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The cubical homology of trace monoids

This article contains an overview of the results of the author's study in the field of algebraic topology used in computer science. The relationship between the cubical homology groups of generalized tori and homology groups of partial trace monoid actions is described. Algorithms for computing the homology groups of asynchronous systems, Petri nets, and Mazurkiewicz trace languages are shown.

The main results of the paper were reported at the International conference «Toric Topology and Automorphic Functions» (September, 5-10th, 2011, Khabarovsk, Russia).

Key words: *semicubical set, homology of small categories, free partially commutative monoid, trace monoid, asynchronous transition system, Petri nets, trace languages.*

Introduction

Trace monoids have found many applications in computer science [3], [19]. M. Bednarczyk [2] studied and applied the category of asynchronous systems. The author has proved that any asynchronous system can be regarded as a partial trace monoid with action on a set. It allows us to build homology theory for the category of asynchronous systems and Petri nets [9]. It should be noted that the homology theory was introduced and studied for higher dimensional automata in [6]. E. Haucourt [7] applied the Baues-Wirsching homology.

The paper is a survey of the author's results on the homology groups of models for concurrency. We study the relationship between the cubical homology of generalized tori and homology of a trace monoid action on a set. We build the algorithms for computing the homology groups of asynchronous systems, elementary Petri nets, and Mazurkiewicz trace languages. It allows us to solve the problem posed in [9, Open problem 1] constructing an algorithm for computing homology groups of the elementary Petri nets.

1. Trace monoids and their partial actions

This section is devoted to the basic definitions and the problems that have arisen.

1.1. Notations

Let Set be a category of all sets and maps and let Ab be a category of all Abelian groups and homomorphisms. We denote by $\mathbb{Z}$ the additive group of integers. Let $\mathbb{N}$ denotes the set of nonnegative integers or the free monoid $\{1, a, a^2, \cdots\}$ generated by one element. Given a category $\mathcal{A}$, we denote the opposite category by $\mathcal{A}^{op}$. Let $\mathrm{Ob}\,\mathcal{A}$ denotes the class of all objects and $\mathrm{Mor}\,\mathcal{A}$

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the class of all morphisms in category $\mathcal{A}$. Given objects $a, b \in \mathrm{Ob}\,\mathcal{A}$, we denote by $\mathcal{A}(a,b)$ the set of all morphisms $a \to b$. For any small category $\mathscr{C}$, functors $F : \mathscr{C} \to \mathcal{A}$ will be called *diagrams of objects in $\mathcal{A}$ on $\mathscr{C}$*. In this case, along with the notation $F : \mathscr{C} \to \mathcal{A}$ we use the notation $\{F(c)\}_{c\in\mathscr{C}}$. The category $\mathcal{A}^{\mathscr{C}}$ of functors $\mathscr{C} \to \mathcal{A}$ is called a *diagram category*.

Let $\Delta\mathbb{Z} : \mathscr{C} \to \mathrm{Ab}$ be a diagram having the value $\mathbb{Z}$ at each $c \in \mathrm{Ob}\mathscr{C}$ and the value $1_{\mathbb{Z}}$ at each $\alpha \in Mor\mathscr{C}$.

Given a family of Abelian groups $\{A_j\}_{j\in J}$, the direct sum is denoted by $\bigoplus\limits_{j\in J} A_j$. Elements of summands are denoted as pairs (j, g) with $j \in J$ and $g \in A_j$. If $A_j = A$ for all $j \in J$, then this direct sum is denoted by $\bigoplus\limits_{j\in J} A$ or $A^{(p)}$ where $p = |J|$ is a cardinal number.

1.2. Trace monoids

Let E be a set and let $I \subseteq E\times E$ be an arbitrary subset. The set $I \subseteq E\times E$ is an *independence relation* on E if the following conditions are met:

- $(\forall a \in E)(a, a) \notin I$,
- $(\forall a \in E)(\forall b \in E)\ (a, b) \in I \Rightarrow (b, a) \in I.$

Let E^* be a free monoid generated by a set E. It consists of the words in alphabet E. The binary operation is defined as the concatenation of words $(a_1 \cdots a_m, b_1 \cdots b_n) \mapsto a_1 \cdots a_m b_1 \cdots b_n$. The empty word is denoted by 1.

Definition 1.1. *Let I be an independence relation on a set E. A trace monoid (or free partially commutative monoid) $M(E,I)$ is the factor monoid $E^*/(\equiv)$ by a least equivalence relation for which $uabv \equiv ubav$, for all $(a,b) \in I$, $u \in E^*$, $v \in E^*$. Elements $a, b \in E$ for which $(a,b) \in I$ are called commuting generators.*

This definition is more general than the one given in [3] since we do not demand that E should be finite.

For example, if $E = \{a, b\}$, $I = \{(a,b),(b,a)\}$, then $M(E,I) \cong \mathbb{N}^2$ is a free commutative monoid generated by two elements.

If $I = \emptyset$, then $M(E,I) = E^*$.

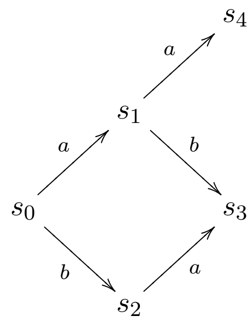
Any element $w = a_1 \cdots a_n \in M(E,I)$ of a trace monoid can be interpreted as finite sequence of instructions $a_1, a_2, \cdots, a_n$ in a program. Relation I consists of pairs (a,b) instructions which can be executed concurrenrly.

1.3. State space

A *partial map* $f : E \rightharpoonup E'$ between sets E and E' is a relation $f \subseteq E \times E'$ for which $(e, e_1') \in f$ & $(e, e_2') \in f$ implies $e_1' = e_2'$. Let $PSet$ be a category of all sets and partial maps between them. Any trace monoid $M(E,I)$ can be considered as a category with the unique object denoted by $o(M(E,I))$.

A *partial trace monoid action* of $M(E,I)$ on a set S is a functor $\mathbf{S} : M(E,I)^{op} \to PSet$ such that its value at $o(M(E,I))$ equals S. We denote $\mathbf{S}(w)(s)$ by $s \cdot w$. A *state space* $(M(E,I), S)$ consists of a trace monoid $M(E,I)$ with a partial action on a set S. A state space $(M(E,I), S)$ is determined by partial maps $(-)\cdot a : S \rightharpoonup S$ corresponding to $a \in E$. Hence, it can be given by a directed graph with vertexes $s \in S$ and labeled edges $s \stackrel{a}{\to} s \cdot e$.

For example, if $E = \{a, b\}$ and $I = \{(a, b), (b, a)\}$, then the directed graph with labeled edges

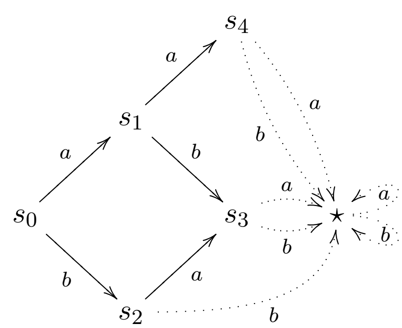


determines the action for which $s_0 \cdot a = s_1$, $s_0 \cdot b = s_2$, $s_1 \cdot a = s_4$, $s_1 \cdot b = s_3$, $s_2 \cdot a = s_3$. But $s_2 \cdot b$, $s_3 \cdot a$, $s_3 \cdot b$, $s_4 \cdot a$, and $s_4 \cdot b$ are not defined.

1.4. Augmented state space

In order to make the action $(M(E, I), S)$ to be total, we add the state $*$ and extend the partial maps $(-) \cdot a : S \rightharpoonup S$ to the (total) maps $(-) \cdot a : S \sqcup \{*\} \to S \sqcup \{*\}$ acting by $s \cdot a = *$ if $s \cdot a$ is not defined. Let $S_* = S \sqcup \{*\}$ and $* \cdot a = *$. Then the pair $(M(E, I), S_*)$ consists of a trace monoid with the total action on the set S_*. This pair is called the state space with an augmentation.

For example, the previous state space gives the augmented state space



Let $(M(E, I), S)$ be a state space. Consider an *augmented state category* $K_*(S)$ as follows. Its class of objects is set $S_* = S \sqcup \{*\}$. Morphisms $s \to s'$ are triples (s, w, s') of $s \in S_*$, $s' \in S_*$, $w \in M(E, I)$.

For any subset $\Sigma \subseteq S_*$, let $K(\Sigma) \subseteq K_*(S)$ denotes a full subcategory with the class of objects Σ. For $\Sigma = S$, $K(S) \subseteq K_*(S)$ will be called a *state category*.

1.5. Homology groups of a small category

Let $\mathscr{C}$ be a small category and let $F : \mathscr{C} \to \mathrm{Ab}$ be a functor into the category of Abelian groups and homomorphisms.

Definition 1.2. *Let $\mathscr{C}$ be a small category and let $F : \mathscr{C} \to \mathrm{Ab}$ be a functor into the category of Abelian groups and homomorphisms. Let $C_\diamond(\mathscr{C}, F)$ denotes a chain complex of Abelian groups*

$$C_n(\mathscr{C}, F) = \bigoplus_{c_0 \to \cdots \to c_n} F(c_0), \quad n \geqslant 0,$$

and homomorphisms $d_n = \sum_{i=0}^{n}(-1)^i d_i^n : C_n(\mathscr{C}, F) \to C_{n-1}(\mathscr{C}, F)$, $n > 0$, *where* $d_i^n(c_0 \stackrel{\alpha_1}{\to} c_1 \stackrel{\alpha_2}{\to} \cdots \stackrel{\alpha_n}{\to} c_n, a) =$

$$\begin{cases} (c_1 \stackrel{\alpha_2}{\to} \cdots \stackrel{\alpha_n}{\to} c_n, F(c_0 \stackrel{\alpha_1}{\to} c_1)(a)) \ , & \text{if } i = 0 \\ (c_0 \stackrel{\alpha_1}{\to} \cdots \stackrel{\alpha_{i-1}}{\to} c_{i-1} \stackrel{\alpha_{i+1}\alpha_i}{\to} c_{i+1} \stackrel{\alpha_{i+2}}{\to} \cdots \stackrel{\alpha_n}{\to} c_n, a) \ , & \text{if } 1 \leqslant i \leqslant n-1 \\ (c_0 \stackrel{\alpha_1}{\to} \cdots \stackrel{\alpha_{n-1}}{\to} c_{n-1}, a) \ , & \text{if } i = n \end{cases}$$

For every integer $n \geqslant 0$, *the* n-th homology group $H_n(\mathscr{C}, F)$ of $\mathscr{C}$ with coefficients in F *is the factor groups* $\mathrm{Ker}\,(d_n)/\mathrm{Im}\,(d_{n+1})$.

It is well known that the functors $H_n(C_\diamond(\mathscr{C}, -)) : \mathrm{Ab}^{\mathscr{C}} \to \mathrm{Ab}$ are isomorphic to the left derived functors $\varinjlim_n^{\mathscr{C}}$ of the colimit functor $\varinjlim^{\mathscr{C}} : \mathrm{Ab}^{\mathscr{C}} \to \mathrm{Ab}$.

Hence, the Abelian groups $H_n(\mathscr{C}, F)$ can be defined as homology groups of the complex

$$0 \leftarrow \varinjlim^{\mathscr{C}} P_0 \leftarrow \varinjlim^{\mathscr{C}} P_1 \leftarrow \varinjlim^{\mathscr{C}} P_2 \leftarrow \cdots$$

obtained from a projective resolution

$$0 \leftarrow F \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

of $F \in \mathrm{Ab}^{\mathscr{C}}$ by the application of the functor $\varinjlim^{\mathscr{C}}$.

1.6. Homology of state categories, asynchronous systems and Petri nets

For an arbitrary small category $\mathscr{C}$, let $\Delta\,\mathbb{Z} : \mathscr{C} \to \mathrm{Ab}$ be the functor taking constant values $\mathbb{Z}$ at objects and $1_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ at morphisms of $\mathscr{C}$.

An *asynchronous system* can be defined as a triple $(S, s_0, M(E, I))$ where $(S, M(E, I))$ is a state space and $s_0 \in S$ is a distinguished element [9]. Elements of $S(s_0) = \{s{\cdot}\mu | \mu \in M(E, I)\} \subseteq S$ are *reachable states. Homology groups of an asynchronous system with coefficients in an arbitrary functor* $F : K(S) \to \mathrm{Ab}$ are Abelian groups $\varinjlim_n^{K(S(s_0))} F|_{K(S(s_0))}$.

For a set B, the set of all its subsets is denoted by 2^B.

A *CE net* [9] or *Petri net* [24] is a quintuple $(B, E, pre, post, s_0)$ consisting of finite sets B and E, the maps $pre, post : E \to 2^B$, and a subset $s_0 \subseteq B$.

Let $\mathcal{N} = (B, E, pre, post, s_0)$ be a CE net. Relation $I \subseteq E \times E$ is defined as a set of all pairs (a, b) for which $(pre(a) \cup post(a)) \cap (pre(b) \cup post(b)) = \emptyset$. We assign to every element $e \in E$ a partial map $(-){\cdot}e : 2^B \rightharpoonup 2^B$ which is defined as $s{\cdot}e = (s \setminus pre(e)) \cup post(e)$ for all $s \subseteq B$ meeting the condition $(pre(e) \subseteq s)$ & $(post(e) \cap s = \emptyset)$ [19]. This defines a partial action of $M(E, I)$ on set 2^B. Assuming $S = 2^B$, we get an asynchronous system $(S, s_0, M(E, I))$, which corresponds to the CE net $\mathcal{N} = (B, E, pre, post, s_0)$. The homology groups of $H_n(\mathcal{N})$ were defined in [9] as $\varinjlim_n^{K(S(s_0))} \Delta\,\mathbb{Z}$ where $S(s_0)$ is a set of all reachable states.

For computing the groups $H_1(K(S), \Delta\,\mathbb{Z})$, an algorithm was built in [9]. It is suitable for the calculation of $H_1(\mathcal{N})$. The following question was formulated in [9].

Problem 1. *Constructing an algorithm for computing the integral homology groups of CE nets.*

By the definition of $H_n(\mathcal{N})$, this problem will be solved when we find an algorithm to compute the homology groups $H_n(K(S), \Delta\,\mathbb{Z})$ for the state categories. Problem 1 could not be solved for a long time. We present a way to solve this problem. Detailed proof is published in the preprint [12].

Let $M(E, I)$ be a trace monoid. Its generators $a, b \in E$ are called to be *independent* if $(a, b) \in I$. In [9], it was proved that if $M(E, I)$ does not contain triples of pairwise independent generators, then $H_n(K_*(S), \Delta\,\mathbb{Z}) = 0$ for $n > 2$. The following conjecture was put forward in [9].

Problem 2. *Let $n > 0$ be the maximal number of pairwise independent generators. Prove that $H_k(K_*(S), F) = 0$ for any $k > n$ and for any functor $F : K_*(S) \to \mathrm{Ab}$.*

In the case of finite E, the conjecture was proved by L. Yu. Polyakova [23]. A complete solution of Problem 2 is given in [10].

2. Semicubical sets and generalized tori

Let me remind you of the definition of semicubical set and its geometric realization. We introduce generalized tori and assign a semicubical set to every partial trace monoid action.

2.1. Semicubical sets

Let $\square_+$ be the category of posets $\mathbb{I}^n$, $n \in \mathbb{N}$, where $\mathbb{I}$ is the set $\{0, 1\}$ ordered by $0 < 1$. Morphisms in $\square_+$ are increasing maps admitting a decomposition in the composition of maps $\delta_i^{k,\varepsilon} : \mathbb{I}^{k-1} \to \mathbb{I}^k$, $1 \leqslant i \leqslant k$, $\varepsilon \in \mathbb{I}$ defined as $\delta_i^{k,\varepsilon}(x_1, \cdots, x_{k-1}) = (x_1, \cdots, x_{i-1}, \varepsilon, x_i, \cdots, x_{k-1})$.

A *semicubical set* is any functor $X : \square_+^{op} \to \mathrm{Set}$. In [6], it is called *precubical set*. Morphisms between semicubical sets are defined as natural transformations. Any semicubical set can be given by a pair $(X_n, \partial_i^{n,\varepsilon})$ consisting of sequence of sets $(X_n)_{n\in\mathbb{N}}$ and a family of maps $\partial_i^{n,\varepsilon} : X_n \to X_{n-1}$, defined for $1 \leqslant i \leqslant n$, $\varepsilon \in \{0, 1\}$, and satisfying to the condition

$$\partial_i^{n-1,\alpha} \circ \partial_j^{n,\beta} = \partial_{j-1}^{n-1,\beta} \circ \partial_i^{n,\alpha}\ , \text{ for } \alpha, \beta \in \{0, 1\}, n \geqslant 2 \text{ and } 1 \leqslant i < j \leqslant n.$$

These maps will be equal $\partial_i^{k,\varepsilon} = X(\delta_i^{k,\varepsilon})$.

Semicubical objects in an arbitrary category $\mathcal{A}$ are defined similarly as functors $\square_+^{op} \to \mathcal{A}$.

2.2. Geometric realization

Let $X \in \mathrm{Set}^{\square_+^{op}}$ be a semicubical set. Its *geometric realization* [4] is defined as the topological quotient space

$$|X|_{\square_+} = \coprod_{n\in\mathbb{N}} X_n \times [0, 1]^n / \equiv$$

with respect to the smallest equivalence relation satisfying

$$(\partial_i^{n,\nu} x, t_1, \cdots, t_{n-1}) \equiv (x, t_1, \cdots, t_{i-1}, \nu, t_i, \cdots, t_{n-1}),$$

for all $n \geqslant 0$, $\nu \in \{0, 1\}$, $1 \leqslant i \leqslant n$, $t_i \in [0, 1]$. Geometric realization determines the functor $|-|_{\square_+}$ assigning to every morphism of semicubical sets $f : X \to Y$ the continuous map $|f|_{\square_+} : |X|_{\square_+} \to |Y|_{\square_+}$ such that $|f|_{\square_+}(x, t_1, \cdots, t_n) = (f(x), t_1, \cdots, t_n)$. The functor $|-|_{\square_+}$ can be constructed from the functor $H : \square_+ \to \mathrm{Top}$, $H(\mathbb{I}^n) = [0, 1]^n$, as in [5, Prop. II.1.3] by extending to the category of semicubical sets. It follows from [5, Prop. II.1.3] that $|-|_{\square_+}$ preserves colimits.

2.3. Generalized tori

For a trace monoid $M(E, I)$ with a total order relation $<$ on E, the *generalized torus* $T(E, I)$ is a semicubical set $(T_n(E, I), \partial_i^{n,\varepsilon})$ such that

$$T_n(E, I) = \{(a_1, \cdots, a_n) \in E^n : a_i < a_j \ \&\ (a_i, a_j) \in I \text{ for all } 1 \leqslant i < j \leqslant n\}$$

and $\partial_i^{n,\varepsilon}(a_1, \cdots, a_n) = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$, for all $n \geqslant 0$, $1 \leqslant i \leqslant n$, $\varepsilon \in \{0, 1\}$.

For example, if $E = \{a_1, \cdots, a_n\}$ ordered by $a_1 < a_2 < \cdots a_n$ with I consisting of all pairs (a_i, a_j) for which $i \neq j$, then the geometric realization $|T(E, I)|_{\square_+}$ is homeomorphic to the usual n-dimensional torus.

2.4. Semicubical set of a state set

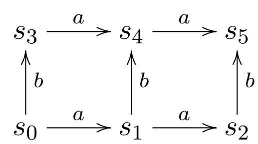
Let $(M(E,I),S)$ be a state space with a total relation $<$ on E. Assign the semicubical set $Q(E,I,S)$ to it with

$$Q_n(E,I,S) = \{(x,a_1,\cdots,a_n) \in S_* \times T_n(E,I) | \\ a_i < a_j \ \& \ (a_i,a_i) \text{ for all } 1 \leqslant i < j \leqslant n\}.$$

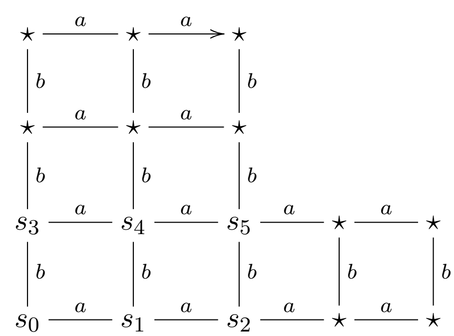
with the boundary maps $\partial_i^{n,\varepsilon}(x,a_1,\cdots,a_n) = (x \cdot a_i^{\varepsilon}, a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n)$ for $1 \leqslant i \leqslant n$, $n \geqslant 1$, $\varepsilon \in \{0,1\}$. Here $a^0 = 1$ and $a^1 = a$.

For any state space $(M(E,I),S)$, a set of all triples $(s,a,s') \in S \times E \times S$ for which $s \cdot a = s'$ is denoted by Tran.

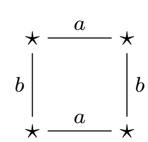
Example 2.1. Consider the state space consisting of $S = \{s_0,s_1,s_2,s_3,s_4,s_5\}$, $E = \{a,b\}$, $I = \{(a,b),(b,a)\}$. Elements in Tran are triples (s,e,s') corresponding to arrows $s \xrightarrow{e} s'$ in the following diagram:



The topological space $|Q(E,I,S)|_{\square_+}$ can be obtained from the union of unit squares



by identifying the vertexes $\star$ with each other, and by identifying the segments $\star \overset{a}{\text{——}} \star$ with each other, and with similar identifications for the segments $\star \overset{b}{\text{——}} \star$ and squares



Geometric realization can be interpreted as the topological space of intermediate states of computational processes.

2.5. Homology groups of semicubical sets

To solve Problems 1 and 2, we need some information from the article [15].

Given a semicubical set $X \in \mathrm{Set}^{\square_+^{op}}$, let $\square_+/X$ be the category with objects $\sigma \in \coprod_{n\in\mathbb{N}} X_n$. Its morphisms between $\sigma \in X_m$ and $\tau \in X_n$ are triples (α,σ,τ), $\alpha \in \square_+(\mathbb{I}^m,\mathbb{I}^n)$, satisfying the relation $X(\alpha)(\tau) = \sigma$. *Homological system on a semicubical set* X is an arbitrary functor $F : (\square_+/X)^{op} \to \mathrm{Ab}$.

Given a semicubical set X and a homological system F, consider Abelian groups $C_n(X,F) = \bigoplus\limits_{\sigma \in X_n} F(\sigma)$. Let $d_i^{n,\varepsilon} : C_n(X,F) \to C_{n-1}(X,F)$ be the homomorphisms

$$\bigoplus_{\sigma \in X_n} F(\sigma) \xrightarrow{d_i^{n,\varepsilon}} \bigoplus_{\sigma \in X_{n-1}} F(\sigma)$$

defined on the direct summands for $1 \leqslant i \leqslant n$, $\varepsilon \in \mathbb{I} = \{0,1\}$, $\sigma \in X_n$, $f \in F(\sigma)$ by the equation

$$d_i^{n.\varepsilon}(\sigma, f) = (\partial_i^{n,\varepsilon}(\sigma), F(\delta_i^{n,\varepsilon}, \partial_i^{n,\varepsilon}(\sigma), \sigma)(f)) \,.$$

For $n \geqslant 0$, the *homology groups* $H_n(X,F)$ *of semicubical set* X *with coefficients in* F are defined as homology of the complex $C_\diamond(X,F)$ consisting of the groups $C_n(X,F) = \bigoplus\limits_{\sigma \in X_n} F(\sigma)$ and differentials $d_n = \sum\limits_{i=1}^{n} (-1)^i (d_i^{n,1} - d_i^{n,0})$. Abelian groups $H_n(X, \Delta\,\mathbb{Z})$ are called the nth *integral* homology groups.

Proposition 2.1. *[15, Theorem 4.3] For any semicubical set X and a homological system F on X there are isomorphisms $\varinjlim_n^{(\square_+/X)^{op}} F \cong H_n(X,F)$, for all $n \geqslant 0$.*

Proposition 2.2. *[13, Prop. 2] For an arbitrary semicubical set X and integer $n \geqslant 0$, the group $H_n(X, \Delta\,\mathbb{Z})$ is isomorphic to the nth singular homology group of the topological space $|X|_{\square_+}$.*

3. Homology of factorization category

In [17], Leech cohomology groups of monoids were introduced. In this section, we study and apply Leech cohomology and homology groups for trace monoids.

3.1. Factorization category

Let $\mathscr{C}$ be a small category. Given $\alpha \in \operatorname{Mor}\mathscr{C}$, we denote its codomain by $\operatorname{cod}\alpha$ its codomain and its domain by $\operatorname{dom}\alpha$.

The *factorization category* $Fact(\mathscr{C})$ has objects $\operatorname{Ob}(Fact(\mathscr{C})) = \operatorname{Mor}\mathscr{C}$, and for every $\alpha, \beta \in \operatorname{Mor}(\mathscr{C})$ each element of $Fact(\mathscr{C})(\alpha,\beta)$ is determined by a pair (f,g) of $f,g \in \operatorname{Mor}(\mathscr{C})$ making commutative the diagram

$$\begin{array}{ccc} \operatorname{cod}\alpha & \xrightarrow{g} & \operatorname{cod}\beta \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ \operatorname{dom}\alpha & \xleftarrow[f]{} & \operatorname{dom}\beta \end{array}$$

For example, any monoid M considered as a small category with a unique object has a factorization category $Fact(M)$ such that $\operatorname{Ob}(Fact(M)) = M$. Morphisms are given by quadruples $\alpha \overset{(f,g)}{\to} \beta$ of $f, \alpha, \beta, g \in M$ satisfying $g\alpha f = \beta$.

3.2. Leech homology of generalized tori

Leech homology groups of monoid M with coefficients in functor $F : Fact(M)^{op} \to \mathrm{Ab}$ are defined as the groups $H_n(Fact(M)^{op}, F)$, $n \geqslant 0$.

Given trace monoid $M(E,I)$, let $\mathcal{S} : \square_+/T(E,I) \to Fact(M(E,I))$ be the functor assigning to each $(a_1, \cdots, a_n) \in \operatorname{Ob}\square_+/T(E,I)$ the object $a_1 \cdots a_n \in M(E,I) = \operatorname{Ob} Fact(M(E,I))$. Each morphism of the category $\square_+/T(E,I)$ can be decomposed into a composition of morphisms of the form $(\delta_i^{n,\varepsilon}),(a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n),(a_1,\cdots,a_n)$. Therefore, it suffices to define $\mathcal{S}$ on the morphisms of this kind. Let

$$\mathcal{S}(\delta_i^{n,\varepsilon},(a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n),(a_1,\cdots,a_n)) = \\ (a_1\cdots a_{i-1}a_{i+1}\cdots a_n \stackrel{(a^{1-\varepsilon},a^{\varepsilon})}{\longrightarrow} a_1\cdots a_n)$$

where $a^0=1$, and $a^1=a$.

Theorem 3.1. *[14] If E does not contain infinite subsets of pairwise independent elements, then there are natural in $F\in \mathrm{Ab}^{Fact(M(E,I))^{op}}$ isomorphisms*

$$H_n(Fact(M(E,I))^{op},F)\cong H_n(T(E,I),F\circ\mathcal{S}^{op}).$$

In the case of a finite set E, this theorem allows us to construct a finite complex for computing the Leech homology groups.

3.3. Global dimension of a trace monoid

Cohomologies of small categories are defined as right derived functors of $\varprojlim_{\mathscr{C}}:\mathrm{Ab}^{\mathscr{C}}\to\mathrm{Ab}$.

Let $\mathscr{C}$ be a small category and let $F:\mathscr{C}\to\mathrm{Ab}$ be a functor. The category $\mathrm{Ab}^{\mathscr{C}}$ has enough injectives. Hence there is an injective resolution $0\to F\to F^0\to F^1\to F^2\to\cdots$. The functor $\varprojlim_{\mathscr{C}}:\mathrm{Ab}^{\mathscr{C}}\to\mathrm{Ab}$ leads to a complex

$$0\stackrel{d^{-1}}{\to}\varprojlim_{\mathscr{C}}F^0\stackrel{d^0}{\to}\varprojlim_{\mathscr{C}}F^1\stackrel{d^1}{\to}\varprojlim_{\mathscr{C}}F^2\to\cdots$$

The *nth cohomology group of $\mathscr{C}$ with coefficients in F* is defined as $H^n(\mathscr{C},F)=\mathrm{Ker}\,d^n/\mathrm{Im}\,d^{n-1}$.

Given semicubical set X and a functor $G:\square_+/X\to\mathrm{Ab}$, define *cohomology groups* $H^n(X,G)$ of X with coefficients in G similarly to homology groups of semicubical set. It is easy to see that $H^n(X,G)\cong H^n(\square_+/X,G)$.

The proof of [14, Theorem 2.2] contains the assertion that for each $\alpha\in\mathrm{Ob}\,Fact(M(E,I))$, $H_n(\mathcal{S}/\alpha,\Delta\,\mathbb{Z})=0$ for $n>0$, and $H_0(\mathcal{S}/\alpha,\Delta\,\mathbb{Z})=\mathbb{Z}$. Hence, it follows from the Oberst Theorem [11, Prop. 1] the following assertion.

Theorem 3.2. *For any functors $F:Fact(M(E,I))\to\mathrm{Ab}$ and for all $n\geqslant 0$, there are isomorphisms $H^n(Fact(M(E,I)),F)\cong H^n(T(E,I),F\circ\mathcal{S})$.*

Given Abelian category $\mathcal{A}$, its *global dimension* $\mathrm{gl.dim}\,\mathcal{A}$ is a supremum of $n\geqslant 0$ for which the functors $\mathrm{Ext}^n(-,=)$ are not equal to 0. Let $\mathscr{C}$ be a small *cancellative* category in the sense of [8]. By [8, Theorem 4.2], its Hochschild-Mitchell dimension $\dim\mathscr{C}$ equals cohomological dimension of $Fact(\mathscr{C})$. For any Abelian category $\mathcal{A}$ with exact coproducts, Mitchell proved the inequality $\mathrm{gl.dim}\,\mathcal{A}^{\mathscr{C}}\leqslant\dim\mathscr{C}+\mathrm{gl.dim}\,\mathcal{A}$ [20]. It follows from [8, Theorem 5.1] that this inequality is true for $\mathcal{A}$ with coproducts and enough projectives. It follows from Theorem 3.2 that $\dim M(E,I)\le n$ when E does not contains $n+1$ pairwise independent elements. If $M(E,I)$ contains n pairwise independent generators, then the free commutative monoid $\mathbb{N}^n$ is a retract of $M(E,I)$. It follows from [20, Prop. 11.6] the ineguality $\mathrm{gl.dim}\,\mathcal{A}^{M(E,I)}\geqslant\mathrm{gl.dim}\,\mathcal{A}^{\mathbb{N}^n}$. It leads us to the following generalization of Hilbert's Syzygy Theorem.

Theorem 3.3. *[11] Let $\mathcal{A}$ be an Abelian category with coproducts and let $M(E,I)$ be a trace monoid. If a maximal cardinality of pairwise independent elements of E equals $n<\infty$, then*

$$\mathrm{gl.dim}\,\mathcal{A}^{M(E,I)}=n+\mathrm{gl.dim}\,\mathcal{A}$$

in each of the following cases:

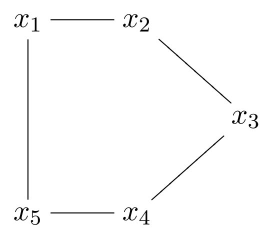
(i) $\mathcal{A}$ has exact coproducts (i.e. $\mathcal{A}$ satisfies to the axiom AB4),

(ii) $\mathcal{A}$ has enough projectives.

In the first case *(i)*, this is proved in [11]. For proof for the second case *(ii)* will be published in Sib. Math. J.

Conjecture 1. *This is true for all Abelian categories with coproducts.*

Example 3.1. Let k be a field and $E = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of variables. Suppose that the independence relation $I \subset E \times E$ is given by the following graph with vertexes E and edges I:



The noncommutative polynomial ring in five variables is denoted by $k\langle x_1, x_2, x_3, x_4, x_5\rangle$. Let (I) be the ideal of $k\langle x_1, x_2, x_3, x_4, x_5\rangle$ generated by polynomials $x_u x_v - x_v x_u$ for which $(x_u, x_v) \in I$, $1 \leqslant u, v \leqslant 5$. The maximal number of independent variables equals 2. By Theorem 3.3, we have

$$\text{gl.dim}\, k\langle x_1, x_2, x_3, x_4, x_5\rangle/(I) = 2.$$

3.4. Homology of augmented state category

Let us consider the functor cod : $Fact(\mathscr{C}) \to \mathscr{C}$, $\alpha \mapsto \operatorname{cod}(\alpha)$, $(\alpha \xrightarrow{(f,g)} \beta) \mapsto g$. For any $c \in \operatorname{Ob}\mathscr{C}$, $H_n(\operatorname{cod}/c, \Delta\,\mathbb{Z}) = 0$ for all $n > 0$ and $H_0(\operatorname{cod}/c, \Delta\,\mathbb{Z}) = \mathbb{Z}$.

Proposition 3.4. *Given a small category $\mathscr{C}$ and a functor $F : \mathscr{C}^{op} \to \mathrm{Ab}$, there exist isomorphisms $\varinjlim_n^{\mathscr{C}^{op}} F \cong \varinjlim_n^{Fact(\mathscr{C})^{op}} F \circ \operatorname{cod}^{op}$ for all $n \geqslant 0$.*

Given a state space $(M(E,I), S_*)$ and a functor $F : K_*(S) \to \mathrm{Ab}$ there are isomorphisms $H_n(K_*(S), F) \cong H_n(M(E,I)^{op}, \overline{F})$ where $\overline{F} = \bigoplus\limits_{x \in S_*} F(x)$ is Abelian group with the right action $(x, f) \cdot \mu = (x\mu, F(x \xrightarrow{\mu} x\mu)(f))$. By Proposition 3.4 and Theorem 3.1 we obtain the following complex for the computing the homology of the state space.

Theorem 3.5. *[14] If $M(E,I)$ contains no infinite subsets of pairwise independent generators, then $H_n(K_*(S), F)$ are isomorphic to nth homology groups of the complex*

$$0 \leftarrow \bigoplus_{x \in S_*} F(x) \overset{d_1}{\leftarrow} \bigoplus_{(x,a_1) \in Q_1(E,I,S)} F(x) \overset{d_2}{\leftarrow} \bigoplus_{(x,a_1,a_2) \in Q_2(E,I,S)} F(x) \leftarrow \cdots$$
$$\cdots \leftarrow \bigoplus_{(x,a_1,\cdots,a_{n-1}) \in Q_{n-1}(E,I,S)} F(x) \overset{d_n}{\longleftarrow} \bigoplus_{(x,a_1,\cdots,a_n) \in Q_n(E,I,S)} F(x) \leftarrow \cdots,$$

with differentials

$$d_n(x, a_1, \cdots, a_n, f) = \sum_{s=1}^{n} (-1)^s ((x \cdot a_s, a_1, \cdots, \widehat{a_s}, \cdots, a_n, F(x \xrightarrow{a_s} x \cdot a_s)(f)) - (x, a_1, \cdots, \widehat{a_s}, \cdots, a_n, f))$$

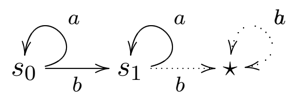
So, we have the following solution of Problem 2.

Corollary 3.6. *[10] If the cardinality of pairwise generators of $M(E,I)$ not greater than n, then $H_k(K_*(S),F)=0$ for all $k>n$.*

In addition, we have a complex of finitely generated Abelian groups for calculating the integral homology $H_n(K_*(S),\Delta\,\mathbb{Z})$ of augmented state category. This result has found applications [16].

Example 3.2. Let us consider a state space $\Sigma=(S,E,I,\mathrm{Tran}\,)$, $S=\{s_0,s_1\}$, $E=\{a,b\}$, $I=\{(a,b),(b,a)\}$, $\mathrm{Tran}\ =\{(s_0,a,s_0),(s_0,b,s_1),(s_1,a,s_1)\}$. The set consists of two elements with the partial action of the free commutative monoid generated by a and b. Let us calculate the groups $H_n(K_*(S),\Delta\,\mathbb{Z})$.

We add the state $\star$



and write down the matrixes of differentials. Since $|S_*|=3$, $|Q_1(E,I,S_*)|=6$, $|Q_2(E,I,S_*)|=3$, the complex consists of Abelian groups

$$0\leftarrow\mathbb{Z}^3\overset{d_1}{\leftarrow}\mathbb{Z}^6\overset{d_2}{\leftarrow}\mathbb{Z}^3\leftarrow 0$$

The differential $d_1(s,e)=-s\cdot e+s$ is defined by the matrix:

$$\begin{array}{cc} & \begin{array}{cccccc}(s_0,a) & (s_0,b) & (s_1,a) & (s_1,b) & (*,a) & (*,b)\end{array}\\ \begin{array}{c}s_0\\ s_1\\ \star\end{array} & \left(\begin{array}{cccccc}+1-1 & +1 & 0 & 0 & 0 & 0\\ 0 & -1 & +1-1 & +1 & 0 & 0\\ 0 & 0 & 0 & -1 & -1+1 & -1+1\end{array}\right)\end{array}$$

The differential $d_2(s,e_1,e_2)=-(s*e_1,e_2)+(s,e_2)+(s*e_2,e_1)-(s,e_1)$ has the matrix:

$$\begin{array}{cc} & \begin{array}{ccc}(s_0,a,b) & (s_1,a,b) & (\star,a,b)\end{array}\\ \begin{array}{c}(s_0,a)\\ (s_0,b)\\ (s_1,a)\\ (s_1,b)\\ (\star,a)\\ (\star,b)\end{array} & \left(\begin{array}{ccc}-1 & 0 & 0\\ -1+1 & 0 & 0\\ +1 & -1 & 0\\ 0 & -1+1 & 0\\ 0 & +1 & +1-1\\ 0 & 0 & -1+1\end{array}\right)\end{array}$$

Using the reduction of these matrices to Smith normal form, we obtain $H_0(K_*(S),\Delta\,\mathbb{Z})=\mathbb{Z}$, $H_1(K_*(S),\Delta\,\mathbb{Z})=\mathbb{Z}^2$, $H_2(K_*(S),\Delta\,\mathbb{Z})=\mathbb{Z}^1$, and $H_n(K_*(S),\Delta\,\mathbb{Z})=0$ for all $n\geqslant 3$.

3.5. Homology of Mazurkiewicz trace languages

Given $v,w\in M(E,I)$, we let $v\leqslant w$ if there exists $u\in M(E,I)$ such that $vu=w$. This relation makes $M(E,I)$ into a partially ordered set, which we denote by $P(E,I)$. A *trace language* is any set of traces.

Definition 3.3. *A set $L\subseteq M(E,I)$ is* prefix closed *if for all $v\in M(E,I)$ and $w\in L$ the relation $v<w$ implies $v\in L$.*

Let $L\subseteq M(E,I)$ be a prefix closed trace language. We have the pair $(M(E,I),L)$ consisting of a trace monoid with the following partial action for $v\in L$, $\mu\in M(E,I)$.

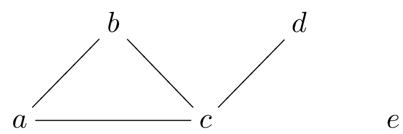
$$v\cdot\mu=\begin{cases} v\mu, & \text{if }\ v\mu\in L\\ undefined, & otherwise.\end{cases}$$

For any functor $F:K_*(L)\to\mathrm{Ab}$, we can consider the homology groups $H_n(K_*(L),F)$. The groups $H_n(K_*(L),\Delta\,\mathbb{Z})$ are called *integral homology groups.*

3.6. Homology groups of the poset of traces

Given prefix closed language $L \subseteq M(E,I)$, let $\mathbb{Z}[L] : P(E,I) \to \mathrm{Ab}$ be a functor with values $\mathbb{Z}[L](v) = \mathbb{Z}$ for $v \in L$ and $\mathbb{Z}[L](v) = 0$, otherwise. For $u \leqslant v \in L$, we will define $\mathbb{Z}[L](u \leqslant v) = 1_{\mathbb{Z}}$. We study the homology groups $H_n(P(E,I), \mathbb{Z}[L])$ of the poset $P(E,I)$ and their relationship with $H_n(K_*(L), \Delta\,\mathbb{Z})$.

Let p_n denotes the cardinality of the set of n-cliques in the graph (E,I). In particular, $p_0 = 1$ as the number of empty subsets in E, $p_1 = |E|$. For example, if (E,I) is the graph



then $p_0 = 1$, $p_1 = 5$, $p_2 = 4$, $p_3 = 1$.

Theorem 3.7. *[13]* $H_n(K_*(L), \Delta\,\mathbb{Z}) \cong H_n(P(E,I), \mathbb{Z}[L]) \oplus \mathbb{Z}^{(p_n)}$.

Given a partially ordered set P, let $\widetilde{H_n}(P)$ be the reduced singular homology of the classifying space $B(P)$. It is not hard to see that $H_n(P(E,I), \mathbb{Z}[L]) \cong \widetilde{H}_{n-1}(P(E,I) \setminus L)$ for $n \geqslant 1$.

Corollary 3.8. *[13]* $H_n(K_*(L), \Delta\,\mathbb{Z}) \cong \widetilde{H}_{n-1}(P(E,I) \setminus L) \oplus \mathbb{Z}^{(p_n)}$ *for all* $n \geqslant 1$.

We see that $H_1(K_*(L), \Delta\,\mathbb{Z})$ is a free Abelian group.

Conjecture 2. *For any trace monoid $M(E,I)$ with partial action on a set S, the Abelian group $H_1(K_*(S), \Delta\,\mathbb{Z})$ is free.*

The following assertions on prefix closed trace languages are proved in [13]:

- If $I = \{(a,b) \in E \times E | a \neq b\}$ and hence $M(E,I)$ is commutative, then $H_n(P(E,I), \mathbb{Z}[L]) = 0$ for all $n \geqslant 1$.
- If $I = \emptyset$ and hence $M(E,I)$ is free, then $H_n(P(E,I), \mathbb{Z}[L]) = 0$ for all $n \geqslant 2$.
- For arbitrary finitely generated Abelian groups A_1, A_2, ..., A_n with free A_1, there exists a trace monoid $M(E,I)$ such that $H_n(P(E,I), \mathbb{Z}[\{1\}]) \cong A_k$ for all $1 \leqslant k \leqslant n$.

3.7. Baues-Wirsching homology of the state category

Let $M(E,I)$ be an arbitrary trace monoid and let X be a right $M(E,I)$-set. It should be remembered that $K(X)$ denotes the category of states with objects $x \in X$ and morphisms $x \xrightarrow{\mu} x\mu$ for $x \in X$ and $\mu \in M(E,I)$. Considering $M(E,I)$ as a category with a unique object we can define a functor $U : K(X) \to M(E,I)$ assigning to each morphism $x \xrightarrow{\mu} x\mu$ the morphism $\mu \in M(E,I)$. Applying the functor $Fact$ to U, we can consider a functor $Fact(U) : Fact(K(X)) \to Fact(M(E,I))$. For any functor $F : Fact(K(X))^{op} \to \mathrm{Ab}$, there exists its Kan extension $\mathrm{Lan}^{Fact(U)^{op}} : Fact(K(M(E,I)))^{op} \to \mathrm{Ab}$ [18].

Theorem 3.9. *[12] Given functor $F : Fact(K(X))^{op} \to \mathrm{Ab}$, there exist isomorphisms*

$$H_n(Fact(K(X))^{op}, F) \cong H_n(Fact(M(E,I))^{op}, \mathrm{Lan}^{Fact(U)^{op}} F)$$

for all $n \geqslant 0$.

3.8. The solution of Problem 1

Let $(M(E,I), S)$ be a trace monoid with a partial action on S and let $K(S) \subset K_*(S)$ be the state category defined in 1.4. Let $\mathbb{Z}S$ denotes the free Abelian group generated by S. Let $\overline{Q}_n(E,I,S) = \{(s, a_1, \cdots, a_n) \in S \times T_n(E,I) | sa_1 \cdots a_n \neq \star\}$.

Theorem 3.10. *[12] Given a state space $(M(E,I),S)$, the groups $H_n(K(S),\Delta\,\mathbb{Z})$ are isomorphic to the homology groups of the complex*

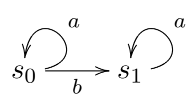
$$0 \leftarrow \mathbb{Z}(S) \overset{d_1}{\leftarrow} \mathbb{Z}\overline{Q}_1(S,E,I) \overset{d_2}{\leftarrow} \mathbb{Z}\overline{Q}_2(S,E,I) \leftarrow \cdots$$
$$\cdots \leftarrow \mathbb{Z}\overline{Q}_{n-1}(S,E,I) \overset{d_n}{\leftarrow} \mathbb{Z}\overline{Q}_n(S,E,I) \leftarrow \cdots$$

with differentials

$$d_n(s,a_1,\cdots,a_n) = \sum_{i=1}^{n}(-1)^i(sa_i,a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n) - \sum_{i=1}^{n}(-1)^i(s,a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n)$$

Let us consider an example of computing the homology groups of a state category.

Example 3.4. Let $M(E,I)$ be a commutative trace monoid generated by two elements and let us suppose that S consists of two elements. That is $E=\{a,b\}$, $I=\{(a,b),(b,a)\}$, $S=\{s_0,s_1\}$. The generators act by $s_0a=s_0$, $s_0b=s_1$, $s_1a=s_1$ as it is shown in the following picture.



The complex consists of Abelian groups

$$C_0=\mathbb{Z}\{s_0,s_1\},\quad C_1=\mathbb{Z}\{(s_0,a),(s_0,b),(s_1,a)\},\quad C_2=\mathbb{Z}\{(s_0,a,b)\}.$$

We have a complex $0\leftarrow \mathbb{Z}^2 \overset{d_1}{\leftarrow} \mathbb{Z}^3 \overset{d_2}{\leftarrow} \mathbb{Z} \leftarrow 0 \leftarrow 0 \leftarrow \cdots$. The differential d_1 is described by the following matrix.

$$\begin{array}{c} \\ s_0 \\ s_1 \end{array}\begin{array}{c} \begin{array}{ccc} (s_0,a) & (s_0,b) & (s_1,a) \end{array} \\ \begin{pmatrix} 1-1 & 1 & 0 \\ 0 & -1 & 1-1 \end{pmatrix}\end{array}$$

The differential d_2 has the following matrix.

$$\begin{array}{c} \\ (s_0,a) \\ (s_0,b) \\ (s_1,a) \end{array}\begin{array}{c} (s_0,a,b) \\ \begin{pmatrix} -1 \\ -1+1 \\ +1 \end{pmatrix}\end{array}$$

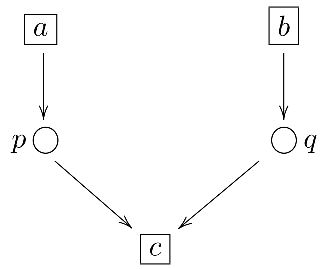
Using the reduction to Smith normal forms, we get

$$H_0(K(S),\Delta\,\mathbb{Z})=\mathbb{Z},\ H_1(K(S),\Delta\,\mathbb{Z})=\mathbb{Z},\ H_n(K(S),\Delta\,\mathbb{Z})=0 \text{ for all } n\geqslant 2.$$

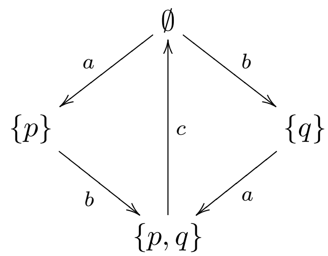
3.9. Homology groups of CE nets

For the computing the homology groups of a finite CE net, we first construct the state space $(M(E,I),S(s_0))$. Then we can compute $H_n(K(S(s_0)),\Delta\,\mathbb{Z})$ by the method described above.

Let, for example, $\mathcal{N}$ be the following CE net.



The corresponding trace monoid $M(E, I)$ is defined by $E = \{a, b, c\}$ and $I = \{(a, b), (b, a)\}$. The set of states S consists of all subsets $s \subseteq \{p, q\}$. The corresponding asynchronous system $(M(E, I), S, s_0)$ is defined by $s_0 = \emptyset$ and a partial action of $M(E, I)$ shown in the following figure.



That is $\emptyset \cdot a = \{p\}$, $\emptyset \cdot b = \{q\}$, $\{p\} \cdot b = \{p, q\}$, $\{q\} \cdot a = \{p, q\}$, and $\{p, q\} \cdot c = \emptyset$. All states are admissible. Hence $S(s_0) = S$. The complex consists of the Abelian groups

$$\begin{aligned} C_0 &= \mathbb{Z}\{\emptyset, \{p\}, \{q\}, \{p, q\}\} \cong \mathbb{Z}^4, \\ C_1 &= \mathbb{Z}\{(\emptyset, a), (\emptyset, b), (\{p\}, b), (\{q\}, a), (\{p, q\}, c)\} \cong \mathbb{Z}^5, \\ C_2 &= \mathbb{Z}\{(\emptyset, a, b)\} \cong \mathbb{Z}. \end{aligned}$$

The differential $d_1(s, e) = -s \cdot e + s$ has the following matrix.

$$\begin{array}{c} \\ \emptyset \\ \{p\} \\ \{q\} \\ \{p, q\} \end{array} \begin{array}{c} \begin{array}{ccccc} (\emptyset, a) & (\emptyset, b) & (\{p\}, b) & (\{q\}, a) & (\{p, q\}, c) \end{array} \\ \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{array} \right) \end{array}$$

We have $d_2(\emptyset, a, b) = -(\emptyset \cdot a, b) + (\emptyset, b) + (\emptyset \cdot b, a) - (\emptyset, a)$. Hence, the matrix of d_2 is described by the matrix

$$\begin{array}{c} \\ (\emptyset, a) \\ (\emptyset, b) \\ (\{p\}, b) \\ (\{q\}, a) \\ (\{p, q\}, c) \end{array} \begin{array}{c} (\emptyset, a, b) \\ \left(\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{array} \right) \end{array}$$

We have the following complex for the computing $H_n(\mathcal{N})$ for all $n \geqslant 0$.

$$0 \leftarrow \mathbb{Z}^4 \overset{d_1}{\leftarrow} \mathbb{Z}^5 \overset{d_2}{\leftarrow} \mathbb{Z} \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

Using the Smith normal forms, we get $H_0(\mathcal{N}) = \mathbb{Z}$, $H_1(\mathcal{N}) = \mathbb{Z}$, and $H_n(\mathcal{N}) = 0$, for all $n \geqslant 2$.

4. Conclusion

The author believes that the results will help in the study of the Goubault homology of asynchronous systems as the homology groups $H_n(K(S), \mathbb{Z}^{\varepsilon})$, $\varepsilon \in \{0,1\}$, with coefficients in some suitable systems of Abelian groups. You can explore the n-deadlocks for asynchronous systems. It is possible to find homological signs for the existence of bisimilar equivalence between asynchronous systems, Petri nets, and trace languages.

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ABSTRACT

Статья содержит обзор результатов автора в области алгебраической топологии, применяемой в компьютерных науках. Описана связь между кубическими группами гомологий обобщенных торов и групп гомологий моноида трасс, действующего частично на множестве. Описаны алгоритмы вычисления групп гомологий асинхронных систем, сетей Петри и трассовых языков Мазуркевича. Основные результаты статьи доложены на секционном докладе Международной конференции «Торическая топология и автоморфные функции» (5-10 сентября 2011 г., г. Хабаровск, Россия).

Key words: *полукубическое множество, гомологии малых категорий, свободный частично коммутативный моноид, асинхронная система переходов, сети Петри, языки трасс.*