MSC2010 Primary 55P15, 57R19

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A homotopy-theoretic rigidity property of Bott manifolds

The rigidity conjecture in toric topology posits that two toric manifolds are diffeomorphic if and only if their integral cohomology rings are isomorphic as graded rings. Only a few low dimensional cases have been resolved. We weaken the conjecture to one concerning homotopy type rather than diffeomorphism, and show that the weaker conjecture holds for Bott manifolds, once enough primes have been inverted. In particular, show that the rational homotopy type of a Bott manifold is determined by its rational cohomology ring.

The material in this paper was inspired by the mathematics discussed at the International conference «Toric Topology and Automorphic Functions» (September, 5-10th, 2011, Khabarovsk, Russia).

Key words: *Bott manifold, rigidity.*

1. Introduction

There has been a great deal of interest recently in the rigidity of toric manifolds. A *toric variety* X of dimension n is a normal complex algebraic variety with an action of an n-dimensional algebraic torus $(\mathbb{C}^*)^n$ having a dense orbit. A *toric manifold* is a compact smooth toric variety. Masuda [5] showed that the variety type of a toric manifold is determined by its equivariant cohomology algebra over $H^*(B(\mathbb{C}^*)^n)$. Following this it was natural to ask to what extent the diffeomorphism type of the manifold is distinguished by ordinary cohomology.

The rigidity conjecture: Two toric manifolds are diffeomorphic if and only if their integral cohomology rings are isomorphic as graded rings.

Some partial results have been obtained in the case when the toric manifold is a Bott manifold or a generalized Bott manifold. A *Bott tower* of height n is a sequence of manifolds

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_0} B_0 = \text{point}$$

where $B_k = P(\underline{\mathbb{C}} \oplus \zeta_{k-1})$ is the projectivization of a complex line bundle ζ_{k-1} over B_{k-1} and a trivial line bundle $\underline{\mathbb{C}}$, and the fibre of each map π_k for $1 \leq k \leq n$ is S^2. A *Bott manifold* of height n is the total space B_n in a Bott tower. A *generalized Bott tower* is a sequence of manifolds as above where B_k is the projectivization of a Whitney sum of complex line bundles, and the fibre at each stage is $\mathbb{C}P^{n_k}$ for some nonnegative integer n_k. The total space B_n is a *generalized Bott manifold.*

Masuda and Panov [6] showed that in the case of a Bott manifold, if $H^*(B_n) \cong H^*(\prod_{i=1}^n S^2)$ then B_n is diffeomorphic to $\prod_{i=1}^n S^2$. Choi, Masuda and Suh [3] improved this to the case of

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generalized Bott manifolds: if $H^*(B_n) \cong H^*(\prod_{i=1}^n \mathbb{C}P^{n_i})$ then B_n is diffeomorphic to $\prod_{i=1}^n \mathbb{C}P^{n_i}$. For the non-product cases, in [3] it was shown that the rigidity conjecture holds for generalized Bott manifolds of height 2 and for Bott manifolds of height 3, and it has recently been announced by Choi [2] that the conjecture also holds for Bott manifolds of height 4. A thorough discussion of progress to date can be found in [4].

In this paper we consider a weaker version of the rigidity conjecture.

The homotopy-theoretic rigidity conjecture: Two toric manifolds are homotopy equivalent if and only if their integral cohomology rings are isomorphic as graded rings.

There are advantages in weakening from a diffeomorphism to a homotopy equivalence. A positive answer in the homotopy-theoretic case would give strong evidence for a positive answer in the diffeomorphism case. A negative answer in the homotopy-theoretic case implies a negative answer in the diffeomorphism case. Further, the case-by-case progress to date in the diffeomorphism rigidity conjecture is *ad hoc*. We will obtain systematic results for the homotopy-theoretic rigidity conjecture.

We prove the following. Let $p_1, \ldots, p_l, \ldots$ be the primes in $\mathbb{N}$, listed in increasing order. Let $P_l = \{p_1, \ldots, p_l\}$. Let R_l be the ring of integers localized away from P_l.

Theorem 1.1. *Let B_n and B'_n be Bott manifolds of height n. Suppose that all spaces and maps are localized away from P_l. If $n < 2p_{l+1} - 1$, then the following are equivalent:*

(a) *there is a ring isomorphism $H^*(B_n; R_l) \cong H^*(B'_n; R_l)$;*

(b) *there is a homotopy equivalence $B_n \simeq B'_n$.*

Theorem 1.1 implies that the homotopy-theoretic rigidity conjecture holds for Bott manifolds, provided enough primes have been inverted. The localization hypothesis is convenient but may not necessary. It is used to eliminate potential obstructions in constructing a homotopy equivalence $B_n \simeq B'_n$ given an isomorphism $H^*(B_n; R_l) \cong H^*(B'_n; R_l)$. It is useful to observe that Theorem 1.1 implies that the homotopy-theoretic rigidity conjecture holds for $n \leq 4$ provided 2 is inverted, and it holds for $n \leq 8$ provided both 2 and 3 are inverted. More emphatically, Theorem 1.1 implies that the homotopy-theoretic rigidity conjecture holds rationally: there is a rational homotopy equivalence $B_n \simeq B'_n$ if and only if there is an isomorphism of graded rings $H^*(B_n : \mathbb{Q}) \cong H^*(B'_n : \mathbb{Q})$.

The positive result in Theorem 1.1 raises many questions. Can the hypothesis regarding localization away from P_l be removed? As this may be difficult, perhaps a starting point is to ask whether the dimensional range $n < 2p_{l+1} - 1$ can be improved. Can Theorem 1.1 be generalized to any toric manifold? An intitial test case is whether it holds for generalized Bott manifolds. Does knowing that the homotopy type of a Bott manifold is determined by its integral cohomology ring imply that the diffeomorphism type is also determined? These questions and more deserve further investigation.

2. Some properties of Bott manifolds

In this section we describe some properties of Bott manifolds which will be used to prove Theorem 1.1. Note that we do *not* localize in this section. We begin with some general information.

Unless otherwise stated, cohomology is taken with $\mathbb{Z}$ coefficients. Let B_n be a Bott manifold of height n. As in [6], the generators in cohomology can be chosen so that there is an isomorphism

$$H^*(B_n) \cong \mathbb{Z}[x_1, \ldots, x_n]/\sim$$

where each x_k is of degree 2, and the relations are $x_1^2 = 0$ and for $1 < k \leq n$,

$$x_k^2 = \Sigma_{i=1}^{k-1} a_{i,k} x_i \otimes x_k$$

for some coefficients $a_{i.k} \in \mathbb{Z}$. In particular, observe that $H^*(B_n)$ is concentrated in even degrees, and that each relation is a linear combination of degree 4 elements. Let $(B_n)_4$ be the 4-skeleton of B_n. Then the description of $H^*(B_n)$ implies that $(B_n)_4$ consists of one zero-cell, n two-cells, and m four-cells, where $m = \binom{n}{2}$. So there is a cofibration

$$\bigvee_{j=1}^{m} S^3 \xrightarrow{f} \bigvee_{i=1}^{n} S^2 \longrightarrow (B_n)_{(4)}$$

for some map f.

Baues considered CW-complexes consisting of only 2-cells and 4-cells in generality. A $(2,4)$-*complex* is a CW-complex C which is the mapping cone of a map

$$a\colon \bigvee_{i=1}^{m} S^3 \longrightarrow \bigvee_{i=1}^{n} S^2$$

for some integers m and n. Baues [1, Proposition 1.2.3] proved the following.

Proposition 2.1. *Let C and C' be two $(2,4)$-complexes with corresponding attaching maps a and a'. Then the following are equivalent:*

(a) *there is a ring isomorphism $H^*(C) \cong H^*(C')$;*

(b) *the attaching maps a and a' are homotopic;*

(c) *there is a homotopy equivalence $C \simeq C'$.*

In our case, observe that if B_n is a Bott manifold then its 4-skeleton $(B_n)_4$ is a $(2,4)$-complex. As a consequence of Proposition 2.1 we obtain the following.

Lemma 2.2. *Let B_n and B'_n be Bott manifolds. Then the following are equivalent:*

(a) *there is a ring isomorphism $H^*(B_n) \cong H^*(B'_n)$;*

(b) *there is a ring isomorphism $H^*((B_n)_4) \cong H^*((B'_n)_4)$;*

(c) *there is a homotopy equivalence $(B_n)_4 \simeq (B'_n)_4$.*

Proof. Part (a) clearly implies part (b), and parts (b) and (c) are equivalent by Proposition 2.1. It remains to show that part (b) implies part (a). Let $s\colon H^*((B_n)_4) \longrightarrow H^*((B'_n)_4)$ be a ring isomorphism. Note that this is an isomorphism on the degree 2 and degree 4 cohomology of B_n and B'_n. Since the generators of $H^*(B_n)$ and $H^*(B'_n)$ are in degree 2 and the relations are in degree 4, by multiplicatively extending s we obtain a ring homomorphism $\sigma\colon H^*(B_n) \longrightarrow H^*(B'_n)$. Since s induces an isomorphism of generating sets, the same is true of σ. Therefore as σ is a ring homomorphism, it must be a ring isomorphism. □

Next, we establish a homotopy decomposition.

Lemma 2.3. *Let B_n be a Bott manifold of height n. For $1 \leq k \leq n$, each fibration $S^2 \longrightarrow B_k \xrightarrow{\pi_k} B_{k-1}$ in the tower has the property that π_k has a right homotopy inverse. Consequently, there is a homotopy equivalence*

$$\Omega B_n \simeq \prod_{i=1}^{n} \Omega S^2.$$

Proof. Recall that B_n is defined as a sequence of manifolds

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_0} B_0 = \text{point}$$

where $B_k = P(\underline{\mathbb{C}} \oplus \zeta_{k-1})$ is the projectivization of a complex line bundle ζ_{k-1} over B_{k-1} and a trivial line bundle $\underline{\mathbb{C}}$. At each stage of the sequence there is a fibration $S^2 \longrightarrow B_k \xrightarrow{\pi_k} B_{k-1}$. The map π_k has a section given by restricting $\underline{\mathbb{C}}$ to the $+1$ fibre. The existence of this section implies that after looping there is a homotopy equivalence $\Omega B_k \simeq \Omega S^2 \times \Omega B_{k-1}$. Since $B_1 = S^2$, if we inductively assume that $\Omega B_{k-1} \simeq \prod_{i=1}^{k-1} \Omega S^2$, then we obtain $\Omega B_k \simeq \prod_{i=1}^{k} \Omega S^2$. The lemma now follows by induction. $\square$

It will be useful to regard B_n as a CW-complex and filter it by its skeleta. Recall that $H^*(B_n)$ is concentrated in even degrees, specifically, in degrees $2k$ for $1 \leq k \leq n$. For $1 \leq k \leq n$, let M_k be the $2k$-skeleton of B_n. Then $M_1 = \bigvee_{i=1}^n S^2$, $M_n = B_n$, and for $1 \leq k \leq n-1$, there are cofibrations

$$\bigvee_{i=1}^{s_k} S^{2k+1} \xrightarrow{f_k} M_k \longrightarrow M_{k+1}$$

where $s_k = \binom{n}{k+1}$ is the number of vector space generators of H^{2k+2}, and f_k is the map attaching the $(2k+2)$-cells to B_n.

Let $m_k \colon M_k \longrightarrow B_n$ be the skeletal inclusion. Define the space Q_k and the map φ_k by the homotopy fibration

$$Q_k \xrightarrow{\varphi_k} M_k \xrightarrow{m_k} B_n.$$

We determine some properties of this fibration.

Lemma 2.4. *For $1 \leq k \leq n$, the map $\Omega M_k \xrightarrow{\Omega m_k} \Omega B_n$ has a right homotopy inverse. Consequently, there is a homotopy equivalence $\Omega M_k \simeq \Omega B_n \times \Omega Q_k$.*

P r o o f. First consider m_1. We begin by constructing a different map $J \colon \bigvee_{i=1}^n S^2 \longrightarrow B_n$ with the property that ΩJ has a right homotopy inverse, and then compare J to m_1. Start with the fibration $S^2 \xrightarrow{i_n} B_n \xrightarrow{\pi_n} B_{n-1}$. By Lemma 2.3, Ωi_n has a left homotopy inverse $r_n \colon \Omega B_n \longrightarrow \Omega S^2$. Let $j_n = i_n$. Next, consider the fibration $S^2 \xrightarrow{i_{n-1}} B_{n-1} \xrightarrow{\pi_{n-1}} B_{n-2}$. Since π_n has a section, i_{n-1} lifts to a map $j_{n-1} \colon S^2 \longrightarrow B_n$. By Lemma 2.3, Ωi_{n-1} has a left homotopy inverse. Therefore Ωj_{n-1} has a left homotopy inverse $r_{n-1} \colon \Omega B_n \longrightarrow \Omega S^2$ which factors through $\Omega \pi_n$. In particular, observe that $r_{n-1} \circ \Omega j_n$ is null homotopic since it factors through $\Omega \pi_n \circ \Omega j_n = \Omega \pi_n \circ \Omega i_n$, and the latter composite is two consecutive maps in a homotopy fibration. Now iterate for $1 \leq k < n-1$. Consider the fibration $S^2 \xrightarrow{i_k} B_k \xrightarrow{\pi_k} B_{k-1}$. Since each map in the composite $g_k \colon B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_{k+1}} B_{k+1} \xrightarrow{\pi_k} B_k$ has a section, we obtain a section $B_k \longrightarrow B_n$ for g_k, implying that i_k lifts to a map $j_k \colon S^2 \longrightarrow B_n$. By Lemma 2.3, Ωi_k has a left homotopy inverse. Thus Ωj_k has a left homotopy inverse $r_k \colon \Omega B_n \longrightarrow \Omega S^2$ which factors through Ωg_k. In particular, $r_l \circ \Omega j_k$ is null homotopic for each $k < l \leq n$.

Taking the wedge sum of the maps j_k for $1 \leq k \leq n$ we obtain a map $J \colon \bigvee_{i=1}^n S^2 \longrightarrow B_n$. Let $t_k \colon S^2 \longrightarrow \bigvee_{i=1}^n S^2$ be the inclusion of the k^{th}-wedge summand. Observe that $J \circ t_k \simeq j_k$. After looping the maps t_k can be multiplied to give a map $T \colon \prod_{k=1}^n \Omega S^2 \longrightarrow \Omega(\bigvee_{i=1}^n S^2)$. Taking the product of the maps Ωr_k we obtain a map $R \colon \Omega B_n \longrightarrow \prod_{k=1}^n \Omega S^2$. The fact that $r_k \circ \Omega j_k$ is a homotopy equivalence while $r_l \circ \Omega j_k \simeq *$ for $k < l \leq n$ implies that the composite

$$\prod_{k=1}^n \Omega S^2 \xrightarrow{T} \Omega(\bigvee_{i=1}^n S^2) \xrightarrow{J} \Omega B_n \xrightarrow{R} \prod_{k=1}^n \Omega S^2$$

is a homotopy equivalence. Thus ΩJ has a right homotopy inverse.

Finally observe that as each i_k induces the projection onto a generator in cohomology, the map J, which lifts all the i_k's to B_n, induces an isomorphism on H^2. Thus, up to a self-equivalence of $\bigvee_{i=1}^n S^2$, J is homotopic to the inclusion m_1 of the 2-skeleton into B_n. Therefore as Ωm_1 has a right homotopy inverse, so does ΩJ.

The remaining cases are easier. Consider the skeletal inclusion $M_k \xrightarrow{m_k} B_n$. Observe that the skeletal inclusion $\bigvee_{i=1}^n S^2 = M_1 \xrightarrow{m_1} B_n$ factors through the skeletal inclusion $M_k \xrightarrow{m_k} B_n$. Since Ωm_1 has a right homotopy inverse $\Omega B_n \longrightarrow \Omega M_1$, the composite $\Omega B_n \longrightarrow \Omega M_1 \longrightarrow \Omega M_k$ is a right homotopy inverse for Ωm_k. $\square$

Our last task in this section is to relate the fibration $Q_k \xrightarrow{\varphi_k} M_k \xrightarrow{m_k} B_n$ to the cofibration $\bigvee_{i=1}^{s_k} S^{2k+1} \xrightarrow{f_k} M_k \longrightarrow M_{k+1}$. We exclude the $k = n$ case as $M_n = B_n$ so $Q_n \simeq *$.

Lemma 2.5. *For $1 \leq k < n$, there is a homotopy commutative diagram*

$$\begin{tikzcd} & \bigvee_{i=1}^{s_k} S^{2k+1} \arrow[dl, "\lambda_k"'] \arrow[d, "f_k"] \\ Q_k \arrow[r, "\varphi_k"] & M_k \end{tikzcd}$$

where λ_k induces an isomorphism on π_{2k+1}.

Proof. Consider the following diagram

$$\begin{tikzcd} \bigvee_{i=1}^{s_k} S^{2k+1} \arrow[r, "f_k"] \arrow[d, "\lambda_k"] & M_k \arrow[r] \arrow[d, equal] & M_{k+1} \arrow[d, "m_{k+1}"] \\ Q_k \arrow[r, "\varphi_k"] & M_k \arrow[r, "m_k"] & B_n \end{tikzcd} \tag{1}$$

where the map λ_k is to be defined momentarily. The right square commutes since all the maps are skeletal inclusions. As the top row is a homotopy cofibration and the bottom row is a homotopy fibration, the composite $m_k \circ f_k$ is null homotopic, so f_k lifts through φ_k to a map $\lambda_k \colon \bigvee S^{2k+1} \longrightarrow Q_k$. Thus the entire diagram homotopy commutes. Note that the homotopy class of λ_k is uniquely determined, since any two choices would have a difference which lifted through the fibration connecting map $\Omega B_n \longrightarrow Q_k$, but this map is null homotopic by Lemma 2.4. The Blakers-Massey Theorem implies that the cofibration along the top row and the fibration along the bottom row are equivalent in dimensions $< 2k+1$. But it is dimension $2k+1$ that we care about, so we need to look more closely at how the two rows compare in this boudary dimension.

We will compare long exact sequences induced by the cofibration along the top and bottom rows of (1). First, let $q \colon M_{k+1} \longrightarrow \bigvee_{i=1}^{s_k} S^{2k+2}$ be the cofibration connecting map. The homotopy cofibration along the top row in (1) induces a long exact sequence

$$\cdots \longrightarrow H_{2k+2}(M_{k+1}) \xrightarrow{\delta} H_{2k+1}(\bigvee_{i=1}^{s_k} S^{2k+1}) \longrightarrow H_{2k+1}(M_k) \longrightarrow H_{2k+1}(M_{k+1}) \longrightarrow \cdots$$

where δ is the connecting map. Explicitly, δ is the composite

$$H_{2k+2}(M_{k+1}) \xrightarrow{q_*} H_{2k+2}(\bigvee_{i=1}^{s_k} S^{2k+2}) \xrightarrow{\cong} H_{2k+1}(\bigvee_{i=1}^{s_k} S^{2k+1})$$

where the right map is the inverse to the suspension map.

Next, consider the homotopy fibration $Q_k \xrightarrow{\varphi_k} M_k \xrightarrow{m_k} B_n$. Notice that as $H^*(B_n)$ is concentrated in even degrees, the fact that M_k is the $2k$-skeleton of B_n means that it also the $(2k+1)$-skeleton. Thus a Serre spectral sequence calculation immediately shows that Q_k is $2k$-connected. So as B_n is 1-connected, the Serre exact sequence for this fibration is of the form

$$H_{2k+2}(Q_k) \longrightarrow H_{2k+2}(M_k) \longrightarrow H_{2k+2}(B_n) \xrightarrow{\partial} H_{2k+1}(Q_k) \longrightarrow H_{2k+1}(M_k) \longrightarrow \cdots$$

where ∂ is the boundary map. The map ∂ is the transgression in the Serre spectral sequence, which in this case can be made explicit as follows. Restrict B_n to its $(2k+2)$-skeleton M_{k+1}. As $H^*(B_n)$ is concentrated in even degrees, so is its dual $H_*(B_n)$. In particular, $H_{2k+3}(B_n) \cong 0$, so the skeletal inclusion $M_{k+1} \xrightarrow{m_{k+1}} B_n$ induces an isomorphism on H_{2k+2}. Thus ∂ is determined by the composite $H_{2k+2}(M_{k+1}) \xrightarrow{q_*} H_{2k+2}(\bigvee_{i=1}^{s_k} S^{2k+2}) \xrightarrow{\cong} H_{2k+1}(\bigvee_{i=1}^{s_k} S^{2k+1})$ where the right map is the inverse to the suspension map. In particular, $\partial = \delta$.

Hence the morphism of long exact sequences in homology induced by (1) is actually an equivalence in dimensions $\leq 2k+1$. Thus λ_k induces an isomorphism in H_{2k+1}. Since $\bigvee_{i=1}^{s_k} S^{2k+1}$ and Q_k are both $2k$-connected, the Hurewicz Theorem therefore implies that λ_k induces an isomorphim on π_{2k+1}. $\square$

3. Some localized properties of Bott manifolds

Recall that P_l is the set consisting of the first l primes, and R_l is the ring of integers localized away from P_l. We begin with a localized version of Proposition 2.1, which is a straightforward consequence of [1, Theorem 1.3.8].

Proposition 3.1. *Let C and C' be two $(2,4)$-complexes with corresponding attaching maps a and a'. Then the following are equivalent:*

(a) *there is a ring isomorphism $H^*(C; R_l) \cong H^*(C'; R_l)$;*

(b) *localized away from P_l, the attaching maps a and a' are homotopic*

(c) *localized away from P_l, there is a homotopy equivalence $C \simeq C'$.*

Note that as $H^*(B_n)$ is torsion-free, so is $H^*(B_n; R_l)$. Therefore, arguing just as in Lemma 2.2, we obtain the following.

Lemma 3.2. *Let B_n and B'_n be Bott manifolds. Then the following are equivalent:*

(a) *there is a ring isomorphism $H^*(B_n; R_l) \cong H^*(B'_n; R_l)$;*

(b) *there is a ring isomorphism $H^*((B_n)_4; R_l) \cong H^*((B'_n)_4; R_l)$;*

(c) *localized away from P_l, there is a homotopy equivalence $(B_n)_4 \simeq (B'_n)_4$.*

Next, we turn to the homotopy groups of Bott manifolds. We begin integrally. The homotopy decomposition in Lemma 2.3 immediately implies the following.

Corollary 3.3. *Let B_n be a Bott manifold of height n. Then for each $m \geq 1$ there is an isomorphism $\pi_m(B_n) \cong \oplus_{i=1}^n \pi_m(S^2)$.*

Next, we recall some information about the homotopy groups of S^2. Classically, Serre showed that there is a homotopy equivalence $\Omega S^2 \simeq S^1 \times \Omega S^3$. Consequently, since S^1 is the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$, we obtain an isomorphism $\pi_m(S^2) \cong \pi_m(S^3)$ for every $m > 2$. In particular, $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$, and $\pi_m(S^2)$ is a torsion group for every $m > 3$. These torsion groups were calculated through a range by Toda [7, 8]. Let p be a prime. Then for $3 < m \leq 4p-3$ the p-component of $\pi_m(S^3)$ is as follows:

$$\pi_m(S^3)_{(p)} \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & m = 2p, 4p-3 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We now use Toda's calculations to obtain the following statement about the *odd dimensional* homotopy groups of S^2.

Lemma 3.4. *Localize away from P_l. If $3 < 2m+1 < 4p_{l+1}-3$ then $\pi_{2m+1}(S^2) \cong 0$.*

Proof. For any prime p, by (2) the least dimensional nonvanishing homotopy group of S^3 occurs in dimension $2p$, and the least dimensional nonvanishing homotopy group of S^3 in an odd dimension occurs in dimension $4p-3$. So if we localize away from P_l, the least dimensional nonvanishing homotopy group of S^3 in an odd dimension occurs in dimension $4p_{l+1}-3$. The lemma now follows from the fact that $\pi_k(S^2) \cong \pi_k(S^3)$ for any $k > 2$. $\square$

Let B_n be a Bott manifold of height n. By Corollary 3.3, for any $m > 3$ there is an isomorphism $\pi_m(B_n) \cong \oplus_{i=1}^n \pi_m(S^2)$. Hence Lemma 3.4 immediately implies the following.

Lemma 3.5. *Let B_n be a Bott manifold of height n. Localize away from P_l. If $3 < 2m+1 < 4p_{l+1}-3$ then $\pi_{2m+1}(B_n) \cong 0$.*

Note that the condition on homotopy groups in Lemma 3.5 holds for any n. In the next Lemma the dimension of B_n does play a role. Recall the homotopy fibration $R_k \xrightarrow{\varphi_k} M_k \xrightarrow{m_k} B_n$ induced by including the $2k$-skeleton M_k into B_n.

Corollary 3.6. *Localize away from P_l. If $n < 2p_{l+1}-1$ then for any $2 \le k < n$, the map $Q_k \xrightarrow{\varphi_k} M_k$ induces an isomorphism on π_{2k+1}.*

Proof. As $2 \le k \le n-1$, we have $3 < 2k+1 \le 2n-1$. Since $n < 2p_{l+1}-1$, we have $2n-1 < 4p_{l+1}-3$. Thus Lemma 3.5 implies that $\pi_{2k+1}(B_n) \cong 0$. Therefore the homotopy equivalence $\Omega M_k \simeq \Omega B_n \times \Omega Q_k$ in Lemma 2.4 implies that $\pi_{2k+1}(R_k) \cong \pi_{2k+1}(M_k)$, with the isomorphism induced by φ_k. $\square$

Combining Lemma 2.5 and Corollary 3.6 we immediately obtain the following.

Corollary 3.7. *Localize away from P_l. If $n < 2p_{l+1}-1$ then for any $2 \le k < n$, the map $\bigvee_{i=1}^{s_k} S^{2k+1} \xrightarrow{f_k} M_k$ induces an isomorphism π_{2k+1}.*

4. The proof of Theorem 1.1

We now combine the results of the previous two sections to prove Theorem 1.1.

Proof of Theorem 1.1. Fix n and let p_{l+1} be the smallest prime $> \frac{n+1}{2}$. Localize spaces away from P_l.
Part (b) implies part (a). This is clear, since a homotopy equivalence $B_n \simeq B_n'$ induces an isomorphism $H^*(B_n : R_l) \cong H^*(B_n; R_l)$.

Part (a) implies part (b). Suppose that $H^*(B_n; R_l) \cong H^*(B_n'; R_l)$. By Lemma 3.2, this implies that there is a homotopy equivalence $(B_n)_4 \longrightarrow (B_n')_4$. In terms of the skeletal filtrations on B_n and B_n', we have a homotopy equivalence $g_2 \colon M_2 \longrightarrow M_2'$. Assume inductively that there is a homotopy equivalence $M_k \xrightarrow{g_k} M_k'$. We wish to show that there is a homotopy equivalence $M_{k+1} \xrightarrow{g_{k+1}} M_{k+1}'$. If so, then by induction we obtain a homotopy equivalence $B_n = M_n \xrightarrow{g_n} M_n' = B_n'$, proving the theorem.

To construct g_{k+1}, for each $1 \le k \le n-1$, we will show that there is a homotopy commutative diagram

$$\begin{tikzcd} \bigvee_{i=1}^{s_k} S^{2k+1} \arrow[r, "f_k"] \arrow[d, "h_k"] & M_k \arrow[d, "g_k"] \\ \bigvee_{i=1}^{s_k} S^{2k+1} \arrow[r, "f_k'"] & M_k' \end{tikzcd} \tag{3}$$

where h_k is a homotopy equivalence. Granting this, we obtain a homotopy cofibration diagram

$$\begin{tikzcd}
\bigvee_{i=1}^{s_k} S^{2k+1} \arrow[r, "f_k"] \arrow[d, "h_k"] & M_k \arrow[r] \arrow[d, "g_k"] & M_{k+1} \arrow[d, "g_{k+1}"] \\
\bigvee_{i=1}^{s_k} S^{2k+1} \arrow[r, "f'_k"] & Mk' \arrow[r] & M'_k
\end{tikzcd}$$

for some induced map g_{k+1} of cofibres. This cofibration diagram induces a morphism of long exact sequences of homology groups. Since h_k and g_k are homotopy equivalences, they induce isomorphisms in homology. So when the five-lemma is applied to the morphism of long exact sequences of homology groups, we obtain that $(g_{k+1})_*$ is also an isomorphism. Hence g_{k+1} is a homotopy equivalence.

It remains to show the existence of (3) for each $2 \le k \le n-1$. Fix k and consider the composite $\bigvee S^{2k+1} \xrightarrow{f_k} M_k \xrightarrow{g_k} M'_k \xrightarrow{m'_k} B'_n$. By hypothesis, $n < 2p_{l+1} - 1$. So Corollary 3.6 implies that the fibre $Q'_k \xrightarrow{\varphi'_k} M'_k$ of m'_k induces an isomorphism on π_{2k+1}. That is, m'_k induces the zero map on π_{2k+1}. Thus $m_k \circ g_k \circ f_k$ is null homotopic. Therefore we obtain a lift

$$\begin{tikzcd}
\bigvee_{i=1}^{s_k} S^{2k+1} \arrow[r, "f_k"] \arrow[d, "\gamma_k"] & M_k \arrow[d, "g_k"] & \\
Q'_k \arrow[r, "\varphi'_k"] & M'_k \arrow[r, "m'_k"] & B'_n.
\end{tikzcd}$$

for some map γ_k. Since γ_k represents a homotopy class in $\pi_{2k+1}(Q'_k)$ and Lemma 2.5 states that the map $\bigvee_{i=1}^{s_k} S^{2k+1} \xrightarrow{\lambda'_k} Q'_k$ induces an isomorphism on π_{2k+1}, we have γ_k factoring as a composite $\bigvee_{i=1}^{s_k} S^{2k+1} \xrightarrow{h_k} \bigvee_{i=1}^{k} S^{2k+1} \xrightarrow{\lambda'_k} Q'_k$ for some map h_k. By Lemma 2.5, the composite $\bigvee_{i=1}^{s_k} S^{2k+1} \xrightarrow{\lambda'_k} Q'_k \xrightarrow{\varphi'_k} M'_k$ is homotopic to f'_k. Thus the previous homotopy commutative diagram can be refined to a homotopy commutative diagram

$$\begin{tikzcd}
\bigvee_{i=1}^{s_k} S^{2k+1} \arrow[r, "f_k"] \arrow[d, "h_k"'] & M_k \arrow[d, "g_k"] \\
\bigvee_{i=1}^{s_k} S^{2k+1} \arrow[r, "f'_k"] & M'_k.
\end{tikzcd} \tag{4}$$

Applying π_{2k+1} to (4) we obtain a commutative diagram

$$\begin{tikzcd}
\pi_{2k+1}(\bigvee_{i=1}^{s_k} S^{2k+1}) \arrow[r, "(f_k)_*"] \arrow[d, "(h_k)_*"] & \pi_{2k+1}(M_k) \arrow[d, "(g_k)_*"] \\
\pi_{2k+1}(\bigvee_{i=1}^{s_k} S^{2k+1}) \arrow[r, "(f'_k)_*"] & \pi_{2k+1}(M'_k).
\end{tikzcd}$$

By Corollary 3.7 both $(f_k)_*$ and $(f'_k)_*$ are isomorphisms. By inductive hypothesis, g_k is a homotopy equivalence so $(g_k)_*$ is an isomorphism. The commutativity of the diagram therefore implies that $(h_k)_*$ is also an isomorphism. Hence h_k is a homotopy equivalence. Therefore (4) establishes the $k > 2$ case of (3), as required, thereby completing the induction. $\square$

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Submitted December 28, 2011

Терио С. Гомотопическое свойство жёсткости для многообразий Ботта. Дальневосточный математический журнал. 2012. Т. 12. № 1. С. 89–97.

АННОТАЦИЯ

Гипотеза жёсткости в торической топологии утверждает, что два торических многообразия диффеоморфны тогда и только тогда, когда их кольца целочисленных когомологий изоморфны как градуированные кольца. Гипотеза доказана лишь для некоторых случаев малой размерности. Мы рассматриваем ослабленный вариант гипотезы, в котором диффеоморфизм заменяется на гомотопическую эквивалентность, и показываем, что в этой ослабленной версии гипотеза верна для многообразий Ботта, если обратить достаточное количество простых чисел. В частности, мы показываем, что рациональный гомотопический тип многообразия Ботта определяется его рациональным кольцом когомологий.
Материалы статьи своим появлением обязаны дискуссиям, проходившим на Международной конференции «Торическая топология и автоморфные функции» (5-10 сентября 2011 г., г. Хабаровск, Россия).
Ключевые слова: *многообразие Ботта, жёсткость.*