau}_p = \Phi_0 + \Phi_1 + \Phi_2$$

$$\tilde{\tau}_{p,1} = \frac{1}{p}(\Phi_0\Phi_1 + \Phi_1\Phi_2) + \frac{p^2-1}{p^3}\Phi_0\Phi_2$$

$$\tilde{\tau}_{p,2} = \frac{1}{p^3}\Phi_0\Phi_2$$

$g = 3$:

$$\tilde{\tau}_p = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3$$

$$\tilde{\tau}_{p,1} = \frac{1}{p}(\Phi_0\Phi_1 + \Phi_1\Phi_2 + \Phi_2\Phi_3) + \frac{p^2-1}{p^3}(\Phi_0\Phi_2 + \Phi_1\Phi_3) + \frac{p^3-1}{p^4}\Phi_0\Phi_3$$

$$\tilde{\tau}_{p,2} = \frac{1}{p^3}(\Phi_0\Phi_2 + \Phi_1\Phi_3) + \frac{p^3-1}{p^6}\Phi_0\Phi_3$$

$$\tilde{\tau}_{p,3} = \frac{1}{p^6}\Phi_0\Phi_3$$

Table 4. Coefficients $\mathfrak{h}_i$ of the Hecke polynomial.

$g = 2$:

$$\mathfrak{h}_0 = p^6 = p^6\mathfrak{h}_4$$

$$\mathfrak{h}_1 = -p^3\tau_p = p^3\mathfrak{h}_3$$

$$\mathfrak{h}_2 = p(\tau_{p,1} + p^2 + 1)$$

$$\mathfrak{h}_3 = -\tau_p$$

$$\mathfrak{h}_4 = 1$$

$g = 3$:

$$\mathfrak{h}_0 = p^{24} = p^{24}\mathfrak{h}_8$$

$$\mathfrak{h}_1 = -p^{18}\tau_p = p^{18}\mathfrak{h}_7$$

$$\mathfrak{h}_2 = p^{13}[\tau_{p,1} + (p^2+1)\tau_{p,2} + (-p^5 - p^3 + 2p^2 + 1)] = p^{12}\mathfrak{h}_6$$

$$\mathfrak{h}_3 = -p^9[\tau_p\tau_{p,2} + \tau_p] = p^6\mathfrak{h}_5$$

$$\mathfrak{h}_4 = p^6[\tau_p^2 + \tau_{p,2}^2 + (-2p+2)\tau_{p,2} - 2p\tau_{p,1} + p^6 + 2p^4 - 2p^3 - 2p + 1]$$

$$\mathfrak{h}_5 = -p^3[\tau_p\tau_{p,2} + \tau_p]$$

$$\mathfrak{h}_6 = p[\tau_{p,1} + (p^2+1)\tau_{p,2} + (-p^5 - p^3 + 2p^2 + 1)]$$

$$\mathfrak{h}_7 = -\tau_p$$

$$\mathfrak{h}_8 = 1$$

Table 5. Numbers $\tilde{C}(\mathfrak{d}, k)$.

$\tilde{C}(\mathfrak{d}, k)$ depend only on q_1 — the quantity of ones in $\mathfrak{d}$ — and k; particularly, they don't depend on g.

k	0	1	2	3
q_1				
0	1			
1	$1-p^{-1}$	p^{-1}		
2	$2-2p^{-1}$	$p^{-1}-p^{-3}$	p^{-3}	
3	$3-4p^{-1}+p^{-4}$	$3p^{-1}-p^{-2}-p^{-3}-p^{-4}$	$p^{-3}-p^{-6}$	p^{-6}

Table 6. Explicit values of $\alpha_G(\chi)(\tau_{p,i})$.

We denote $\sigma_i=\sigma_i(b_*)$.

$g=2$: $\alpha_G(\chi)(\tau_{p,1})=p^{-1}\sigma_1\sigma_2+(p^{-1}-p^{-3})\sigma_2+p^{-1}\sigma_1$

$g=3$: $\alpha_G(\chi)(\tau_{p,1})=p^{-1}(\sigma_1+\sigma_1\sigma_2+\sigma_2\sigma_3)+(p^{-1}-p^{-3})(\sigma_1\sigma_3+\sigma_2)+(p^{-1}-p^{-4})\sigma_3$

$\alpha_G(\chi)(\tau_{p,2})=p^{-3}(\sigma_1\sigma_3+\sigma_2)+(p^{-3}-p^{-6})\sigma_3$

Table 7. Polynomials Y_i.

$Y_1=-1$
$Y_2=p^3-p^2+p-1$
$Y_3=-p^7+p^6-p^5+p^4+p^3-2p^2+p-1$

8. Some properties of $\mathcal{M}_P$, P of two simplest types.[2]

(a) $P=B$, i.e. P of the first type, all $\mathfrak{b}_i=1$.

In this case $\mathcal{M}_P$ is (generally) irreducible, of weight $\frac{g(g+1)}{2}$, the packet Π_P consists of 2^{g-1} representations $\pi_{\mathfrak{c}}$ where $\mathfrak{c}$ runs over the set of all subsets of $1,\dots,g$ factorized by the equivalence relation: a subset is equivalent to the complementary subset. The partition (0.1.1) is trivial (i.e. consists of one set). For any $\mathfrak{c}\in\Pi_P$ $\quad\mathfrak{S}(\mathfrak{c},P)$ is trivial, and the Hodge number $h^{i,j}(\mathcal{M}_P)$ ($i+j=\frac{g(g+1)}{2}$) is equal to the quantity of subsets of $1,\dots,g$ such that the sum of elements of this subset is i, i.e.

$$\sum_{i=0}^{\frac{g(g+1)}{2}} h^{i,j}(\mathcal{M}_P)t^i=\prod_{i=1}^{g}(t^i+1)$$

(b) P of the second type, all $\mathfrak{b}_i=1$.

In this case $\mathcal{M}_P$ is the sum of 2 (generally) irreducible submotives M^-, M^+ of weights $\frac{g(g+1)}{2}-1$, $\frac{g(g+1)}{2}+1$ respectively, the packet Π_P consists of 2^{g-2} representations $\pi_{\mathfrak{c}}$ where $\mathfrak{c}$ runs over the set of all subsets of $2,\dots,g$ factorized like in (a). (0.1.1) is also trivial. For any $\mathfrak{c}\in\Pi_P$ $\quad\mathfrak{S}(\mathfrak{c},P)$ is the irreducible $\mathfrak{sl}_2$-module of dimension 2, and the Hodge number $h^{i,j}(M^-)$ ($i+j=\frac{g(g+1)}{2}-1$) is equal to the quantity of subsets of $2,\dots,g$ such that the sum of elements of this subset is i, i.e.

$$\sum_{i=0}^{\frac{g(g+1)}{2}-1} h^{i,j}(M^-)t^i=\prod_{i=2}^{g}(t^i+1)$$

8a. Hecke polynomial for the case M^-, $g=3$.

We consider numbers b_1, b_2, b_3 for M^-. We have $b_1=p$, we denote $s_1=b_2+b_3$, $s_2=b_2b_3$. Roots of Hecke polynomial for M^- are a_0, a_0b_2, a_0b_3, $a_0b_2b_3$, i.e. this polynomial is

$$\mathrm{fr}^{\,4}-a_0(1+s_1+s_2)\mathrm{fr}^{\,3}+a_0^2(s_1+s_1s_2+2s_2)\mathrm{fr}^{\,2}-a_0(1+s_1+s_2)p^5\mathrm{fr}\;+p^{10}$$

[2]Written by a request of a referee. This is an elementary corollary of results of Section 4.

(recall that $a_0^2 s_2 = p^5$).

In terms of τ_p, $\tau_{p,1}$ this polynomial is

$\mathrm{fr}^{\ 4} - \frac{\tau_p}{p+1}\mathrm{fr}^{\ 3} + \frac{p^2}{(p^2+1)(p-1)}[-(\frac{\tau_p}{p+1})^2 + \tau_{p,1} + p^6 - 2p^5 + 2p^4 - 2p^3 + 2p^2]\mathrm{fr}^{\ 2} - \frac{p^5\tau_p}{p+1}\mathrm{fr}\ + p^{10}$.

It can be also obtained by taking the Hecke polynomial for $g=3$, substituting $\tau_{p,2}$ from (4.5) and factorizing the obtained expression.

9. Another way to find relations between $\mathfrak{m}_p$, $\mathfrak{m}_{p,i}$.

Here we give this method only for the case P of Appendix, **8(b)**. Let $\alpha_1, \dots, \alpha_{2^g-1}$ be the Weil numbers of M^-, so $p\alpha_1, \dots, p\alpha_{2^g-1}$ are the Weil numbers of M^+. We denote the Weil numbers of $\mathcal{M}_P$ by $\gamma_1, \dots, \gamma_{2^g}$ (the union of sets $\alpha_1, \dots, \alpha_{2^g-1}$ and $p\alpha_1, \dots, p\alpha_{2^g-1}$), so we have

$$\sigma_i(\gamma_*) = \sum_{j=0}^{i} p^j \sigma_{i-j}(\alpha_*)\sigma_j(\alpha_*) \tag{A1}$$

Further, $\sigma_i(\gamma_*) = \mathfrak{h}_i$. From now we consider only the case $g=3$.

Taking values of $\mathfrak{h}_i$ from Table 4 and taking into consideration that $\sigma_3(\alpha_*) = p^5\sigma_1(\alpha_*)$, $\sigma_4(\alpha_*) = p^{10}$ we get from (A1), $i = 1, 2$:

$\sigma_1(\alpha_*) = \frac{\mathfrak{m}_p}{p+1}$;

$\sigma_2(\alpha_*) = \frac{p(\mathfrak{m}_{p,1} + (p^2+1)\mathfrak{m}_{p,2} - p^5 - p^3 + 2p^2 + 1 - \frac{\mathfrak{m}_p^2}{(p+1)^2})}{p^2+1}$

Further, the equality for $\mathfrak{h}_3$ is equivalent to $\mathfrak{m}_p A = 0$ and the equality for $\mathfrak{h}_4$ is equivalent to $AB = 0$, where the common multiple A is the left hand side of (4.5):

$A = \frac{\mathfrak{m}_p^2}{(p+1)^2} + \sum_{i=1}^3 Y_i \mathfrak{m}_{p,i} = (p^3 - p^2 + p - 1)\mathfrak{m}_{p,2} + (-p^7 + p^6 - p^5 + p^4 + p^3 - 2p^2 + p - 1) + \frac{\mathfrak{m}_p^2}{(p+1)^2} - \mathfrak{m}_{p,1}$

and

$B = -\frac{\mathfrak{m}_p^2}{(p+1)^2} + \sum_{i=1}^3 Y_i^+ \mathfrak{m}_{p,i} = (p^3 + p^2 + p + 1)\mathfrak{m}_{p,2} + (p^7 + p^6 + p^5 + p^4 + p^3 + 2p^2 + p + 1) - \frac{\mathfrak{m}_p^2}{(p+1)^2} + \mathfrak{m}_{p,1}$, where Y_i^+ are polynomials in p whose coefficients are the absolute values of the ones of Y_i.

(4.5) shows that $\mathfrak{m}_{p,*}$ satisfy the condition $A = 0$ (but not $\mathfrak{m}_p = B = 0$).

Substituting the condition $A = 0$ to the formula for $\sigma_2(\alpha_*)$, we can slightly simplify it:

$\sigma_2(\alpha_*) = p^2(\mathfrak{m}_{p,2} - p^4 + p^3 - p^2 + 1)$.

Notation Index

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B	Borel subgroup, 2.1	$B(*)$	Proof of 4.3
$\mathfrak{b}_i$	0.2, 3.1.1	c_i	3.3
c_*	coefficients, after 4.3.6	$\mathfrak{c}$	3.3
$\mathfrak{C}$	3.3	d_i	3.5
D	3.5	D_i	3.5
$\mathfrak{d}$	Line between (4.3.1) and (4.3.2)	E_g	the unit matrix

$E_{ij}, E_{i,j}$	the elementary matrix
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l	length of an element of Ω
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$S(n)$	permutations group
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$\mathfrak{S}(\mathfrak{c}, P)$	0.3; 3.5
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V_i	1.5
V_P	after 4.3
$\mathfrak{w}$	3.4
W	2.6
x_I	2.6
X	Shimura variety, Section 0.
χ	2.1

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Логачев Д.Ю. Соотношения между гипотетическими собственными значениями операторов Гекке на подмотивах многообразий Зигеля. Дальневосточный математический журнал. 2012. Т. 12. № 1. С. 60–85.

АННОТАЦИЯ

Существуют гипотетические соотношения между L-функциями подмотивов многообразий Шимуры и автоморфными представлениями соответствующих редуктивных групп, принадлежащие Ленглендсу – Артуру. В настоящей работе эти соотношения используются для получения явных соотношений между собственными числами p-операторов Гекке (образующих p-алгебры Гекке многообразия X) на пространствах когомологий некоторых таких подмотивов в случае, когда X — многообразие Зигеля. Этот результат также является гипотетическим: методы подсчета точек на редукциях X, основанные на формуле следа Сельберга, не используются.

Полученные соотношения оказываются линейными, коэффициенты в них являются многочленами от p и удовлетворяют простой рекуррентной формуле. Аналогичный результат может быть легко получен для произвольного многообразия Шимуры.

Представленный результат есть промежуточный шаг в обобщении теоремы Колывагина о конечности группы Тэйта – Шафаревича эллиптических кривых аналитического ранга 0 или 1 над $\mathbb{Q}$ на случай подмотивов других многообразий Шимуры, в частности, многообразий Зигеля рода 3, см. [9].

Идея доказательства: с одной стороны, упомянутые формулы Ленглендса – Артура дают (гипотетические) соотношения на числа Вейля подмотива; с другой стороны, отображение Сатаке позволяет преобразовывать эти соотношения в соотношения на собственные числа p-операторов Гекке на X.

Статья также содержит обзор некоторых близких вопросов, например, явного нахождения полиномов Гекке многообразия X. В приложении содержатся таблицы для случаев $g = 2, 3$.

Ключевые слова: *многообразия Зигеля, подмотивы, соответствия Гекке,числа Вейля, отображение Сатаке.*