FAR EASTERN MATHEMATICAL JOURNAL. 2012. Vol. 12. № 2. P. 237–254

UDC 511.7
MSC2010 11J13

© Nikolay Moshchevitin[1]

On certain Littlewood-like and Schmidt-like problems in inhomogeneous Diophantine approximations

We give several results related to inhomogeneous approximations to two real numbers and badly approximable numbers. Our results are related to classical theorems by A. Khintchine [7] and to an original method invented by Y. Peres and W. Schlag [13].

Key words: *Diophantine approximation, Littlewood conjecture, Peres – Schlag's method, badly approximable numbers.*

1. Functions and parameters

In all what follows, $||\cdot||$ is the distance to the nearest integer. All functions here are non-negative valued functions in real non-negative variables.

Consider strictly increasing functions $\omega_1(t), \omega_2(t)$. Let $\omega_1^*(t)$ be the inverse function to $\omega_1(t)$, that is

$$\omega_1^*(\omega_1(t)) = t.$$

Suppose that another function in two variables $\Omega(x, y)$ satisfies the condition

$$\begin{cases} xy \le \omega\left(\frac{z}{x}\right), \\ x \le z \end{cases} \implies x \le \Omega(y, z), \quad \forall\, x, y, z \in \mathbb{Z}_+. \tag{1}$$

This condition may be rewritten as

$$\begin{cases} x\omega_1^*(x \cdot y) \le z, \\ x \le z \end{cases} \implies x \le \Omega(y, z), \quad \forall\, x, y, z \in \mathbb{Z}_+. \tag{2}$$

Suppose that the functions $\phi(t), \phi_2(t), \phi_2(t), \psi_1(t), \psi_2(t)$, increase as $t \to \infty$ and

$$\phi(0) = \phi_1(0) = \phi_2(0) = \psi_1(0) = \psi_2(0) = 0. \tag{3}$$

Suppose that $\psi_j(t)$, $j = 1, 2$ are strictly increasing functions and that $\psi_j^*(t)$ is the inverse function of $\psi_j(t)$, that is

$$\psi_j^*(\psi_j(t)) = t \;\; \forall\, t \in \mathbb{R}_+, \quad j = 1, 2.$$

For a positive $\varepsilon > 0$ and integers ν, μ define

$$\delta_\varepsilon^{[1]}(\mu, \nu) = \psi_2^*\left(\frac{\varepsilon}{\phi(2^\nu)\psi_1(2^{-\mu-1})}\right), \tag{4}$$

[1] Lomonosov Moscow State University, Leninskie Gory, Moscow, 119991, Russian Federation. E-mail: moshchevitin@gmail.com

$$\delta_\varepsilon^{[2]}(\nu) = \psi_2^* \left(\frac{\varepsilon}{\phi_2(2^\nu)} \right). \tag{5}$$

Suppose that $A > 1$. For functions $\omega_1(t), \omega_2(t), \phi(t), \psi_1(t), \psi_2(t)$ we consider the following sum:

$$S_{A,\varepsilon}^{[1]}(X) = \sum_{X \le \nu < A(X+1)} \sum_{1 \le \mu \le \log_2(\omega_2(2^{\nu+1}))+1} \delta_\varepsilon^{[1]}(\mu, \nu) \cdot \max\left(\Omega(2^{\mu-1}, 2^{\nu+1}), 2^{\nu-\mu}, 1\right) \tag{6}$$

For functions $\omega_1(t), \omega_2(t), \phi_1(t), \phi_2(t), \psi_1(t), \psi_2(t)$ we consider another sum:

$$S_{A,\varepsilon}^{[2]}(X) = \sum_{X \le \nu < A(X+1)} \delta_\varepsilon^{[2]}(\nu) \cdot \max\left(\Omega(1/2r_\varepsilon(\nu), 2^{\nu+1}), 2^\nu r_\varepsilon(\nu), 1\right), \tag{7}$$

where

$$r_\varepsilon(\nu) = \psi_1^* \left(\frac{\varepsilon}{\phi_1(2^\nu)} \right). \tag{8}$$

2. Main results

Here we formulate two new results – Theorems 1,2. Proofs of these theorems are given in Sections 6, 7, 8. Section 4 below is devoted to certain examples of applications of Theorem 1. Section 5 deals with applications of Theorem 2. In Section 3, we discuss Khintchine's theorems and some of their extensions.

Theorem 1. *Suppose that functions $\psi_1(t), \psi_2(t), \phi(t)$ are increasing. Suppose that (3) is valid. Suppose that for certain $A > 1, \varepsilon > 0, X_0 \ge 0$ all the functions satisfy the conditions*

$$\log_2 \left(\frac{X}{2\psi_2^* \left(\frac{\varepsilon}{\phi(X)\psi_1(1/2)} \right)} \right) \le (A-1)\log_2 X, \quad \forall X \ge X_0, \tag{9}$$

and

$$\sup_{X \ge X_0} S_{A,\varepsilon}^{[1]}(X) \le \frac{1}{2^9}. \tag{10}$$

Consider two real numbers α, η such that

$$\inf_{x \ge X_0} \omega_1(x) \cdot ||x\alpha|| \ge 1 \tag{11}$$

and

$$\inf_{x \ge X_0} \omega_2(x) \cdot ||x\alpha - \eta|| \ge 1 \tag{12}$$

Then for any sequence of real numbers $\eta_1, \eta_2, ..., \eta_x, ...$ there exists a real number β such that

$$\inf_{x \ge X_0} \phi(x)\psi_1(||x\alpha - \eta||)\psi_2(||x\beta - \eta_x||) \ge \varepsilon. \tag{13}$$

A simpler version of the theorem was announced in [4] (Theorem 8 from [4]). Some inhomogeneous results in special case were announced in [9] (see Appendix from [9]).

The following Theorem 2 generalizes a result from [10].

Theorem 2. *Consider a real number α satisfying (11). Let η be an arbitrary real number. Suppose that*

$$\log_2\left(\frac{X}{2\psi_2^*\left(\frac{\varepsilon}{\phi_2(X)}\right)}\right) \le (A-1)\log_2 X, \quad \forall X \ge X_0, \tag{14}$$

and

$$\sup_{X\ge X_0} \; S_{A,\varepsilon}^{[2]}(X) \le \frac{1}{2^9}. \tag{15}$$

Then for any sequence of real numbers $\eta_1, \eta_2, ..., \eta_x, ...$ there exists a real number β such that

$$\inf_{x\ge X_0} \; \max\left(\phi_1(x)\cdot\psi_1(||x\alpha-\eta||),\; \phi_2(x)\cdot\psi_2(||x\beta-\eta_x||)\,\right) \ge \varepsilon. \tag{16}$$

Remark. The method under consideration enables one to obtain results about intersections. Suppose that $j\in\{1,2\}$. Given two different collections of functions

$$\omega_1^j(t), \omega_2^j(t), \psi_1^j(t), \psi_2^j(t), \phi^j(t), \sigma_1^j(t), \sigma_2^j(t),$$

two sequences $\{\eta_x^j\}_{x=1}^{\infty}$ and two couples of reals α^j, η^j satisfying the conditions specified (with more restrictions on constants) it is easy to prove the existence of a real β such that the conclusions (13, 16) (or even both of them) are valid for both values of $j\in\{1,2\}$. A simpler example of such a result was proved in [10]. Moreover the method can give lower bound for Hausdorff dimension of the sets.

3. Khintchine's theorems and their extensions

In [7] A. Khintchine proved the following result.

Theorem A. *There exists an absolute constant γ such that for any real α there exists a real η such that*

$$\inf_{x\in\mathbb{Z}_+} \; x\cdot||x\alpha-\eta|| \ge \gamma. \tag{17}$$

One can find this theorem in the books [5] (Ch. 10) and [14] (Ch. 4). The best known value of γ probably is due to H. Godwin [6]. From [19] we know that for every $\alpha\in\mathbb{R}$ the set of all η for which there exists a positive constant γ such that (17) is true is a $1/2$-winning set.

From Khintchine's theorem it follows that there exist reals α, η such that inequalities (11), (12) are valid with

$$\omega_1(t) = \omega_2(t) = \gamma t$$

with an absolute positive constant γ.

Here we formulate an immediate corollary to Khintchine's Theorem A.

Corollary 1.

(i) *Suppose that reals α_1 and α_2 are linearly dependent over $\mathbb{Z}$ together with 1. Then there exist reals η_1, η_2 such that*

$$\inf_{x\in\mathbb{Z}_+} \; x\cdot||x\alpha_1-\eta_1||\cdot||x\alpha_2-\eta_2|| > 0.$$

(ii) *Suppose that α_1 is a badly approximable number satisfying*

$$\inf_{x\in\mathbb{Z}_+} \; x\cdot||x\alpha_1|| > 0.$$

Suppose that α_2 is linearly dependent with α_1 and 1. Then there exists η such that

$$\inf_{x\in\mathbb{Z}_+} x\cdot ||x\alpha_1||\cdot||x\alpha_2-\eta|| > 0.$$

Quite similar result was obtained recently by U. Shapira [17] by means of dynamical systems. We would like to note here that two papers by E. Lindenstrauss and U. Shapira [8, 18] related to the topic appeared very recently.

Proof of Corollary 1.

As α_1, α_2 are linearly dependent, we have integers A_1, A_2, B, not all zero, such that

$$A_1\alpha_1 + A_2\alpha_2 + B = 0.$$

From Khintchine's Theorem A we can deduce that there exists *uncountably many* η satisfying the conclusion of the theorem. (From [19] we know that the corresponding set is a winning set and hence is uncountable and dense). So we may find η_1, η_2 satisfying

$$\inf_{x\in\mathbb{Z}_+} x\cdot ||\alpha_i x-\eta_i|| \geq \delta, \quad i=1,2. \tag{18}$$

and

$$||A_1\eta_1 + A_2\eta_2|| \geq \delta$$

with some positive δ. (For the statement (ii) one can take $\eta_1 = 0, \eta_2 = \eta$.) Then

$$\delta \leq ||A_1\eta_1+A_2\eta_2|| = ||A_1(\alpha_1 x-\eta_1)+A_2(\alpha_2 x-\eta_2)|| \leq A\cdot\max_{i=1,2}||\alpha_i x-\eta_i||, \quad A=\max_{i=1,2}|A_i|. \tag{19}$$

Take a positive integer x. From (19) we see that one of the quantities $||\alpha_i x-\eta_i|| i=1,2$ is not less than δ/A. To the other quantity we may apply lower bound from (19). This gives

$$x\cdot||x\alpha_1||\cdot||x\alpha_2-\eta|| \geq \delta^2/A.$$

Corollary 1 is proved.

For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ we define a function

$$\Psi_\alpha(t) = \min_{(x_1,x_2)\in\mathbb{Z}^2\setminus\{(0,0)\}, \max|x_i|\leq t} ||\alpha_1 x_1 + \alpha_2 x_2||.$$

Now we formulate another two theorems from Khintchine's paper [7].

Theorem B. *Given a function $\varphi(t)$ decreasing to zero there exist α_1, α_2 linearly independent over $\mathbb{Z}$ together with 1 such that*

$$\Psi_\alpha(t) \leq \varphi(t)$$

for all t large enough.

Theorem C. *Given a function $\psi(t)$ increasing to infinity there exist reals α_1, α_2 linearly independent over $\mathbb{Z}$ together with 1 and reals η_1, η_2 such that*

$$\inf_{x\in\mathbb{Z}_+} \psi(x)\cdot\max_{i=1,2}||\alpha_i x-\eta_i|| > 0.$$

In fact A. Khintchine deduces Theorem C from Theorem B. In the fundamental paper [7] A. Khintchine states also two additional general results. One of them is as follows.

Theorem D. *Given a tuple of real numbers (η_1, η_2) and given a function $\psi(t)$ increasing to infinity there exist reals α_1, α_2 linearly independent over $\mathbb{Z}$ together with 1 such that*

$$\inf_{x\in\mathbb{Z}_+} \psi(x)\cdot\max_{i=1,2}||\alpha_i x-\eta_i|| > 0.$$

On the other hand, by a result of J. Tseng [19], we know that for any real α the set

$$\mathcal{B} = \{\eta : \inf_{x\in\mathbb{Z}_+} x \cdot ||\alpha x - \eta|| > 0\}$$

is an 1/2-winning set in $\mathbb{R}$. It follows that the sets

$$\mathcal{B}_1 = \{(\eta_1, \eta_2) : \ \eta_1 \in \mathcal{B}, \eta_2 \in \mathbb{R}\}, \qquad \mathcal{B}_2 = \{(\eta_1, \eta_2) : \ \eta_1 \in \mathbb{R}, \eta_2 \in \mathcal{B}\ \}$$

are 1/2-winning sets in $\mathbb{R}^2$.

In the paper [11] N. Moshchevitin proved a general result. The theorem below is a particular case of this result.

Theorem E. *Suppose that $\psi(t)$ is a function increasing to infinity as $t \to +\infty$. Suppose that for any $w \geq 1$ we have the inequality*

$$\sup_{x\geq 1} \frac{\psi(wx)}{\psi(x)} < +\infty. \tag{20}$$

Let $\rho(t)$ be the function inverse to the function $t \mapsto 1/\psi(t)$ Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ be such that

$$\Psi_\alpha(t) \leq \rho(t).$$

Then the set

$$\mathcal{B}^{[\psi]} = \{(\eta_1, \eta_2) : \ \inf_{x\in\mathbb{Z}_+} \psi(x) \cdot \max_{i=1,2} ||\alpha_i x - \eta_i|| > 0\}$$

is an 1/2-winning set in $\mathbb{R}^2$.

From the theory of winning sets (see [15]) we know that a countable intersection of α-winning set is also an α-winning set. In particular the set

$$\mathcal{B}^{[\psi]} \cap \mathcal{B}_1 \cap \mathcal{B}_2$$

is an 1/2-winning set in $\mathbb{R}^2$. Moreover every α-winning set has full Hausdorff dimension and hence is not empty. Thus we deduce the following result.

Theorem 3. *Suppose that $\psi(t)$ is a function increasing to infinity as $t \to +\infty$. Suppose that (20) is valid. Then there exist real numbers α_1, α_2 linearly independent over $\mathbb{Z}$ together with 1 and real numbers η_1, η_2 such that*

$$\inf_{x\in\mathbb{Z}_+} x\psi(x) \cdot ||\alpha_1 x - \eta_1|| \cdot ||\alpha_2 x - \eta_2|| > 0.$$

A proof immediately follows from the fact that $\mathcal{B}^{[\psi]} \cap \mathcal{B}_1 \cap \mathcal{B}_2 \neq \varnothing$. Let (α_1, α_2) be the tuple from Theorem C applied to $\varphi(t) = \rho(t)$. Take $(\eta_1, \eta_2) \in \mathcal{B}^{[\psi]} \cap \mathcal{B}_1 \cap \mathcal{B}_2$. Take positive integer x. One of the values $||\alpha_i x - \eta_i||$ should be greater than $\varepsilon/\psi(x)$ where ε depends on $\alpha_1, \alpha_2, \eta_1, \eta_2$ only. Then the other one is greater than ε'/x where ε' depends on $\alpha_1, \alpha_2, \eta_1, \eta_2$ only. Theorem 3 is proved.

Theorem 3 may be compared with the main result from the paper [17]. It does not answer the following question, already posed in [3].

Problem. *Let α and β be real numbers with $1, \alpha, \beta$ being linearly independent over the rationals. Let α_0, β_0 and γ be real numbers. To prove or to disprove that*

$$\inf_{q\neq 0} |q| \cdot \|q\alpha - \alpha_0\| \cdot \|q\beta - \beta_0\| = 0$$

and/or that

$$\inf_{(x,y)\neq(0,0)} \|x\alpha + y\beta - \gamma\| \cdot \max\{|x|,1\} \cdot \max\{|y|,1\} = 0.$$

The following two theorems by U. Shapira from the paper [17] worth noting in the context of this problem.

Theorem F. *Almost all (in the sense of Lebesgue measure) pairs* $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ *satisfy the following property: for every pair* $(\eta_1, \eta_2) \in \mathbb{R}^2$ *one has*

$$\liminf_{q\to\infty} q\ ||q\alpha_1 - \eta_1||\ ||q\alpha_1 - \eta_2|| = 0.$$

Theorem G. *The conclusion of Theorem F is true for numbers* α_1, α_2 *which form together with* 1 *a basis of a totally real algebraic field of degree* 3.

Also we would like to refer to one more Khintchine's result (see [7], Hilfssatz 4)

Theorem H. *Given* $c \in (0,1)$ *there exists* $\Gamma > 0$ *with the following property. For any* $\alpha \in \mathbb{R}$ *there exists* $\beta \in \mathbb{R}$ *such that*

$$\max(cx|\alpha x - y|,\ \Gamma|\beta x - z|) \geq 1,$$

where maximum is taken over integers $x > 0, y, z,\ (x,y) = 1$. *In other words if*

$$|\alpha x - y| \leq \frac{1}{cx},\ \ (x,y) = 1$$

then

$$||\beta x|| \geq \frac{1}{\Gamma}.$$

At the end of this section we want to refer to wondeful recent result by D. Badziahin, A. Pollington and S. Velani from the paper [1]. In this paper they solve famous W.M. Schmidt's conjecture [16].

Theorem I. *Let* $u, v \geq 0, u + v = 1$. *Suppose that*

$$\inf_{x\in\mathbb{Z}_+} x^{\frac{1}{u}}||\alpha x|| > 0. \tag{21}$$

Then the set

$$B_u(\alpha) = \{\beta \in \mathbb{R} :\ \inf_{x\in\mathbb{Z}_+} \max(x^u||\alpha x||,\ x^v||\beta x||)\ > 0\}$$

has full Hausdorff dimension.

Here we should note that the main result from [1] shows for a given α under the condition (21) that intersections of sets of the form $B_u(\alpha)$ for a finite collection of different values of u has full Hausdorff dimension. An explicit version of the original proof invented by D. Badziahin, A. Pollington and S. Velani was given in [12], in the simplest case $u = 1/2$.

Recently D. Badziahin [2] proved the following result.

Theorem J. *The set*

$$\{(\alpha,\beta) \in \mathbb{R}^2 :\ \inf_{x\in\mathbb{Z}, x\geq 3} x\log x\ \log\log x\, ||\alpha x||\, ||\beta x|| > 0\}$$

has Hausdorff dimension equal to 2.

Moreover if α *is a badly approximable number then the set*

$$\{\beta \in \mathbb{R} :\ \inf_{x\in\mathbb{Z}, x\geq 3} x\log x\ \log\log x\, ||\alpha x||\, ||\beta x|| > 0\}$$

has Hausdorff dimension equal to 1.

We think that the method from [1, 2] cannot be generalized for inhomogeneous setting.

4. Examples to Theorem 1

Here we give several special choices of parameters in Theorem 1 and deduce several corollaries.

Example 1. Put

$$\omega_1(t) = \omega_2(t) = \gamma t$$

with some positive $\gamma > 1$. Then

$$\omega_1^*(t) = \frac{t}{\gamma}$$

and we may take in (1)

$$\Omega(y, z) = \sqrt{\frac{1}{\gamma}\frac{z}{y}}.$$

Put

$$\psi_1(t) = \psi_2(t) = t, \quad \phi(t) = t \cdot \ln^2 t.$$

Then

$$\psi_2^*(t) = t.$$

So

$$\delta_\varepsilon^{[1]}(\mu, \nu) = 2 \cdot \varepsilon \cdot \frac{2^{\mu-\nu}}{\nu^2} \tag{22}$$

and

$$S_{A,\varepsilon}^{[1]}(X) = \ 2 \cdot \varepsilon \cdot \sum_{X \le \nu < A(X+1)} \ \sum_{1 \le \mu \le \nu + \log_2 \gamma + 2} \frac{2^{\mu-\nu}}{\nu^2} \max\left(\sqrt{\frac{1}{\gamma}\, 2^{\nu-\mu+2}}, 2^{\nu-\mu}, 1\right) \le$$

$$\le 4 \cdot \varepsilon \cdot \sum_{X \le \nu < A(X+1)} \left(\sum_{1 \le \mu \le \nu} \frac{1}{\nu^2} + \sum_{\nu+1 \le \mu \le \nu+2+\log_2 \gamma} \frac{2^{\mu-\nu}}{\nu^2} \right) \le 8 \cdot \varepsilon \cdot \sum_{X \le \nu < A(X+1)} \left(\frac{1}{\nu} + \frac{4\gamma}{\nu^2} \right) \le$$

$$\le 16 \varepsilon \ln(2A)$$

for X_0 large enough ($X_0 \ge \gamma/\varepsilon$). Put $A = 4$. Then the condition (9) is satisfied provided $\frac{X_0}{\ln^2 X_0} \ge \frac{1}{\varepsilon}$. Thus we obtain the following results.

Corollary 1.1. *Let $\eta_x, x = 1, 2, 3, ..$ be a sequence of reals. Given positive $\varepsilon \le 2^{-14}$ and a badly approximable real α such that*

$$||\alpha x|| \ge \frac{1}{\gamma x} \ \ \forall x \in \mathbb{Z}_+, \ \ \gamma > 1,$$

there exist $X_0 = X_0(\varepsilon, \gamma)$ and a real β such that

$$\inf_{x \ge X_0} \ x \ln^2 x \cdot ||x\alpha|| \cdot ||x\beta - \eta_x|| \ \ge \varepsilon.$$

Corollary 1.2. *Let $\eta_x, x = 1, 2, 3, ..$ be a sequence of reals. Given positive $\varepsilon \le 2^{-14}$ and real α, η such that simultaneously*

$$||\alpha x|| \ge \frac{1}{\gamma x} \ \ \forall x \in \mathbb{Z}_+, \ \ \gamma > 1$$

and

$$||\alpha x - \eta|| \ge \frac{1}{\gamma x} \ \ \forall x \in \mathbb{Z}_+, \ \ \gamma > 1,$$

there exist $X_0 = X_0(\varepsilon, \gamma)$ *and a real* β *such that*

$$\inf_{x \ge X_0} x \ln^2 x \cdot ||x\alpha - \eta|| \cdot ||x\beta - \eta_x|| \ge \varepsilon.$$

From Khintchine's Theorem A we deduce the following result.

Corollary 1.3. *Let* $\eta_x, x = 1, 2, 3, ..$ *be a sequence of reals. Given positive* $\varepsilon \le 2^{-14}$ *and a real* α *such that*

$$||\alpha x|| \ge \frac{1}{\gamma x} \ \ \forall x \in \mathbb{Z}_+, \ \ \gamma > 1,$$

there exist $X_0 = X_0(\varepsilon, \gamma)$ *and real* η, β *such that*

$$\inf_{x \ge X_0} x \ln^2 x \cdot ||x\alpha - \eta|| \cdot ||x\beta - \eta_x|| \ge \varepsilon.$$

Example 2. Put

$$\omega_1(t) = \omega_2(t) = t \ln t$$

Then

$$\omega_1^*(t) \asymp \frac{t}{\ln t}$$

and we may take in (1)

$$\Omega(y, z) = c\sqrt{\frac{z \ln z}{y}}$$

with small positive c.

Put

$$\psi_1(t) = \psi_2(t) = t, \quad \phi(t) = t \cdot \ln^2 t.$$

Then

$$\psi_2^*(t) = t,$$

and again $\delta_\varepsilon^{[1]}(\mu, \nu)$ satisfies (22). Now

$$S_{A,\varepsilon}^{[1]}(X) \ll \varepsilon \cdot \sum_{X \le \nu < A(X+1)} \ \sum_{1 \le \mu \le \nu + \log_2(\nu+1)+2} \frac{2^{\mu-\nu}}{\nu^2} \max\left(\sqrt{2^{\nu-\mu}\nu}, 2^{\nu-\mu}, 1\right) \ll$$

$$\ll \varepsilon \cdot \sum_{X \le \nu < A(X+1)} \ \sum_{1 \le \mu \le \nu + \log_2(\nu+1)+2} \frac{2^{\mu-\nu}}{\nu^2} \max\left(\sqrt{2^{\nu-\mu}\nu}, 2^{\nu-\mu}\right) \ll$$

$$\ll \varepsilon \cdot \sum_{X \le \nu < A(X+1)} \left(\sum_{1 \le \mu \le \nu} \frac{1}{\nu^2} + \sum_{\nu - \log_2(\nu+1) \le \mu \le \nu + \log_2(\nu+1)+2} \frac{2^{\frac{\mu-\nu}{2}}}{\nu^{3/2}} \right) \ll \varepsilon \ln 2A,$$

for X_0 large enough. Put $A = 4$. Then for X_0 large enough the inequality (9) is valid. Thus we obtain the following results.

Corollary 2.1. *There exists an absolute positive constant* ε_0 *with the following property. Let* $\eta_x, x = 1, 2, 3, ..$ *be a sequence of reals. Given positive* $\varepsilon \le \varepsilon_0$ *and a real* α *such that for all* $x \ge X_1$ *one has*

$$||\alpha x|| \ge \frac{1}{x \ln x} \ ,$$

there exist $X_0 = X_0(\varepsilon, X_1)$ *and a real* β *such that*

$$\inf_{x \ge X_0} x \ln^2 x \cdot ||x\alpha|| \cdot ||x\beta - \eta_x|| \ge \varepsilon.$$

Corollary 2.1 is a more general statement than Corollary 1.1.

Corollary 2.2. *Let $\eta_x, x = 1,2,3,..$ be a sequence of reals. Given positive ε small enough and real α,η such that for all $x \ge X_1$ simultaneously*

$$||\alpha x|| \ge \frac{1}{x\ln x},$$

and

$$||\alpha x-\eta|| \ge \frac{1}{x\ln x},$$

there exist $X_0 = X_0(\varepsilon, X_1)$ and a real β such that

$$\inf_{x\ge X_0} \; x\ln^2 x\cdot ||x\alpha-\eta||\cdot||x\beta-\eta_x|| \ge \varepsilon.$$

Example 3. Put

$$\omega_1(t) = t\ln^2 t, \quad \omega_2(t) = \gamma t, \gamma > 1.$$

Then

$$\omega_1^*(t) \asymp \frac{t}{\ln^2 t}$$

and we may take in (1)

$$\Omega(y,z) = c\sqrt{\frac{z}{y}}\ln z$$

with small positive c.

Put

$$\psi_1(t) = \psi_2(t) = t, \quad \phi(t) = t\cdot\ln^2 t.$$

Then

$$\psi_2^*(t) = t,$$

and again $\delta_\varepsilon^{[1]}(\mu,\nu)$ satisfies (22). So

$$S_{A,\varepsilon}^{[1]}(X) \ll \varepsilon\cdot\sum_{X\le\nu<A(X+1)}\;\sum_{1\le\mu\le\nu+\log_2\gamma+2}\frac{2^{\mu-\nu}}{\nu^2}\max\left(\sqrt{2^{\nu-\mu}}\cdot\nu, 2^{\nu-\mu}\right) \ll$$

$$\ll \varepsilon\cdot\sum_{X\le\nu<A(X+1)}\left(\sum_{1\le\mu\le\nu}\frac{1}{\nu^2} + \sum_{\nu-2\log_2(\nu+1)\le\mu\le\nu+\log_2\gamma+2}\frac{2^{\frac{\mu-\nu}{2}}}{\nu}\right) \ll$$

$$\ll \varepsilon\cdot\sum_{X\le\nu<A(X+1)}\;\sum_{1\le\mu\le\nu}\frac{1+\sqrt{\gamma}}{\nu} \ll \varepsilon(1+\sqrt{\gamma})\ln 2A,$$

for X_0 large enough. Again for $A = 4$ and X_0 large enough the inequality (9) is valid. Thus we obtain the following results. This result is a more general statement than Corollary 1.2.

Corollary 3.1. *Let $\eta_x, x = 1,2,3,..$ be a sequence of reals. Let $\gamma > 1$. Suppose that the product $\varepsilon\sqrt{\gamma}$ is small enough. Suppose that for certain real α,η and for $x \ge X_1$ simultaneously one has*

$$||\alpha x|| \ge \frac{1}{x\ln^2 x}$$

and

$$||\alpha x-\eta|| \ge \frac{1}{\gamma x}.$$

Then there exist $X_0 = X_0(\varepsilon, \gamma, X_1)$ and a real β such that

$$\inf_{x \geq X_0} x \ln^2 x \cdot ||x\alpha - \eta|| \cdot ||x\beta - \eta_x|| \geq \varepsilon.$$

Now from Khintchine's Theorem A we deduce a result which is more general than Corollary 1.3.

Corollary 3.2. *Let $\eta_x, x = 1, 2, 3, ..$ be a sequence of reals. Given positive ε small enough and a real α such that*

$$||\alpha x|| \geq \frac{1}{x \ln^2 x},$$

there exist $X_0 = X_0(\varepsilon)$ and real η, β such that

$$\inf_{x \geq X_0} x \ln^2 x \cdot ||x\alpha - \eta|| \cdot ||x\beta - \eta_x|| \geq \varepsilon.$$

Example 4. Put

$$\omega_1(t) = \omega_2(t) = \gamma t$$

with some positive $\gamma > 1$. Then as in Example 1 we have

$$\omega_1^*(t) = \frac{t}{\gamma}, \quad \Omega(y, z) = \sqrt{\frac{1}{\gamma}\frac{z}{y}}.$$

Suppose that $0 \leq a < 1$. Put

$$\psi_1(t) = t \cdot (\log_2 1/t)^a, \quad \psi_2(t) = t, \quad \phi(t) = t \cdot \log_2^{2-a} t.$$

Then

$$\psi_2^*(t) = t$$

and

$$\delta_\varepsilon^{[1]}(\mu, \nu) = 2 \cdot \varepsilon \cdot \frac{2^{\mu-\nu}}{\nu^{2-a}(\mu+1)^a}.$$

Now

$$S_{A,\varepsilon}^{[1]}(X) \ll 4\varepsilon \cdot \sum_{X \leq \nu < A(X+1)} \sum_{1 \leq \mu \leq \nu + \log_2 \gamma + 2} \frac{2^{\mu-\nu}}{\nu^{2-a}(\mu+1)^a} \max\left(\sqrt{2^{\nu-\mu}/\gamma}, 2^{\nu-\mu}, 1\right) \leq$$

$$\leq 4 \cdot \varepsilon \cdot \sum_{X \leq \nu < A(X+1)} \left(\sum_{1 \leq \mu \leq \nu} \frac{1}{\nu^{2-a}(\mu+1)^a} + \sum_{\nu+1 \leq \mu \leq \nu+2+\log_2 \gamma} \frac{2^{\mu-\nu}}{\nu^{2-a}(\mu+1)^a} \right) \leq$$

$$\leq \frac{8\varepsilon}{1-a} \sum_{X \leq \nu < A(X+1)} \left(\frac{1}{\nu} + \frac{4\gamma}{\nu^2}\right) \leq \frac{32\varepsilon \ln(2A)}{1-a}$$

for $X_0 \geq \gamma/\varepsilon$. Put again $A = 4$. Then the condition (9) is satisfied provided $\frac{X_0}{\ln^2 X_0} \geq \frac{1}{\varepsilon}$. Thus we obtain the following results (compare with Theorem 3 from [4]).

Corollary 4.1. *Let $\eta_x, x = 1, 2, 3, ..$ be a sequence of reals. Suppose that $0 \leq a < 1$. Given positive $\varepsilon \leq \frac{1}{2^{20}(1-a)}$ and a badly approximable real α such that*

$$||\alpha x|| \geq \frac{1}{\gamma x} \;\; \forall x \in \mathbb{Z}_+, \;\; \gamma > 1,$$

there exist $X_0 = X_0(\varepsilon, \gamma)$ *and a real* β *such that*

$$\inf_{x \geq X_0} x(\log_2 x)^{2-a} \cdot (\log_2 1/||x\alpha||)^a \cdot ||x\alpha|| \cdot ||x\beta - \eta_x|| \geq \varepsilon.$$

Corollary 4.2. *Let* $\eta_x, x = 1, 2, 3, ..$ *be a sequence of reals. Given positive* $\varepsilon \leq \frac{1}{2^{20}(1-a)}$ *and real* α, η *such that simultaneously*

$$||\alpha x|| \geq \frac{1}{\gamma x}, \quad ||\alpha x - \eta|| \geq \frac{1}{\gamma x} \ \forall x \in \mathbb{Z}_+, \ \gamma > 1,$$

there exist $X_0 = X_0(\varepsilon, \gamma)$ *and a real* β *such that*

$$\inf_{x \geq X_0} x(\log_2 x)^{2-a} \cdot (\log_2 1/||x\alpha - \eta||)^a \cdot ||x\alpha - \eta|| \cdot ||x\beta - \eta_x|| \geq \varepsilon.$$

Corollary 4.3. *Let* $\eta_x, x = 1, 2, 3, ..$ *be a sequence of reals. Given positive* $\varepsilon \leq \frac{1}{2^{20}(1-a)}$ *and a real* α *such that*

$$||\alpha x|| \geq \frac{1}{\gamma x} \ \forall x \in \mathbb{Z}_+, \ \gamma > 1,$$

there exist $X_0 = X_0(\varepsilon, \gamma)$ *and real* η, β *such that*

$$\inf_{x \geq X_0} x(\log_2 x)^{2-a} \cdot (\log_2 1/||x\alpha - \eta||)^a \cdot ||x\alpha - \eta|| \cdot ||x\beta - \eta_x|| \geq \varepsilon.$$

Of course one can deduce other corollaries of a similar type from Theorem 1. For example one may deduce statements which are more general than Corollaries 4.1 - 4.3 in the same manner as it was done in Examples 2,3.

5. Examples to Theorem 2

Here we consider some corollaries related to special choices of parameters in Theorem 2.

Example 5. Let $u, v > 0, \ u + v = 1$. Put

$$\omega_1(t) = \frac{\gamma t^{\frac{1}{u}}}{(\ln t)^u}, \ \gamma > 1.$$

Then we may take in (1)

$$\Omega(y, z) = c \left(\frac{z}{y^u (\ln z)^{u^2}} \right)^{\frac{1}{1+u}}$$

with small positive c (we take into account that $x \ll z^{1/2} \ln z$) .

Put

$$\psi_1(t) = \psi_2(t) = t, \quad \phi_1(t) = (t \log_2 t)^u, \ \phi_2(t) = (t \log_2 t)^v.$$

Then

$$\psi_1^* = \psi_2^*(t) = t,$$

and

$$\delta_\varepsilon^{[2]}(\nu) = \frac{\varepsilon}{(\nu 2^\nu)^v}, \quad r_\varepsilon(\nu) = \frac{\varepsilon}{(\nu 2^\nu)^u}.$$

So

$$S_{A,\varepsilon}^{[2]}(X) \ll \varepsilon \cdot \sum_{X \leq \nu < A(X+1)} \frac{1}{(\nu 2^\nu)^v} \cdot \frac{2^{(1-u)\nu}}{\nu^u} \ll \varepsilon \cdot \sum_{X \leq \nu < A(X+1)} \sum_{1 \leq \mu \leq \nu} \frac{1}{\nu} \ll \varepsilon \ln 2A,$$

So we get

Corollary 5.1. *Suppose that $u, v > 0$, $u + v = 1$. Let $\eta_x, x = 1, 2, 3, ..$ be a sequence of reals. Let η be an arbitrary real number. Let $\gamma > 0$. Suppose that ε is small enough. Suppose that for certain real α and for $x \geq X_1$ one has*

$$||\alpha x|| \geq \frac{\gamma (\ln x)^u}{x^{1/u}}.$$

Then there exist $X_0 = X_0(\varepsilon, \gamma, X_1)$ and a real β such that

$$\inf_{x \geq X_0} \max((x \ln x)^u \cdot ||x\alpha - \eta||, \ (x \ln x)^v \cdot ||x\beta - \eta_x||) \geq \varepsilon.$$

Example 6. Put

$$\omega_1(t) = \gamma t \ln t, \ \ \gamma > 1.$$

Then we may take in (1)

$$\Omega(y, z) = c\sqrt{\frac{z \ln z}{y}}$$

with small positive c.

Put

$$\psi_1(t) = \psi_2(t) = t, \quad \phi_1(t) = \Delta t, \ \ \phi_2(t) = (\log_2 t)^{3/2}, \quad \Delta > 0.$$

Then

$$\psi_1^* = \psi_2^*(t) = t,$$

and

$$\delta_\varepsilon^{[2]}(\nu) = \frac{\varepsilon}{\nu^{3/2}}, \quad r_\varepsilon(\nu) = \frac{\varepsilon}{\Delta 2^\nu}.$$

So

$$S_{A,\varepsilon}^{[2]}(X) \ll \varepsilon \cdot \sum_{X \leq \nu < A(X+1)} \frac{\Omega(1/2^{\nu+1})}{\nu^{3/2}} \ll \frac{\varepsilon^{3/2}}{\Delta^{1/2}} \cdot \sum_{X \leq \nu < A(X+1)} \sum_{1 \leq \mu \leq \nu} \frac{1}{\nu} \ll \frac{\varepsilon^{3/2}}{\Delta^{1/2}} \ln A,$$

So we get

Corollary 6.1. *Let $\eta_x, x = 1, 2, 3, ..$ be a sequence of reals. Let η be an arbitrary real number. Let $\Delta > 1$. Suppose that $2^{20}\varepsilon^3 \leq \Delta$. Suppose that for certain real α and for all positive integers x one has*

$$||\alpha x|| \geq \frac{\gamma}{x \ln x}.$$

Then there exist $X_0 = X_0(\gamma)$ and a real β such that

$$\inf_{x \geq X_0} \max(\Delta x \cdot ||x\alpha - \eta||, \ (\ln x)^{3/2} \cdot ||x\beta - \eta_x||) \geq \varepsilon.$$

In other words for this β if

$$||x\alpha - \eta|| \leq \frac{\varepsilon}{\Delta x}$$

then

$$||x\beta - \eta_x|| \geq \frac{\varepsilon}{(\ln x)^{3/2}}.$$

6. Sets of integers.

Consider sets

$$A_{\nu,\mu} = \{x \in \mathbb{Z}_+ : \ 2^\nu \le x < 2^{\nu+1}, \ 2^{-\mu-1} < ||\alpha x - \eta|| \le 2^{-\mu}\},$$

$$A_\nu(t) = \{x \in \mathbb{Z}_+ : \ 2^\nu \le x < 2^{\nu+1}, \ ||\alpha x - \eta|| \le t\},$$

Now we deduce an upper bound for the cardinality of the set $A_{\nu,\mu}$.

Lemma 1. *Under the condition (11) one has*

$$\operatorname{card} A_{\nu,\mu} \le 2^3 \max\left(\Omega(2^{\mu-1}, 2^{\nu+1}), 2^{\nu-\mu}, 1\right).$$

Proof. For $a \in A_{\nu,\mu}$ define integer y from the condition

$$||x\alpha - \eta|| = |x\alpha - \eta - y|.$$

Case 1^0. All integer points $z = (x, y), x \in A_{\nu,\mu}$ form a convex polygon Π of positive measure $\operatorname{mes}\Pi > 0$. Then

$$\operatorname{card} A_{\nu,\mu} \le 6 \operatorname{mes}\Pi \le 6 \cdot 2^{\nu+1-\mu} < 2^{\nu-\mu+3}. \tag{23}$$

Case 2^0. All integer points $z = (x, y), x \in A_{\nu,\mu}$ lie on the same line. Then all these points are of the form

$$z_0 + l z_1, \ \ z_j = (x_j, y_j), \ \ 0 \le l \le L.$$

Now we see that

$$|\alpha L x_1 - L y_1| \le 2^{-\mu+1}$$

and

$$|\alpha x_1 - y_1| \le 2^{-\mu+1} L^{-1}.$$

From (11) we have

$$\omega_1(x_1) \ge 2^{\mu-1} L.$$

So

$$x_1 \ge \omega_1^*(2^{\mu-1} L)$$

and

$$L x_1 \le 2^{\nu+1}.$$

We conclude that

$$L \le 2^{\nu+1}, \quad L \cdot \omega_1^*(2^{\mu-1} L) \le 2^{\nu+1}.$$

So by (2) we have

$$\operatorname{card} A_{\nu,\mu} \le L + 1 \le \Omega(2^{\mu-1}, 2^{\nu+1}) + 1. \tag{24}$$

We take together (23,24) to obtain

$$\operatorname{card} A_{\nu,\mu} \le \max\left(\Omega(2^{\mu-1}, 2^{\nu+1}), 2^{\nu-\mu}, 1\right).$$

Lemma is proved.

The next lemma deals with the cardinality of $A_\nu(t)$.

Lemma 2. *Under the condition (11) one has*

$$\operatorname{card} A_\nu(t) \le 2^2 \max\left(\Omega(1/2t, 2^{\nu+1}), 2^\nu t, 1\right).$$

Proof. The proof is quite similar to the proof of Lemma 1. We should consider two similar cases 1^0 and 2^0. In the **Case 1^0** we deduce the bound

$$\text{card}\, A_\nu(t) \le 2^{\nu+2} t.$$

In the **Case 2^0** we see that

$$L \le 2^{\nu+1}, \quad L \cdot \omega_1^* \left(\frac{L}{2t}\right) \le 2^{\nu+1}.$$

By (2) we have

$$\text{card}\, A_{\nu,\mu} \le L + 1 \le \Omega(1/2t, 2^{\nu+1}) + 1.$$

Lemma 2 follows.

7. Lemmas about fractional parts

Put

$$\sigma_\varepsilon^{[1]}(x) = \sigma_{\varepsilon,\alpha,\gamma}^{[1]}(x) = \psi_2^* \left(\frac{\varepsilon}{\phi(x)\psi_1(||x\alpha - \gamma||)}\right), \tag{25}$$

$$\sigma_\varepsilon^{[2]}(x) = \sigma_{\varepsilon,\alpha,\gamma}^{[2]}(x) = \psi_2^* \left(\frac{\varepsilon}{\phi_2(x)}\right). \tag{26}$$

Then from the definitions (25) of $\sigma_\varepsilon^{[1]}(x)$ and $\delta_\varepsilon^{[1]}(\mu,\nu)$ and monotonicity conditions we see that

$$x \in A_{\nu,\mu} \implies \sigma_\varepsilon^{[1]}(x) \le \delta_\varepsilon^{[1]}(\mu,\nu). \tag{27}$$

Consider sums

$$T_{A,\varepsilon}^{[1]}(Y) = \sum_{Y \le x < Y^A} \sigma_\varepsilon^{[1]}(x), \tag{28}$$

(with σ defined in (25)) and

$$T_{A,\varepsilon}^{[2]}(Y) = \sum_{Y \le x < Y^A,\ \phi_1(x)\psi_1(||\alpha x||) \le \varepsilon} \sigma_\varepsilon^{[2]}(x), \tag{29}$$

Lemma 3. *Suppose that (11) and (12) are valid. Then under the condition (10) one has*

$$\sup_{Y \in \mathbb{Z}_+} T_{A,\varepsilon}^{[1]}(Y) \le \frac{1}{2^6}. \tag{30}$$

Proof. Put $X = [\log_2 Y]$. We see that

$$T_{A,\varepsilon}^{[1]}(Y) \le \sum_{X \le \nu < A(X+1)} \sum_{\mu=1}^{\infty} \sum_{x \in A_{\nu,\mu}} \sigma_\varepsilon^{[1]}(x).$$

Note that from (12) it follows that sets $A_{\nu,\mu}$ are empty for $\mu > \log_2(\omega_2(2^{\nu+1})) + 1$. So from (27) we have

$$T_{A,\varepsilon}^{[1]}(Y) \le \sum_{X \le \nu < A(X+1)} \sum_{\mu=1}^{[\log_2(\omega_2(2^{\nu+1}))]+1} \delta_\varepsilon^{[1]}(\mu,\nu) \times \text{card}\, A_{\nu,\mu}. \tag{31}$$

Now from (31) and Lemma 1 we have

$$T_{A,\varepsilon}^{[1]}(Y) \le 2^3 \sum_{X \le \nu < A(X+1)} \sum_{\mu=1}^{[\log_2(\omega_2(2^{\nu+1}))]+1} \delta_\varepsilon^{[1]}(\mu,\nu) \times \max\left(\Omega(2^{\mu-1}, 2^{\nu+1}), 2^{\nu-\mu}, 1\right).$$

Lemma 3 follows from (10).

Lemma 4. *Suppose that (11) is valid. Then under the condition (15) one has*

$$\sup_{Y\in\mathbb{Z}_+} T^{[2]}_{A,\varepsilon}(Y) \le \frac{1}{2^6}. \tag{32}$$

Proof. The proof is quite similar to those of Lemma 3. Put $X=[\log_2 Y]$. Then

$$T^{[2]}_{A,\varepsilon}(Y) \le \sum_{X\le\nu<A(X+1)} \sum_{x\in A_\nu(r_\varepsilon(\nu))} \sigma^{[2]}_\varepsilon(x),$$

where $r_\varepsilon(\nu)$ is defined in (8). Now Lemma 4 immediately follows from (7, 15), Lemma 2 and the inequality $\sigma^{[2]}_\varepsilon(x)\le\delta^{[2]}_\varepsilon(\nu)$ which is valid for $x\in A_\nu(r_\varepsilon(\nu))$.

8. Common PS argument

Here we follow the arguments from the paper [13] by Y. Peres and W. Schlag.

Let $j\in\{1,2\}$. For integers $2\le x, 0\le y\le x$ define

$$E^{[j]}(x,y)=\left[\frac{y+\eta_x}{x}-\frac{\sigma^{[j]}_\varepsilon(x)}{x},\frac{y+\eta_x}{x}+\frac{\sigma^{[j]}_\varepsilon(x)}{x}\right],\ \ E^{[j]}(x)=\bigcup_{y=0}^{x} E^{[j]}(x,y)\bigcap[0,1]. \tag{33}$$

Define

$$l_0=0,\ \ l_x=l^{[j]}_x=[\log_2(x/2\sigma^{[j]}_\varepsilon(x))],\ \ x\in\mathbb{N}. \tag{34}$$

Each segment from the union $E_\alpha(x)$ from (33) can be covered by a dyadic interval of the form

$$\left(\frac{b}{2^{l_x}},\frac{b+z}{2^{l_x}}\right),\ \ z=1,2.$$

Let $A^{[j]}(x)$ be the smallest union of all such dyadic segments which cover the whole set $E^{[j]}(x)$. Put

$$(A^{[j]})^c(x)=[0,1]\setminus A^{[j]}(x).$$

Then

$$(A^{[j]})^c(x)=\bigcup_{\nu=1}^{\tau_x} I_\nu$$

where closed segments I_ν are of the form

$$\left[\frac{a}{2^{l_x}},\frac{a+1}{2^{l_x}}\right],\ \ a\in\mathbb{Z}. \tag{35}$$

We take q_0 to be a large positive integer. In order to prove Theorem 1 it is sufficient to show that for all $q\ge q_0$ the sets

$$B^{[1]}_q=\bigcap_{x=q_0}^{q}(A^{[1]})^c(x)$$

are not empty. Indeed as the sets $B^{[1]}_q$ are closed and nested we see that there exists real β such that

$$\beta\in\bigcap_{q\ge q_0} B^{[1]}_q.$$

One can see that the pair α,β satisfies the conclusion of Theorem 1.

Similarly, in order to prove Theorem 2 it is sufficient to show that for all $q \geq q_0$ the sets

$$B_q^{[2]} = \bigcap_{x \leq q,\ \phi_1(x)\psi_1(||\alpha x||) \leq \varepsilon} (A^{[2]})^c(x)$$

are not empty.

Under the conditions of Theorems 1 and 2 the following statement is valid:

Lemma 5. *Let $j \in \{1, 2\}$. Suppose that ε is small enough. Then for q_0 large enough and for any*

$$q_1 \geq q_0, \quad q_2 = q_1^A, \quad q_3 = q_2^A$$

the following holds. If

$$\text{mes} B_{q_2}^{[j]} \geq \text{mes} B_{q_1}^{[j]}/2 > 0 \tag{36}$$

then

$$\text{mes} B_{q_3}^{[j]} \geq \text{mes} B_{q_2}^{[j]}/2 > 0. \tag{37}$$

Theorems 1, 2 follow from Lemma 5 by induction as the base of the induction obviously follows from the arguments of Lemma's proof.

Proof of Lemma 5. First of all we show that for every $j \in \{1.2\}$ and $x \geq q^A$ where $q \geq q_0$ one has

$$\text{mes}\left(B_q^{[j]} \bigcap A^{[j]}(x)\right) \leq 2^4 \sigma_\varepsilon^{[1]}(x) \times \text{mes} B_q^{[j]}. \tag{38}$$

Indeed as from (34) and from (9) in the case $j = 1$ (or from (14) in the case $j = 2$) it follows that

$$l_x^{[j]} \leq (A-1)\log q, \ \forall x \leq q.$$

We see that $B_q^{[j]}$ is a union

$$B_q^{[j]} = \bigcup_{\nu=1}^{T_q} J_\nu$$

with J_ν of the form

$$\left[\frac{a}{2^l}, \frac{a+1}{2^l}\right], \quad a \in \mathbb{Z}.$$

Note that $A^{[j]}(x)$ consists of the segments of the form (35) and for $x \geq q^A > 2^{l+1}$ (for q_0 large enough) we see that each J_ν has at least two rational fractions of the form $\frac{y}{x}, \frac{y+1}{x}$ inside. So

$$\text{mes}(J_\nu \cap A^{[j]}(x)) \leq 2^4 \sigma_\varepsilon^{[j]}(x) \times \text{mes} J_\nu. \tag{39}$$

Now (38) follows from (39) by summation over $1 \leq \nu \leq T_q$.

To continue we observe that

$$B_{q_3}^{[1]} = B_{q_2}^{[1]} \setminus \left(\bigcup_{x=q_2+1}^{q_3} A^{[1]}(x) \right),$$

and

$$B_{q_3}^{[2]} = B_{q_2}^{[2]} \setminus \left(\bigcup_{q_2+1 \leq x \leq q_3,\, \phi_1(x)\psi_1(||\alpha x||) \leq \varepsilon} A^{[2]}(x) \right).$$

Hence

$$\text{mes} B_{q_3}^{[1]} \geq \text{mes} B_{q_2}^{[1]} - \sum_{x=q_2+1}^{q_3} \text{mes}(B_{q_2}^{[1]} \cap A^{[1]}(x)).$$

At the same time

$$\mathrm{mes}B^{[2]}_{q_3} \geq \mathrm{mes}B^{[2]}_{q_2} - \sum_{q_2+1\leq x\leq q_3,\, \phi_1(x)\psi_1(||\alpha x||)\leq\varepsilon} \mathrm{mes}(B^{[2]}_{q_2} \cap A^{[2]}(x)).$$

As

$$B^{[j]}_{q_2} \cap A^{[j]}(x) \subseteq B^{[j]}_{q_1} \cap A^{[j]}(x)$$

we can apply (38) for every x from the interval $q_1^3 \leq q_2 < x \leq q_3$:

$$\mathrm{mes}(B^{[j]}_{q_2} \cap A^{[j]}(x)) \leq \mathrm{mes}(B^{[j]}_{q_1} \cap A^{[j]}(x)) \leq 2^4\sigma^{[j]}_{\varepsilon}(x) \times \mathrm{mes}B^{[j]}_{q_1} \leq 2^5\sigma^{[j]}_{\varepsilon}(x) \times \mathrm{mes}B^{[j]}_{q_2}$$

(in the last inequality we use the condition (36) of Lemma 2). Now as $\frac{\log_2 q_3}{\log_2 q_2} = A$ the conclusion (37) of Lemma 5 in the case $j = 1$ follows from Lemma 3:

$$\mathrm{mes}B^{[1]}_{q_3} \geq \mathrm{mes}B^{[1]}_{q_2}\left(1 - 2^5 T^{[1]}_{A,\varepsilon}(q_2)\right) \geq \mathrm{mes}B^{[1]}_{q_2}/2.$$

In the case $j = 2$ Lemma 5 follows from Lemma 4 by a similar argument.

9. Acknowledgement

It is my great pleasure to thank Yann Bugeaud for many suggestions and comments.

References

[1] D. Badziahin, A. Pollington, S. Velani, "On a problem in simultaneous Diophantine approximation: Schmidt's conjecture", *Annals of Mathematics*, **174** (2011), 1837–1883.

[2] D. Badziahin, *On multiplicatively badly approximable numbers*, Preprint available at arXiv:1101.1855, 2011.

[3] Y. Bugeaud, "Multiplicative Diophantine approximation", *Dynamical systems and Diophantine Approximation*, Société mathématique de France, Séminaires et Congrès 19, 2009, 107–127.

[4] Y. Bugeaud, N. Moshchevitin, "Badly approximable numbers and Littlewood-type problems", *Mathematical Proceedings of the Cambridge Philosophical Society*, **150**:02 (2011), 215–226.

[5] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts Math. Math. Phys.. V. 45, Cambridge Univ. Press, New York, 1957.

[6] H. J. Godwin, "On the theorem of Khintchine", *Proc. London Math. Soc.* V. 3, 1, 1953, 211–221.

[7] A. Khintchine, "Über eine Klasse linearer diophantischer Approximationen", *Rendiconti Circ. Math. Palermo*, **50**:2 (1926), 170–195.

[8] E. Lindenstrauss, U. Shapira, *Homogeneous orbit closures and applications*, Preprint available at arXiv:1101.3945, 2011.

[9] N. Moshchevitin, "Khintchine's Diophantine singular systems and their applications", *Russian Math. Surveys*, **65**:3 (2010), 433–511.

[10] N. Moshchevitin, "On simultaneously badly approximable numbers", *Bull. London Math. Soc.*, **42**:1 (2010), 149–154.

[11] N. Moshchevitin, "A note on badly approximable affine forms and winning sets", *Moscow Mathematical Journal*, **11**:1 (2011), 129–137.

[12] N. G. Moshchevitin, *Schmidt's conjecture and Badziahin – Pollington – Velani's theorem*, Preprint available at arXiv:1004.4269, 2010.

[13] Y. Peres, W. Schlag, "Two Erdös problems on lacunary sequences: chromatic numbers and Diophantine approximations", *Bull. London Math. Soc.*, **42**:2 (2010), 295–300.

[14] A. M. Rockett, P. Szüsz, *Continued Fractions*, World Scientific Publishing Co., 1992.

[15] W. M. Schmidt, *Diophantine Approximations*, Lect. Not. Math. V. 785, 1980.

[16] W. M. Schmidt, "Open problems in Diophantine approximations", *Approximations Diophantiennes et nombres transcendants' Luminy, 1982*, Progress in Mathematics, Birkhäuser, 1983, 271–289.

[17] U. Shapira, “A solution to a problem of Cassels and Diophantine properties of cubic numbers”, *Annals of Mathematics*, **173**:1 (2011), 543–557.

[18] U. Shapira, *Grids with dense values*, Preprint available at arXiv:1101.3941, 2011.

[19] J. Tseng, “Badly approhimable affine forms and Schmidt games”, *J. Number Theory*, **129**:12 (2009), 3020–3025.

Submitted August 10, 2012

Supported by the grant RFBR №09-01-00371 and by the grant of Russian Government, project 11. G34.31.0053.

Мощевитин Н. Г. О некоторых задачах теории неоднородных диофантовых приближений, связанных с проблемами Литтлвуда и Шмидта. Дальневосточный математический журнал. 2012. Т. 12. № 2. С. 237–254.

АННОТАЦИЯ

Доказывается ряд новых результатов о неоднородных диофантовых приближениях для двух вещественных чисел. Наши теоремы связаны со старыми результатами А. Я. Хинчина [7] и новым подходом, предложенным Ю. Пересом и В. Шлагом [13].

Ключевые слова: *диофантовы приближения, гипотеза Литтлвуда, метод Переса – Шлага, плохо приближаемые числа.*