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A Kernel Smoothing Method for General Integral Equations

In this paper, we reduce the general linear integral equation of the third kind in $L^2(Y,\mu)$, with largely arbitrary kernel and coefficient, to an equivalent integral equation either of the second kind or of the first kind in $L^2(\mathbb{R})$, with the kernel being the linear pencil of bounded infinitely differentiable bi-Carleman kernels expandable in absolutely and uniformly convergent bilinear series. The reduction is done by using unitary equivalence transformations.

Key words: *linear integral equations of the first, second, and third kind, unitary operator, multiplication operator, bi-integral operator, bi-Carleman kernel, Hilbert-Schmidt kernel, bilinear series expansions of kernels*

1. Introduction and Preliminaries

In the theory of general linear integral equations in L^2 spaces, equations with bounded infinitely differentiable bi-Carleman kernels (termed K^∞ kernels) should and do lend themselves well to solution by approximation and variational methods. The question of whether a second-kind integral equation with arbitrary kernel can be reduced to an equivalent one with a K^∞ kernel was positively answered using a unitary-reduction method by the author [18]. In the present paper, our goal is to extend the method in order to deal with a third-kind integral equation ((1) below) with arbitrary measurable kernel and coefficient.

Results obtained are presented with proofs in Section 2. The results say that the general linear integral equation of the third kind in $L^2(Y,\mu)$ can be reduced to an equivalent integral equation either of the second kind (Theorem 3) or of the first kind (Theorem 4) in $L^2(\mathbb{R})$, with the kernel being the linear pencil of K^∞ kernels of Mercer type or of Hilbert-Schmidt K^∞ kernels of Mercer type, respectively. Before we can write down and prove our results, we need to fix the terminology and notation and to give some definitions and preliminary material.

Throughout this paper, $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ and inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. (Y,μ) is a measure space Y equipped with a positive, σ-finite, complete, separable, and nonatomic, measure μ. $L^2(Y,\mu)$ is the Hilbert space of (equivalence classes of) μ-measurable complex-valued functions on Y equipped with the inner product $\langle f,g\rangle_{L^2(Y,\mu)} = \int_Y f(y)\overline{g(y)}\,d\mu(y)$ and the norm $\|f\|_{L^2(Y,\mu)} = \langle f,f\rangle^{1/2}_{L^2(Y,\mu)}$; when μ is the Lebesgue measure on the real line $\mathbb{R}$, $L^2(\mathbb{R},\mu)$ is abbreviated into L^2, and $d\mu(y)$ into dy. $C(X,B)$, where B is a Banach space with norm $\|\cdot\|_B$, is the Banach space (with the norm $\|f\|_{C(X,B)} = \sup_{x\in X}\|f(x)\|_B$) of continuous B-valued functions defined on a locally compact space X and *vanishing at infinity* (that is, given any $f\in C(X,B)$ and $\varepsilon>0$, there exists a compact subset $X(\varepsilon,f)\subset X$ such that

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$\|f(x)\|_B < \varepsilon$ whenever $x \notin X(\varepsilon, f)$). A series $\sum_n f_n$ is *B-absolutely convergent in* $C(X, B)$ if $f_n \in C(X, B)$ ($n \in \mathbb{N}$) and the series $\sum_n \|f_n(x)\|_B$ converges in $C(X, \mathbb{R})$. Given an equivalence class $f \in L^2$ containing a function of $C(\mathbb{R}, \mathbb{C})$, the symbols $[f]$ and $[f]^{(i)}$ are used to denote that function and its ith derivative, if exists. The symbols $\mathbb{C}$ and $\mathbb{N}$ refer to the complex plane and the set of all positive integers, respectively.

Throughout, $\mathfrak{R}(\mathcal{H})$ denotes the Banach algebra of all bounded linear operators acting on $\mathcal{H}$. For an operator T of $\mathfrak{R}(\mathcal{H})$, T^* stands for the adjoint to T (w.r.t. $\langle \cdot, \cdot \rangle_{\mathcal{H}}$), and the family $\mathcal{M}^+(T)$ is defined as the set of all those operators $P \in \mathfrak{R}(\mathcal{H})$ that are positive (that is, $\langle Px, x \rangle_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$) and factorizable as $P = TB$ or as $P = BT$ with $B \in \mathfrak{R}(\mathcal{H})$. A factorization of an operator $T \in \mathfrak{R}(\mathcal{H})$ into the product $T = WV^*$ (where V, $W \in \mathfrak{R}(\mathcal{H})$) is called an $\mathcal{M}$ *factorization* for T provided that VV^*, $WW^* \in \mathcal{M}^+(T)$; cf. [21]. One example of an $\mathcal{M}$ factorization for any $T \in \mathfrak{R}(\mathcal{H})$ is obtained by letting $W = UP$, $V = P$, where P is the positive square root of $|T| := (T^*T)^{\frac{1}{2}}$ and U is the partially isometric factor in the polar decomposition $T = U|T|$. (Indeed: $T = WV^*$, $WW^* = U|T|U^* = TU^* = UT^* \in \mathcal{M}^+(T)$, and $VV^* = |T| = T^*U = U^*T \in \mathcal{M}^+(T)$.)

A bounded linear operator $U \colon \mathcal{H} \to L^2$ is *unitary* if it has range L^2 and $\langle Uf, Ug \rangle_{L^2} = \langle f, g \rangle_{\mathcal{H}}$ for all f, $g \in \mathcal{H}$. An operator $S \in \mathfrak{R}(\mathcal{H})$ is *unitarily equivalent* to an operator $T \in \mathfrak{R}\left(L^2\right)$ if a unitary operator $U : \mathcal{H} \to L^2$ exists such that $T = USU^{-1}$.

A linear operator $T \colon L^2(Y, \mu) \to L^2(Y, \mu)$ is *integral* if there is a complex-valued $\mu \times \mu$-measurable function $\boldsymbol{T}$ (*kernel*) defined on the Cartesian product $Y^2 = Y \times Y$ such that, for each f in $L^2(Y, \mu)$,

$$(Tf)(x) = \int_Y \boldsymbol{T}(x, y) f(y)\, d\mu(y) \quad \text{for } \mu\text{-almost every } x \text{ in } Y;$$

cf. [6], [9]. The integral operator T is *bi-integral* if its adjoint T^* is also an integral operator on $L^2(Y, \mu)$; cf. [9]. The integral and the bi-integral operators are bounded, and need not be compact.

The general linear integral equation of the *third kind* in $L^2(Y, \mu)$ is an equation of the form

$$\boldsymbol{H}(x)\phi(x) - \lambda \int_Y \boldsymbol{K}(x, y)\phi(y)\, d\mu(y) = \psi(x) \qquad \mu\text{-almost everywhere on } Y, \tag{1}$$

where $\boldsymbol{H} \colon Y \to \mathbb{C}$ (the *coefficient* of the equation) is a given bounded μ-measurable function, $\boldsymbol{K} \colon Y^2 \to \mathbb{C}$ (the *kernel* of the equation) is a given kernel of a bi-integral operator $K \in \mathfrak{R}\left(L^2(Y, \mu)\right)$, the scalar $\lambda \in \mathbb{C}$ (a *parameter*) is given, the function ψ of $L^2(Y, \mu)$ is given, and the function ϕ of $L^2(Y, \mu)$ is to be determined. When the coefficient $\boldsymbol{H}$ has the constant value 0 (resp. 1) μ-almost everywhere on Y, the general linear integral equation (1) is referred to as of the *first* (resp. *second*) *kind*.

A *bi-Carleman kernel* $\boldsymbol{T}$ on Y^2 is a kernel for which

$$\int_Y |\boldsymbol{T}(x, y)|^2 d\mu(y) < \infty, \quad \int_Y |\boldsymbol{T}(y, x)|^2 d\mu(y) < \infty \quad \text{for } \mu\text{-almost every } x \text{ in } Y.$$

A *Hilbert-Schmidt kernel* $\boldsymbol{\Gamma}$ on Y^2 is one for which

$$\int_Y \int_Y |\boldsymbol{\Gamma}(x, y)|^2 d\mu(y)\, d\mu(x) < \infty.$$

A K^∞ *kernel* $\boldsymbol{T}$ is a bi-Carleman kernel on $\mathbb{R}^2$, which is subject to the following infinite differentiability requirements:

(i) the function $\boldsymbol{T}$ and all its partial derivatives of all orders are in $C\left(\mathbb{R}^2, \mathbb{C}\right)$,

(ii) the (first) *Carleman function* $\boldsymbol{t}\colon \mathbb{R} \to L^2$, defined via $\boldsymbol{T}$ by $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$, and its (strong) derivatives $\boldsymbol{t}^{(i)}$ of all orders are in $C\left(\mathbb{R}, L^2\right)$,

(iii) the (second) *Carleman function* $\boldsymbol{t'}\colon \mathbb{R} \to L^2$, defined via $\boldsymbol{T}$ by $\boldsymbol{t'}(s) = \boldsymbol{T}(\cdot, s)$, and its (strong) derivatives $(\boldsymbol{t'})^{(j)}$ of all orders are in $C\left(\mathbb{R}, L^2\right)$;

cf. [17], [21]. A K^∞ kernel $\boldsymbol{T}$ is called of *Mercer type* if it induces an integral operator $T \in \mathfrak{R}(L^2)$, with the property that any operator belonging to $\mathcal{M}^+(T)$ is also an integral operator induced by a K^∞ kernel; cf. [21].

Any K^∞ kernel $\boldsymbol{T}$ of Mercer type, along with all its partial and strong derivatives, is entirely recoverable from the knowledge of at least one $\mathcal{M}$ factorization for its associated integral operator T, by means of bilinear series formulae universally applicable on arbitrary orthonormal bases of L^2:

Theorem 1. *Let $T \in \mathfrak{R}\left(L^2\right)$ be an integral operator, with a kernel $\boldsymbol{T}$ that is a K^∞ kernel of Mercer type. Then, for any $\mathcal{M}$ factorization $T = WV^*$ for T and for any orthonormal basis $\{u_n\}$ for L^2, the following formulae hold*

$$\frac{\partial^{i+j}\boldsymbol{T}}{\partial s^i \partial t^j}(s,t) = \sum_n \left[Wu_n\right]^{(i)}(s)\overline{\left[Vu_n\right]^{(j)}(t)}, \tag{2}$$

$$\boldsymbol{t}^{(i)}(s) = \sum_n \overline{\left[Wu_n\right]^{(i)}(s)}Vu_n, \quad \left(\boldsymbol{t'}\right)^{(j)}(t) = \sum_n \overline{\left[Vu_n\right]^{(j)}(t)}Wu_n \tag{3}$$

for all non-negative integers i, j and all s, $t \in \mathbb{R}$, where the series of (2) *converges $\mathbb{C}$-absolutely in $C\left(\mathbb{R}^2, \mathbb{C}\right)$, and the series of* (3) *converge in $C\left(\mathbb{R}, L^2\right)$.*

The theorem is proven in [21]. It can also be seen as a generalization of both Mercer's [13] theorem (about absoluteness and uniformity of convergence of bilinear eigenfunction expansions for continuous compactly supported kernels of positive, integral operators) and Kadota's [8] theorem (about term-by-term differentiability of those expansions while retaining the absolute and the uniform convergence) to various other settings; for details see [14], [21].

The main device for the proof of our reduction theorems in the next section is provided by the following result, which characterizes families incorporating those operators in $\mathfrak{R}(\mathcal{H})$ that can be simultaneously transformed by the same unitary equivalence transformation into bi-integral operators having as kernels K^∞ kernels of Mercer type:

Theorem 2. *Suppose that for an operator family $\{S_\gamma\}_{\gamma\in\mathcal{G}} \subset \mathfrak{R}(\mathcal{H})$ with an index set of arbitrary cardinality there exists an orthonormal sequence $\{e_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \sup_{\gamma\in\mathcal{G}} \left\|S_\gamma e_n\right\|_{\mathcal{H}} = 0, \quad \lim_{n\to\infty} \sup_{\gamma\in\mathcal{G}} \left\|S_\gamma^* e_n\right\|_{\mathcal{H}} = 0. \tag{4}$$

Then there exists a unitary operator $U : \mathcal{H} \to L^2$ such that all the operators $T_\gamma = US_\gamma U^{-1}$ ($\gamma \in \mathcal{G}$) and their linear combinations are bi-integral operators on L^2, whose kernels are K^∞ kernels of Mercer type.

The proof is given in [21]. It provides an explicit procedure to find that unitary operator $U : \mathcal{H} \to L^2$ whose existence the Theorem 2 asserts. The procedure uses no spectral properties of the operators S_γ, other than their joint property imposed in (4), to determine the action of U by specifying two orthonormal bases, of $\mathcal{H}$ and of L^2, one of which is meant to be the image by U of the other; the basis for L^2 may be chosen to be the Lemarié-Meyer wavelet basis [3], [7, Example D, p. 62].

2. Reduction Theorems

Theorem 3. *Suppose that the essential range of the coefficient $\boldsymbol{H}$ in* (1) *contains the point $\alpha \in \mathbb{C}$, that is,*

$$\mu\{y \in Y : |\boldsymbol{H}(y) - \alpha| < \varepsilon\} > 0 \quad \text{for all } \varepsilon > 0. \tag{5}$$

Then equation (1) *is equivalent (via a unitary operator U from $L^2(Y,\mu)$ onto L^2) to an integral equation in L^2, of the form*

$$\alpha f(s) + \int_{\mathbb{R}} (\boldsymbol{T}_0(s,t) - \lambda \boldsymbol{T}(s,t))\, f(t)\, dt = g(s) \quad \text{almost everywhere on } \mathbb{R}, \tag{6}$$

where the function $f(= U\phi)$ of L^2 is to be determined, the function $g(= U\psi)$ of L^2 is given, both the functions $\boldsymbol{T}_0$ and $\boldsymbol{T}$ are K^∞ kernels of Mercer type, not depending on λ, and the function $\boldsymbol{T}_0 - \lambda\boldsymbol{T}$ is also a K^∞ kernel of Mercer type.

Proof. The proof relies primarily on the following observation by Korotkov [11, Corollary 1]: If H is the multiplication operator induced on $L^2(Y,\mu)$ by the coefficient $\boldsymbol{H}$, and I is the identity operator on $L^2(Y,\mu)$, then the two-element family $\{S_1 = H - \alpha I, S_2 = K\}$ of bounded operators on $\mathcal{H} = L^2(Y,\mu)$ satisfies the assumptions of Theorem 2. The construction of Korotkov's sequence $\{e_n\}$ fulfilling (4) for this family is likely to be of practical use and deserves to be expounded in some detail.

If $E \subset Y$ is a μ-measurable set of positive finite measure, the orthonormal sequence of *generalized* Rademacher functions with supports in E will be denoted by $\{R_{n,E}\}_{n=1}^\infty$ and is constructed iteratively through successive bisections of E as follows:

$$R_{1,E} = \frac{1}{\sqrt{\mu E}}\left(\chi_{E_1} - \chi_{E_2}\right)$$

provided $E_1 \sqcup E_2 = E$ with $\mu E_1 = \mu E_2 = \frac{1}{2}\mu E$;

$$R_{2,E} = \frac{1}{\sqrt{\mu E}}\left(\chi_{E_{1,1}} - \chi_{E_{1,2}} + \chi_{E_{2,1}} - \chi_{E_{2,2}}\right)$$

provided $E_{i,1} \sqcup E_{i,2} = E_i$ with $\mu E_{i,k} = \frac{1}{4}\mu E$ for $i,k = 1,2$;

$$R_{3,E} = \frac{1}{\sqrt{\mu E}}(\chi_{E_{1,1,1}} - \chi_{E_{1,1,2}} + \chi_{E_{1,2,1}} - \chi_{E_{1,2,2}} + \chi_{E_{2,1,1}} - \chi_{E_{2,1,2}} + \chi_{E_{2,2,1}} - \chi_{E_{2,2,2}})$$

provided $E_{i,k,1} \sqcup E_{i,k,2} = E_{i,k}$ with $\mu E_{i,k,j} = \frac{1}{8}\mu E$ for $i,k,j = 1,2$; and so on indefinitely (here χ_Z denotes the characteristic function of a set Z and the unions are disjoint). A relevant result due to Korotkov states that

$$\lim_{n\to\infty} \|K^* R_{n,E}\|_{L^2(Y,\mu)} = 0 \tag{7}$$

for any integral operator $K \in \mathfrak{R}\left(L^2(Y,\mu)\right)$ and any μ-measurable $E \subset Y$ with $0 < \mu E < \infty$; see, e.g., the proof of Theorem 3 in [10], or in [12, pp. 108–111].

Let $\{Y_n\}_{n=1}^\infty$ be an ascending sequence of sets of positive finite measure, such that $Y_n \uparrow Y$, let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence of positive reals strictly decreasing to zero, and define $E_n = Y_n \cap \{y \in Y : \varepsilon_{n+1} < |\boldsymbol{H}(y) - \alpha| \le \varepsilon_n\}$ whenever $n \in \mathbb{N}$. Due to the assumption (5), one can always make the sets E_n to have finite nonzero measures by an appropriate choice of Y_n and ε_n ($n \in \mathbb{N}$). Supposing this has been done, let $e_n = R_{k_n,E_n}$, where, for each $n \in \mathbb{N}$, k_n is an index satisfying

$$\|S_2 e_n\|_{L^2(Y,\mu)} + \|S_2^* e_n\|_{L^2(Y,\mu)} = \|K R_{k_n,E_n}\|_{L^2(Y,\mu)} + \|K^* R_{k_n,E_n}\|_{L^2(Y,\mu)} \le 1/n \tag{8}$$

(cf. (7)). Since the E_n's are pairwise disjoint, the e_n's form an orthonormal sequence in $L^2(Y,\mu)$. Moreover, by construction of sets E_n,

$$\|S_1 e_n\|^2_{L^2(Y,\mu)} = \|S_1^* e_n\|^2_{L^2(Y,\mu)} = (1/\mu E_n)\int_{E_n} |\boldsymbol{H}(y)-\alpha|^2\, d\mu(y) \le \varepsilon_n^2. \tag{9}$$

One can now assert from (8), (9) that $\|S_r e_n\|_{L^2(Y,\mu)} \to 0$, $\|S_r^* e_n\|_{L^2(Y,\mu)} \to 0$ as $n\to\infty$, for $r = 1, 2$. By Theorem 2, there is then a unitary operator $U\colon L^2(Y,\mu)\to L^2$ such that the operators $T_0 = US_1U^{-1}$, $T = US_2U^{-1}$ and their linear combinations are bi-integral operators on L^2, whose kernels are K^∞ kernels of Mercer type. This unitary operator can also be used to transform the integral equation (1) into an equivalent integral equation of the form (6) in such a way that $\boldsymbol{T}_0$, $\boldsymbol{T}$, and $\boldsymbol{T}_0-\lambda\boldsymbol{T}$, are just those K^∞ kernels of Mercer type that induce T_0, T, and $T_0-\lambda T$, respectively. In operator notation, such a passage from (1) to (6) looks as follows: $U\psi = U(H-\lambda K)U^{-1}U\phi = U(\alpha I + S_1 - \lambda S_2)U^{-1}U\phi = \alpha f + (T_0-\lambda T)f = g$ where $f = U\phi$, $g = U\psi$. The theorem is proved. □

Theorem 4. *If, with the notation and hypotheses of Theorem 3, $\alpha = 0$, then equation* (1) *is equivalent to a first-kind integral equation in L^2, of the form*

$$\int_{\mathbb{R}} (\boldsymbol{\Gamma}_0(s,t)-\lambda\boldsymbol{\Gamma}(s,t))\, f(t)\, dt = w(s) \quad \textit{almost everywhere on } \mathbb{R}, \tag{10}$$

where the function f of L^2 is to be determined, both the functions $\boldsymbol{\Gamma}_0$ and $\boldsymbol{\Gamma}$ are Hilbert-Schmidt K^∞ kernels of Mercer type, not depending on λ, and the function $\boldsymbol{\Gamma}_0-\lambda\boldsymbol{\Gamma}$ is also a Hilbert-Schmidt K^∞ kernel of Mercer type.

Proof. In this case the equation (6), equivalent to (1), becomes

$$\int_{\mathbb{R}} (\boldsymbol{T}_0(s,t)-\lambda\boldsymbol{T}(s,t))\, f(t)\, dt = g(s) \quad \text{for almost every } s \text{ in } \mathbb{R}. \tag{11}$$

Let $m\in L^2$ be such that $[m]$ is an infinitely differentiable, positive function all whose derivatives $[m]^{(i)}$ belong to $C(\mathbb{R},\mathbb{R})$, and let M be the multiplication operator induced on L^2 by m. Multiply both sides of equation (11) by m, to recast it into an equivalent equation of the form (10), with the same sought-for function $f\in L^2$, the new right side $w = Mg\in L^2$, and the new kernel $\boldsymbol{\Gamma}_0-\lambda\boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}_0(s,t) = [m](s)\boldsymbol{T}_0(s,t)$, $\boldsymbol{\Gamma}(s,t) = [m](s)\boldsymbol{T}(s,t)$. It is to be proved that $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}$ are Hilbert-Schmidt K^∞ kernels. The proof is further given only for $\boldsymbol{\Gamma}$, as the proof for the other kernel $\boldsymbol{\Gamma}_0$ is entirely similar. If $\boldsymbol{t}$ is the associated Carleman function of the K^∞ kernel $\boldsymbol{T}$ (see (ii)), then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\boldsymbol{\Gamma}(s,t)|^2\, dt\, ds = \int_{\mathbb{R}} m^2(s)\|\boldsymbol{t}(s)\|^2_{L^2}\, ds \le \|\boldsymbol{t}\|^2_{C(\mathbb{R},L^2)}\|m\|^2_{L^2} < \infty,$$

implying that $\boldsymbol{\Gamma}$ is a Hilbert-Schmidt kernel and hence induces two Carleman functions $\boldsymbol{\gamma}$, $\boldsymbol{\gamma'}\colon \mathbb{R}\to L^2$ by $\boldsymbol{\gamma}(s) = \overline{\boldsymbol{\Gamma}(s,\cdot)}$, $\boldsymbol{\gamma'}(t) = \boldsymbol{\Gamma}(\cdot,t)$. The series representation of $\boldsymbol{T}$ (see (2) for $i = j = 0$) gives rise to a series representation of $\boldsymbol{\Gamma}$, namely, with the notation of Theorem 1,

$$\boldsymbol{\Gamma}(s,t) = \sum_n [MWu_n](s)\overline{[Vu_n](t)} = [m](s)\sum_n [Wu_n](s)\overline{[Vu_n](t)}$$

for all $(s,t)\in\mathbb{R}^2$. Moreover, for all non-negative integers i, j and all s, $t\in\mathbb{R}$, the following formulae hold

$$\begin{aligned}\frac{\partial^{i+j}\boldsymbol{\Gamma}}{\partial s^i\partial t^j}(s,t) &= \sum_n [MWu_n]^{(i)}(s)\overline{[Vu_n]^{(j)}(t)}\\ &= \sum_{r=0}^{i}\binom{i}{r}[m]^{(i-r)}(s)\left(\sum_n [Wu_n]^{(r)}(s)\overline{[Vu_n]^{(j)}(t)}\right),\end{aligned}$$

$$\gamma^{(i)}(s) = \sum_n \overline{[MWu_n]^{(i)}(s)} Vu_n = \sum_{r=0}^{i} \binom{i}{r} [m]^{(i-r)}(s) \left(\sum_n \overline{[Wu_n]^{(r)}(s)} Vu_n \right),$$

$$(\gamma')^{(j)}(t) = \sum_n \overline{[Vu_n]^{(j)}(t)} MWu_n = M \left(\sum_n \overline{[Vu_n]^{(j)}(t)} Wu_n \right),$$

in as much as the bracketed series above converge $\mathbb{C}$-absolutely in $C(\mathbb{R}^2, \mathbb{C})$, as regards the first formula, and in $C(\mathbb{R}, L^2)$, as regards the last two formulae, by Theorem 1. Therefore

$$\frac{\partial^{i+j}\boldsymbol{\Gamma}}{\partial s^i \partial t^j} \in C(\mathbb{R}^2, \mathbb{C}), \quad \gamma^{(i)}, (\gamma')^{(j)} \in C(\mathbb{R}, L^2)$$

for all non-negative integers i, j, implying that $\boldsymbol{\Gamma}$ is a K^∞ kernel.

If not all of the K^∞ kernels $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}_0 - \lambda\boldsymbol{\Gamma}$ are now of Mercer type, apply Theorem 2 to the two-element family $\{S_1 = MT_0, S_2 = MT\}$ of compact operators on $\mathcal{H} = L^2$; the condition (4) for this family is satisified by any orthonormal sequence $\{e_n\}$ in L^2. The proof of the theorem is now complete. $\square$

3. Remarks

In virtue of Theorems 3 and 4, one can confine one's attention (with no essential loss of generality) to integral equations, whose kernels are K^∞ kernels of Mercer type, depending linearly on a parameter. One of the main technical advantages of dealing with such kernels is that their restrictions to compact rectangles in $\mathbb{R}^2$ are fully amenable to the methods of the classical theory of ordinary (nonsingular) integral equations, and can be applied to approximate the original kernels with respect to $C(\mathbb{R}^2, \mathbb{C})$ and $C(\mathbb{R}, L^2)$ norms. This, for instance, can be used directly to establish an explicit theory of spectral functions for any Hermitian K^∞ kernel ($\boldsymbol{T}(s,t) = \overline{\boldsymbol{T}(t,s)}$) by a development essentially the same as the one given by T. Carleman in [4, pp. 25–51]; see also [24], [23], [1], [2, Appendix I], [5], [25], and [9], for further developments and applications of Carleman's spectral theory. We believe that with regard to K^∞ kernels of Mercer type this Carleman's line of development can be extended far beyond the restrictive assumption of a Hermitian (or normal) kernel; see, e.g., [14]. For the theory of Fredholm determinant and minors, there are some applications in [15], [16], [19], and [20].

The present paper is a slightly edited version of [22].

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АННОТАЦИЯ

Общее линейное интегральное уравнение 3-го рода в $L^2(Y,\mu)$ сводится унитарным преобразованием к эквивалентному интегральному уравнению 1-го или 2-го рода в $L^2(\mathbb{R})$ с ядром, представляющим собой линейный пучок ограниченных, бесконечно дифференцируемых бикарлемановских ядер мерсеровского типа.

Ключевые слова: *линейные интегральные уравнения 1, 2 и 3-го рода, унитарный оператор, оператор умножения, биинтегральный оператор, бикарлемановское ядро, ядро Гильберта – Шмидта, билинейные разложения ядер*