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Inhomogeneous Diophantine approximation on curves with non-monotonic error function

In this paper we prove a convergent part of inhomogeneous Groshev type theorem for non–degenerate curves in Euclidean space where an error function is not necessarily monotonic. Our result naturally incorporates and generalizes the homogeneous measure theorem for non-degenerate curves. In particular, the method of Inhomogeneous Transference Principle and Sprindzuk's method of essential and inessential domains are used in the proof.

Key words: *Inhomogeneous Diophantine approximation, Khintchine theorem, nondegenerate curve.*

Introduction and Statements

In 1998 Kleinbock and Margulis [10] established the Baker–Sprindzuk conjecture concerning homogeneous Diophantine approximation on manifolds. An inhomogeneous version was then proved by Beresnevich and Velani [6]. The theory of inhomogeneous Diophantine approximation on manifolds was started with the result of V. I. Bernik, D. Dickinson and M. Dodson [7]. The significantly stronger Groshev type theory for dual Diophantine approximation on manifolds is established in [3], [4], and [8] for the homogeneous case and in [2] for the inhomogeneous case. In all of these results the error function Ψ was assumed to be monotonic. In 2005 Beresnevich [5] showed that the condition that Ψ is monotonic could be removed for the Veronese curve $\mathcal{V}_n = \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}$; he conjectured that the result should also hold for any non–degenerate curve in Euclidean space. This was proved in [9].

Our main result below is a convergent part of Groshev type theorem for inhomogeneous Diophantine approximation on non–degenerate curves in Euclidean space without monotonicity condition. First some notation is needed. Let $\mathcal{F}_n$ be the set of functions

$$a_n f_n(x) + \ldots + a_1 f_1(x) + a_0,$$

with $n \geq 2$, $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$, and $f_1, f_2, \ldots, f_n$ be $C^{(n)}$ functions from $\mathbb{R} \to \mathbb{R}$ with non-vanishing Wronskian $wr(f_1', \ldots, f_n')(x)$ almost everywhere. For $F \in$

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$\mathcal{F}_n$ define the height of F as $H = H(F) = \max_{0\le j\le n} |a_j|$. The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ is denoted by $\mu(A)$.

Define a real valued function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\theta : \mathbb{R} \to \mathbb{R}$. Denote by $\mathcal{L}_{n,\theta}(\Psi)$ the set of $x \in \mathbb{R}$ such that the inequality

$$|F(x) + \theta(x)| < \Psi(H(F)) \tag{1}$$

has infinitely many solutions $F \in \mathcal{F}_n$.

The main result of this paper is the following statement.

Theorem 1. *Let $n \ge 2$ and $\theta : \mathbb{R} \to \mathbb{R}$ be a function such that $\theta \in C^{(n)}$. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be an arbitrary function (not necessarily monotonic) such that the sum $\sum_{h=1}^{\infty} h^{n-1}\Psi(h)$ converges. Then*

$$\mu(\mathcal{L}_{n,\theta}(\Psi)) = 0.$$

Throughout, the Vinogradov symbol $\ll$ is used so that if K and M are positive real numbers then $K \ll M$ means that there exists $C > 0$ such that $K \le CM$. If $K \ll M$ and $M \ll K$ we write $K \asymp M$.

1. Proof of Theorem 1

First note that since $\sum_{h=1}^{\infty} h^{n-1}\Psi(h)$ converges, $h^{n-1}\Psi(h)$ tends to 0 as $h \to \infty$. Therefore,

$$\Psi(h) = o(h^{-n+1}). \tag{2}$$

The set $S = \{x \in \mathbb{R} : wr(f_1', \dots, f_n')(x) = 0\}$ is closed and of zero measure. Thus $\mathbb{R} \setminus S$ is open and therefore an F_σ set. We can write $\mathbb{R} \setminus S = \cup_{k=1}^{\infty} [a_k, b_k]$. It is therefore sufficient to prove the theorem for a closed interval I. Also, since $|wr(f_1', \dots, f_n')(x)| \ne 0$ almost everywhere we will assume from now on, without loss of generality that

$$|wr(f_1', \dots, f_n')(x)| \ge \varepsilon = \varepsilon(I) > 0 \tag{3}$$

for all x in such an interval I. Since the functions $\mathbf{f} = (f_1, \dots, f_n)$ and θ are $C^{(n)}$ then we can assume that there exists a constant $K_0 = K_0(I, \mathbf{f}, \theta)$ such that

$$\max_{0\le i\le n} \sup_{x\in I} |\mathbf{f}^{(i)}(x)| \le K_0 \quad \text{and} \quad \max_{0\le i\le n} \sup_{x\in I} |\theta^{(i)}(x)| \le K_0. \tag{4}$$

Let $\gamma > 0$ be a fixed real number. As $\gamma \to 0$ the measure of the set of $x \in I$ for which the inequality $|f_s(x)| \le \gamma$ holds for at least for one s, $1 \le s \le n$, also tends to zero. Hence, from now on it is assumed that

$$|f_i(x)| > \gamma, \quad 1 \le i \le n. \tag{5}$$

In what follows define the function t_{ij} as $t_{ij}(x) = f_i(x) f_j^{-1}(x)$. It is shown in Lemma 3 of [1] that if $wr(f_1', \dots, f_n')(x) \ne 0$ almost everywhere then $t_{ij}'(x) \ne 0$ almost everywhere for all $i, j \in \{1, \dots, n\}$. The next lemma relates the size of $|wr(f_1', \dots, f_n')(x)|$ to the size of $|f_i(x) f_j'(x) - f_i'(x) f_j(x)|$.

Lemma 1. *[9] If $|wr(f_1',\dots,f_n')(x)| \geq \varepsilon$ then $|f_i(x)f_j'(x) - f_i'(x)f_j(x)| > \frac{\varepsilon\gamma^2}{2^{n+1}n!K_0^n}$ for all i, j in $\{1,\dots,n\}$.*

From now on, it is therefore assumed without loss of generality that

$$|f_i(x)f_j'(x) - f_i'(x)f_j(x)| \geq \delta_2 = \frac{\varepsilon\gamma^2}{2^{n+1}n!K_0^n} \tag{6}$$

for all $i, j \in \{1,\dots,n\}$ with $i \neq j$.

Lemma 2. *[9] Let $I \subset \mathbb{R}$ be an interval for which $|wr(f_1',\dots,f_n')(x)| \geq \varepsilon$. Let $B_1 \subset \mathbb{R}$ be a set with $\mu(B_1) = 0$ and let $B_2 = \{x \in I : t_{ij}(x) \in B_1\}$, then B_2 also has zero measure.*

For the proof of main result we will need some properties of the functions $F \in \mathcal{F}_n$. The following lemma is a modification and combination of Lemmas 2 and 3 of Pyartli, [11]. We are assuming that (3) holds.

Lemma 3. *Let $F \in \mathcal{F}_n$. For any interval $I_1 \subset I$ with length $|I_1| \leq l = l(\varepsilon(I), K_0)$ there exists i, $1 \leq i \leq n$, such that*

$$|F^{(i)}(x)| > c(l)H(F) \tag{7}$$

for all $x \in I_1$. The number of zeros of $F \in \mathcal{F}_n$ in I_1 does not exceed n.

Using Lemma 3 and (4), we obtain that for $F \in \mathcal{F}_n$ and any interval $I_1 \subset I$ with length $|I_1| \leq l = l(\varepsilon(I), K_0)$ there exists i, $1 \leq i \leq n$, such that

$$|F^{(i)}(x) + \theta^{(i)}(x)| > c_0(l)H(F) \tag{8}$$

for all $x \in I_1$ and sufficiently large $H(F)$. Rolle's theorem and (8) imply that the number of sub–intervals in any interval I_1 with $|I_1| \leq l(\varepsilon(I), K_0)$ where $F + \theta$ is monotonic is at most n, where $F \in \mathcal{F}_n$ and $H(F)$ is sufficiently large.

Every interval I can be written as a finite union of intervals I_1 with $|I_1| \leq l$. Therefore, it is sufficient to prove the theorem for each of these smaller intervals. From now on, we restrict ourselves to such an interval, relabelled I, which without loss of generality satisifies (8).

The proof is now split into two parts and the following two sets are considered. Fix a real number v satisfying

$$0 < v < 1/4. \tag{9}$$

Define,

$$\mathcal{L}_1(n,\theta) = \{x \in I : |F(x)+\theta(x)| < H(F)^{-n+1},\ |F'(x)+\theta'(x)| < H(F)^{-v} \ \text{ i.m. } F \in \mathcal{F}_n\}$$

and

$$\mathcal{L}_2(n,\theta,\Psi) = \{x \in I : |F(x)+\theta(x)| < \Psi(H(F)),\ |F'(x)+\theta'(x)| \geq H(F)^{-v} \ \text{ i.m. } F \in \mathcal{F}_n\}$$

where i.m. should read *for infinitely many.* Clearly, from (2),

$$\mathcal{L}_{n,\theta}(\Psi) \subset \mathcal{L}_1(n,\theta) \cup \mathcal{L}_2(n,\theta,\Psi).$$

It will be shown that each of the sets $\mathcal{L}_1(n,\theta)$ and $\mathcal{L}_2(n,\theta,\Psi)$ has Lebesgue measure zero. Thus, to prove the theorem two different cases concerning the size of $|F'(x)+\theta'(x)|$ are considered. If $x \in \mathcal{L}_{n,\theta}(\Psi)$ then x must satisfy at least one of these cases infinitely often. To prove that each set of x satisfying one of the conditions infinitely often has measure zero, repeated use will be made of the Borel–Cantelli Lemma below.

Lemma 4 (Borel–Cantelli)**.** *Let A_j be a family of Lebesgue measurable sets and let A_∞ be the set of points $x \in \mathbb{R}$ which lie in infinitely many A_j. If $\sum\limits_{j=1}^{\infty} \mu(A_j) < \infty$ then $\mu(A_\infty) = 0$.*

1.1. The case of small derivative

Proposition 1. *Let $n \geq 2$. Then, $\mu(\mathcal{L}_1(n,\theta)) = 0$.*

Доказательство. First $\mathcal{L}_1(n,\theta)$ is written as a lim sup set. For $F \in \mathcal{F}_n$ define

$$B(F) = \{x \in I : \, |F(x)+\theta(x)| < H(F)^{-n+1}, \; |F'(x)+\theta'(x)| < H(F)^{-v}\}.$$

Then

$$\mathcal{L}_1(n,\theta) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} \bigcup_{F \in \mathcal{F}_n^t} B(F),$$

where

$$\mathcal{F}_n^t := \{F \in \mathcal{F}_n, \; 2^t \leq H(F) < 2^{t+1}\}.$$

To prove the proposition it will be shown that a larger set (containing $\mathcal{L}_1(n,\theta)$) has measure zero and then the Inhomogeneous Transference Principle proved in [6] will be used. The Inhomogeneous Transference Principle allows the transfer of zero measure statements for homogeneous lim sup sets to inhomogeneous lim sup sets and is described below.

Inhomogeneous Transference Principle. Most of this section is adapted from [6, Case B]. For our purposes the two countable indexing sets $\mathbf{T}$ and $\mathcal{A}$ from [6] are the sets $\mathbf{T} = \mathbb{N} \cup \{0\}$ and $\mathcal{A} = \mathcal{F}_n$. Throughout, J denotes a finite open interval in $\mathbb{R}$ with closure denoted by $\bar{J}$. Let $\mathcal{H}$ and $\mathcal{I}$ be two maps from $(\mathbb{N} \cup \{0\}) \times \mathcal{F}_n \times \mathbb{R}^+$ into the set of open subsets of $\mathbb{R}$ such that

$$\mathcal{H}(t,F,\epsilon) = \mathcal{I}_0^t(F,\epsilon), \quad \mathcal{I}(t,F,\epsilon) = \mathcal{I}_\theta^t(F,\epsilon).$$

For the specific case considered in this article the sets $\mathcal{I}_0^t(F,\epsilon)$ and $\mathcal{I}_\theta^t(F,\epsilon)$ are defined as follows:

$$\mathcal{I}_\theta^t(F,\epsilon) = \begin{cases} \{x \in I : \; |F(x)+\theta(x)| < 2^{t(-n+1)}\epsilon, \;\; |F'(x)+\theta'(x)| < 2^{-tv}\epsilon\} & \text{if} \quad F \in \mathcal{F}_n^t, \\ \emptyset & \text{else;} \end{cases}$$

and

$$\mathcal{I}_0^t(F,\epsilon)=\begin{cases}\{x\in I:\ |F(x)|<2^{t(-n+1)}\epsilon,\ \ |F'(x)|<2^{-tv}\epsilon\} & \text{if}\quad F\in\bigcup_{s=0}^{t+1}\mathcal{F}_n^s\\ \emptyset & \text{else.}\end{cases} \tag{10}$$

Let $\delta\in\mathbb{R}$ and define the function $\phi_\delta(t)=2^{\delta t}$. Also, define $\Phi=\{\phi_\delta:\ 0\le\delta<v/2\}$. For any $\phi\in\Phi$ define

$$\mathcal{I}_\theta^t(\phi)=\cup_{F\in\mathcal{F}_n}\mathcal{I}_\theta^t(F,\phi(t))=\cup_{F\in\mathcal{F}_n^t}\mathcal{I}_\theta^t(F,\phi(t))$$

and denote by $\Lambda_{\mathcal{I}}(\phi)$ the limsup set

$$\Lambda_{\mathcal{I}}(\phi)=\bigcap_{N=1}^{\infty}\bigcup_{t=N}^{\infty}\mathcal{I}_\theta^t(\phi).$$

In order to use the inhomogeneous transference principle from [6] we also define the homogeneous limsup set

$$\Lambda_{\mathcal{H}}(\phi)=\bigcap_{N=1}^{\infty}\bigcup_{t=N}^{\infty}\mathcal{I}_0^t(\phi).$$

where

$$\mathcal{I}_0^t(\phi)=\cup_{F\in\mathcal{F}_n}\mathcal{I}_0^t(F,\phi(t))=\cup_{s=0}^{t+1}\cup_{F\in\mathcal{F}_n^s}\mathcal{I}_0^t(F,\phi(t)).$$

Clearly, for any $0\le\delta<v/2$ the inclusion

$$\mathcal{L}_1(n,\theta)\subset\Lambda_{\mathcal{I}}(\phi_\delta)$$

holds. The use of the transference principle depends on the two following properties being satisfied.

Intersection Property: Let Φ denote a set of functions $\phi:\mathbb{N}\cup\{0\}\to\mathbb{R}^+$. The triple $(\mathcal{H},\mathcal{I},\Phi)$ is said to satisfy the intersection property if for any $\phi\in\Phi$ there exists $\phi^*\in\Phi$ such that for all but finitely many $t\in\mathbb{N}\cup\{0\}$ and all distinct $F,\tilde{F}\in\mathcal{F}_n$

$$\mathcal{I}_\theta^t(F,\phi(t))\cap\mathcal{I}_\theta^t(\tilde{F},\phi(t))\subset\mathcal{I}_0^t(\phi^*). \tag{11}$$

Contracting Property: Let $\{k_t\}_{t\in\mathbb{N}}$ be a sequence of positive numbers such that

$$\sum_{t\in\mathbb{N}\cup\{0\}}k_t<\infty. \tag{12}$$

The measure μ is said to be *contracting with respect to* $(\mathcal{I},\Phi)$ if for any $\phi\in\Phi$ there exists $\phi^+\in\Phi$ such that for all but finitely many t and all $F\in\mathcal{F}_n$ there exists a collection $C_{t,F}$ of balls B centred in $\bar{J}$ satisfying the following three conditions:

$$\bar{J}\cap\mathcal{I}_\theta^t(F,\phi(t))\subset\bigcup_{B\in C_{t,F}}B, \tag{13}$$

$$\bar{J}\bigcap\bigcup_{B\in C_{t,F}}B\subset\mathcal{I}_\theta^t(F,\phi^+(t)), \tag{14}$$

$$\mu(5B\cap\mathcal{I}_\theta^t(F,\phi(t)))\le k_t\mu(5B). \tag{15}$$

We now state the theorem from [6].

Theorem 2 (Inhomogeneous Transference Principle)**.** *Suppose that $(\mathcal{H},\mathcal{I},\Phi)$ satisfies the intersection property and that μ is contracting with respect to $(\mathcal{I},\Phi)$. If, for all $\phi\in\Phi$, $\mu(\Lambda_{\mathcal{H}}(\phi))=0$ then for all $\phi\in\Phi$, $\mu(\Lambda_{\mathcal{I}}(\phi))=0$.*

First the contracting and intersection properties are verified and then it will be shown that $\mu(\Lambda_{\mathcal{H}}(\phi_\delta))=0$. This will imply using the transference principle that $\Lambda_{\mathcal{I}}(\phi_\delta)$ has measure zero and further that $\mu(\mathcal{L}_1(n,d))=0$ as required.

1.1.1. Verifying the intersection property

Let $t\in\mathbb{N}\cup\{0\}$ and $F,\tilde{F}\in\mathcal{F}_n$ with $F\neq\tilde{F}$. Suppose that

$$x\in\mathcal{I}_\theta^t(F,\phi_\delta(t))\cap\mathcal{I}_\theta^t(\tilde{F},\phi_\delta(t)).$$

Then, the inequalities

$$|F(x)+\theta(x)|<\phi_\delta(t)2^{t(-n+1)}\quad\text{and}\quad|\tilde{F}(x)+\theta(x)|<\phi_\delta(t)2^{t(-n+1)},$$

$$|F'(x)+\theta'(x)|<\phi_\delta(t)2^{-vt}\quad\text{and}\quad|\tilde{F}'(x)+\theta'(x)|<\phi_\delta(t)2^{-vt}$$

hold.

Let $R(x)=(F(x)+\theta(x))-(\tilde{F}(x)+\theta(x))$. Then,

$$\begin{aligned}|R(x)|&<2\phi_\delta(t)2^{t(-n+1)}<\phi_{\delta'}(t)2^{t(-n+1)},\\|R'(x)|&<2^{1-vt}\phi_\delta(t)<2^{-vt}\phi_{\delta'}(t),\end{aligned}$$

for all $t>\frac{1}{v/2-\delta}$ and where $\phi_{\delta'}\in\Phi$. Clearly R cannot be constant for $n\geq2$ and $t\geq2$, so $R\in\cup_{s=0}^{t+1}\mathcal{F}_n^s$. Thus, $x\in\mathcal{I}_0^t(R,\phi_{\delta'}(t))$ and (11) is satisfied with $\phi^*=\phi_{\delta'}$.

1.1.2. Verifying the contracting property

The following definition from [10] will be used.

Definition 1. *Let C and α be positive numbers and $f:I\to\mathbb{R}$ be a function defined on the open interval $I\subset\mathbb{R}$. Then f is called (C,α)–good on I if, for any open interval $B\subset I$ and any $\epsilon>0$,*

$$\mu(\{x\in B:\,|f(x)|<\epsilon\sup_{x\in B}|f(x)|\})\leq C\epsilon^\alpha\mu(B).$$

Several useful facts about (C,α)-*good* functions are listed below.

Lemma 5. *[8, Lemma 3.1] Let $I\subset\mathbb{R}$ and $C,\alpha>0$ be given.*
(i) If f is (C,α)-good on I then so is λf for any $\lambda\in\mathbb{R}$.
(ii) If f_i, $i\in I_0$, are (C,α)-good on I then so is $\sup_{i\in I_0}|f_i|$.
(iii) If f is (C,α)-good on I and $c_1\leq\frac{|f(x)|}{|g(x)|}\leq c_2$ for all $x\in I$, then g is $(C(c_2/c_1)^\alpha,\alpha)$-good on I.
(iv) If f is (C,α)-good on I then f is (C',α')-good on I' for every $C'\geq C$, $\alpha'\leq\alpha$ and $I'\subset I$.

Lemma 6. *[2, Corollary 3] Let U be an open subset of $\mathbb{R}^m$, $\mathbf{x}_0 \in U$ and let $\mathbf{f} = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ be n-nondegenerate at $\mathbf{x}_0$ for some $n \geq 2$. Let $\theta \in C^{(n)}(U)$. Then there exists a neighborhood $V \subset U$ of $\mathbf{x}_0$ and a positive constants C and H_0 such that for any $\mathbf{a} \in \mathbb{R}^n$ satisfying $|\mathbf{a}| \geq H_0$*

(a) $a_0 + \mathbf{a} \cdot \mathbf{f} + \theta$ is $(C, \frac{1}{nm})$-good on V for every $a_0 \in \mathbb{R}$, and

(b) $|\nabla(\mathbf{a} \cdot \mathbf{f} + \theta)|$ is $(C, \frac{1}{m(n-1)})$-good on V.

Here ∇ denotes the gradient operator. Note that in the case $m = 1$ the map $\mathbf{f}$ is nondegenerate iff $wr(f_1', \ldots, f_n')(x) \neq 0$ almost everywhere.

Lemma 7. *[2, Corollary 4] Let $U, \mathbf{x}_0, \mathbf{f}$ and θ be as in Lemma 6. Then for every sufficiently small neighborhood $V \subset U$ of $\mathbf{x}_0$, there exists $H_0 > 1$ such that*

$$\inf_{\substack{(\mathbf{a},a_0)\in\mathbb{R}^{n+1} \\ |\mathbf{a}|\geq H_0}} \sup_{\mathbf{x}\in V} |a_0 + \mathbf{a} \cdot \mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| > 0.$$

Let J be a sufficiently small open interval such that $5J \subset I$. By Lemma 5 and Lemma 6, there exist positive numbers C and H_0 such that for any $\mathbf{t} \in \mathbf{T}$ and $F \in \mathcal{A}$ satisfying $H(F) \geq H_0$ both $F + \theta$ and $F' + \theta'$ are $(C, \frac{1}{n})$-good on $5J$. Similarly, for any $t \in \mathbb{N} \cup \{0\}$ and $F \in \mathcal{F}_n^t$, using the properies of $(C, \frac{1}{n})$-*good* functions from Lemma 5, we have that the function $\mathbf{F}_{t,F} : I \to \mathbb{R}$ given by

$$\mathbf{F}_{t,F}(x) := \max\left\{2^{t(n-1)}2^{-vt}|F(x) + \theta(x)|, |F'(x) + \theta'(x)|\right\}$$

is also $(C, \frac{1}{n})$–good on $5J$. By definition, for $F \in \mathcal{F}_n^t$,

$$\mathcal{I}_\theta^t(F, \phi_\delta(t)) = \begin{cases} \{x \in I : \mathbf{F}_{t,F}(x) < \phi_\delta(t)2^{-vt}\} & \text{if } F \in \mathcal{F}_n^t \\ \emptyset & \text{else} \end{cases}. \tag{16}$$

Next, given $\phi_\delta \in \Phi$ let

$$\phi_\delta^+ := \phi_{\frac{1}{2}(\delta + \frac{v}{2})}.$$

Clearly, $\phi_\delta^+ \in \Phi$ and $\phi_\delta(t) \leq \phi_\delta^+(t)$ for all $t \in \mathbb{N} \cup \{0\}$; therefore,

$$\mathcal{I}_\theta^t(F, \phi_\delta(t)) \subset \mathcal{I}_\theta^t(F, \phi_\delta^+(t)). \tag{17}$$

The collection $C_{t,F}$ will consist of intervals $B(x)$, each centred at a point $x \in J$, which satisfy conditions (12)–(15) for an appropriate sequence k_t; they are constructed in the following way. Let $F \in \mathcal{F}_n$. If $\mathcal{I}_\theta^t(F, \phi_\delta(t)) = \emptyset$ then $C_{t,F} = \emptyset$. Now assume that $\mathcal{I}_\theta^t(F, \phi_\delta(t)) \neq \emptyset$. By the definition of Φ and (9), it follows that

$$\mathcal{I}_\theta^t(F, \phi_\delta^+(t)) \subset \{x \in I : |F(x) + \theta(x)| < 2^{-t(n-\frac{7}{6})}\}.$$

Since $F + \theta$ is $(C, \frac{1}{n})$-good on $5J$ for all sufficiently large $H(F)$ then by definition of (C, α)-good function and Lemma 7 we have

$$\mu(\mathcal{I}_\theta^t(F, \phi_\delta^+(t)) \cap J) \leq \mu(\{x \in J : |F(x) + \theta(x)| < 2^{-t(n-\frac{7}{6})}\}) \ll 2^{-t(1-\frac{7}{6n})}\mu(J)$$

for sufficiently large t. Hence,

$$J \not\subset \mathcal{I}_\theta^t(F, \phi_\delta^+(t)) \tag{18}$$

for sufficiently large t and $n \geq 2$.

By (17) and the fact that $\mathcal{I}_\theta^t(F, \phi_\delta^+(t))$ is open, for every $x \in \bar{J} \cap \mathcal{I}_\theta^t(F, \phi_\delta(t))$ there is an open interval $B'(x)$ containing x such that

$$B'(x) \subset \mathcal{I}_\theta^t(F, \phi_\delta^+(t)).$$

Hence, by (18), and the fact that J is bounded, there exists a scaling factor $\tau \geq 1$ such that the open interval $B(x) := \tau B'(x)$ satisfies

$$\begin{array}{rcl} \bar{J} \cap B(x) & \subset & \mathcal{I}_\theta^t(F, \phi_\delta^+(t)), \\ \bar{J} \cap 5B(x) & \not\subset & \mathcal{I}_\theta^t(F, \phi_\delta^+(t)), \\ 5B(x) & \subset & 5J. \end{array} \tag{19}$$

Let

$$C_{t,F} := \{B(x) : x \in \bar{J} \cap \mathcal{I}_\theta^t(F, \phi_\delta(t))\}.$$

By (19) and the construction, (13) and (14) are automatically satisfied. Consider any interval $B \in C_{t,F}$. By (16) and (19)

$$\sup_{x \in 5B} \mathbf{F}_{t,F}(x) \geq \sup_{x \in \bar{J} \cap 5B} \mathbf{F}_{t,F}(x) \geq \phi_\delta^+(t) 2^{-vt}. \tag{20}$$

On the other hand, by (16)

$$\sup_{x \in \mathcal{I}_\theta^t(F, \phi_\delta(t)) \cap 5B} \mathbf{F}_{t,F}(x) \leq \phi_\delta(t) 2^{-vt}. \tag{21}$$

Let $\delta^* = \frac{1}{4}(v - 2\delta) > 0$. Then, using (20), (21) and the definitions of ϕ_δ and ϕ_δ^+, we obtain

$$\sup_{x \in \mathcal{I}_\theta^t(F, \phi_\delta(t)) \cap 5B} \mathbf{F}_{t,F}(x) \leq 2^{-\delta^* t} \sup_{x \in 5B} \mathbf{F}_{t,F}(x).$$

Since $F \in \mathcal{F}_n^t$ then we have that $H(F) > H_0$ for all $\mathbf{t} \in \mathbf{T}$ with $\mathbf{t}$ sufficiently large. Since $\mathbf{F}_{t,F}$ is a $(C, \frac{1}{n})$-good on $5J$ for sufficiently large $\mathbf{t}$ it follows from (19) and (21), that

$$\begin{array}{rcl} \mu(\mathcal{I}_\theta^t(F, \phi_\delta(t)) \cap 5B) & \leq & \mu(\{x \in 5B : \mathbf{F}_{t,F}(x) \leq 2^{-\delta^* t} \sup\limits_{x \in 5B} \mathbf{F}_{t,F}(x)\}) \\ & \leq & 2^{-\frac{\delta^* t}{n}} C \mu(5B) \end{array} \tag{22}$$

for sufficiently large t. This verifies (15) with $k_t := 2^{-\frac{\delta^* t}{n}} C$ and it is easily seen that the convergence condition (12) is fulfilled.

1.1.3. Establishing $\mu(\Lambda_{\mathcal{H}}(\phi_\delta)) = 0$ for $\delta \in [0, v/2)$

For this Theorem 1.4 of [8] is used. In the notation of that paper take $d = 1$, $U = \mathbb{R}$ and $T_1 = \ldots = T_n = T$, to obtain the next result.

Theorem 3. *[8] Let $x_0 \in I$ and $\mathbf{f} : I \to \mathbb{R}^n$ be n-nondegenerate at x_0. There exists an interval $J \subset I$ containing x_0 such that for any interval $B \subset J$ there exists a constant $E > 0$ such that for any choice of real numbers ω, K, T satisfying the inequalities*

$$0 < \omega \leq 1, \; T \geq 1, \; K > 0, \; \omega K T^{n-1} \leq 1$$

the set

$$S(\omega, K, T) := \left\{ x \in B : \text{ there exists } F \in \mathcal{F}_n \text{ such that } \begin{array}{l} |F(x)| < \omega, \\ |F'(x)| < K, \\ 0 < H(F) < T \end{array} \right\}$$

has measure at most $E\epsilon^{\frac{1}{2n-1}}\mu(B)$, where

$$\epsilon := \max\left(\omega, \left(\omega K T^{n-1}\right)^{\frac{1}{n+1}}\right).$$

Fix $\delta \in [0, v/2)$. It then follows from (10) that

$$\mathcal{I}_0^t(\phi_\delta) = \cup_{s=0}^{t+1} \cup_{F \in \mathcal{F}_n^s} \mathcal{I}_0^t(P, \phi_\delta(t)) = S(\omega, K, T)$$

with $\omega = \phi_\delta(t)2^{t(-n+1)}$, $K = \phi_\delta(t)2^{-vt}$ and $T = 2^{t+2}$. By (9), we have $\epsilon \ll 2^{-\frac{4\delta^* t}{n+1}}$. Thus, Theorem 3 implies that

$$\mu(\mathcal{I}_0^t(\phi_\delta)) \ll 2^{-\beta t},$$

where $\beta := \frac{4\delta^*}{(n+1)(2n-1)}$ is a positive constant. This finally gives that

$$\sum_{t \in \mathbb{N}} \mu(\mathcal{I}_0^t(\phi_\delta)) \ll \sum_{t=0}^{\infty} 2^{-\beta t} < \infty.$$

Therefore, by the Borel-Cantelli lemma $\mu(\Lambda_{\mathcal{H}}(\phi_\delta)) = 0$ for all $\delta \in [0, \frac{v}{2})$. By the inhomogeneous transference principle this further implies that $\mu(\Lambda_{\mathcal{I}}(\phi_\delta)) = 0$ as required. The proposition has now been proved. □

1.2. The case of big derivative

Proposition 2. *Let $n \geq 2$. Then, $\mu(\mathcal{L}_2(n, \theta, \Psi)) = 0$.*

Доказательство. Let $\mathcal{F}_n(H) = \{F \in \mathcal{F}_n : H(F) = H\}$, then $\mathcal{F}_n = \cup_{H=1}^{\infty} \mathcal{F}_n(H)$. Now consider $F \in \mathcal{F}_n(H)$ satisfying $H^{-v} \leq |F'(x) + \theta'(x)|$. For the remaining case we need the following. The set of solutions of (1) in I consists of at most n intervals. Each of these intervals can be further divided into subintervals on which $F' + \theta'$ is also monotonic (at most $n-1$ of them). Each of these new intervals is finally further subdivided into intervals with respect to the value of $F'(x) + \theta'(x)$. Any interval on which $|F'(x) + \theta'(x)| < H^{-v}$ has already been considered. For $F \in \mathcal{F}_n(H)$, let $I_j(F, \theta)$ be one of the remaining intervals; thus, on $I_j(F, \theta)$, $F + \theta$ and $F' + \theta'$ are monotonic and $|F(x) + \theta(x)| < \Psi(H(F))$, $H^{-v} \leq |F'(x) + \theta'(x)|$ for all $x \in I_j(F, \theta)$. The number of $I_j(F,\theta)$ is clearly finite. Let $\bar{I}_j(F,\theta)$ denote the closure of $I_j(F,\theta)$ and $\alpha_{j,F}$ denote a point in $\bar{I}_j(F,\theta)$ such that

$$|F'(\alpha_{j,F})+\theta'(\alpha_{j,F})| = \min_{x\in\bar{I}_j(F)} |F'(x)+\theta'(x)|.$$

For convenience we will use F_θ to denote the function $F(x)+\theta(x)$.

Lemma 8. *[11] Let $a_1, a_2 > 0$. Let ψ be an n-times continuously differentiable function on (b_1,b_2) satisfying $|\psi^{(n)}(x)| \geq a_1$ for all $x \in (b_1,b_2)$. Then*

$$\mu(\{x\in(b_1,b_2) : \psi(x) < a_2\}) \leq c(n)(a_2/a_1)^{1/n}.$$

From Lemma 8 we have

$$\mu(I_j(F,\theta)) \leq c(n)\Psi(H)|F'_\theta(\alpha_{j,F})|^{-1}. \tag{23}$$

It follows from the choice of $\alpha_{j,F}$ that $H^{-v} \leq |F'_\theta(\alpha_{j,F})|$.

Now we are ready to complete the proof of Theorem 1. The three remaining cases in the proof concern different ranges for the size of $F'_\theta(\alpha_{j,F})$.

Case I. For $F\in\mathcal{F}_n(H)$, let $\sigma(F_\theta)$ be the union of intervals $I_j(F,\theta)$ for which $|F'_\theta(\alpha_j)| \geq c_1H^{1/2}$. Hence, $\sigma(F_\theta)$ is the set of $x\in I$ which satisfy $|F_\theta(x)| < \Psi(H)$ and x lies in some interval $I_j(F,\theta)$ for which

$$|F'_\theta(\alpha_{j,F})| \geq c_1H^{1/2}. \tag{24}$$

For every $F\in\mathcal{F}_n(H)$ and every j, where $\alpha_{j,F}\in\sigma(F_\theta)$, and some constant $c_2 = c_2(n)$ define the set $\sigma_{1,j}(F_\theta)$ of points $x\in I$ which satisfy

$$|x-\alpha_{j,F}| < c_2|F'_\theta(\alpha_{j,F})|^{-1}$$

for $\alpha_{j,F}\in\sigma(F_\theta)$. Let $\sigma_1(F_\theta) = \cup_j\sigma_{1,j}(F_\theta)$. From (23), for $H > H_0(c_2)$, the inequality $\sigma(F_\theta)\subset\sigma_1(F_\theta)$ holds and

$$\mu(\sigma(F_\theta)) \leq c(n)c_2^{-1}\Psi(H)\mu(\sigma_1(F_\theta)). \tag{25}$$

For each j with $\alpha_{j,F}\in\sigma(F_\theta)$ develop F as a Taylor series on $\sigma_{1,j}(F_\theta)$ so that

$$F_\theta(x) = F_\theta(\alpha_{j,F}) + F'_\theta(\alpha_{j,F})(x-\alpha_{j,F}) + F''_\theta(\xi_1)(x-\alpha_{j,F})^2/2,$$

where ξ_1 is between x and $\alpha_{j,F}$. Estimate each term in the above equation to obtain

$$\begin{aligned}
|F_\theta(\alpha_{j,F})| &< \Psi(H) < c_2,\\
|F'_\theta(\alpha_{j,F})(x-\alpha_{j,F})| &< c_2,\\
|F''_\theta(\alpha_{j,F})(x-\alpha_{j,F})^2| &< 2nK_0H(c_2|F'_\theta(\alpha_{j,F})|^{-1})^2 = 2nK_0c_2^2c_1^{-2}.
\end{aligned}$$

It is possible to choose $c_2 = c_2(\gamma) < \min\{1/6, \gamma/10\}$ such that $2nK_0c_2c_1^{-2} < 1$. Thus, $|F_\theta(x)| < 3c_2$ for $H > H_0(c_2)$.

Fix the vector $\mathbf{b}_0 = (a_n, a_{n-1}, \ldots, a_{k+1}, H, a_{k-1}, \ldots, a_{l+1}, a_{l-1}, \ldots, a_1, a_0)$, where $a_k = H$, $k\neq l$, and let the subclass of $\mathcal{F}_n(H)$ of functions with the same vector $\mathbf{b}_0$ be denoted by $\mathcal{F}_{n,\mathbf{b}_0}(H)$. The number of different $\mathcal{F}_{n,\mathbf{b}_0}(H)$ is $\ll H^{n-1}$. Let $F,\tilde{F}\in\mathcal{F}_{n,\mathbf{b}_0}(H)$, and assume that they have different coefficients a_l. Also, assume that $\sigma_{1,j}(F_\theta)\bigcap\sigma_{1,i}(\tilde{F}_\theta)\neq\emptyset$, for $F,\tilde{F}\in\mathcal{F}_{n,\mathbf{b}_0}(H)$. Let $l\geq 1$ and

$$R(x)=\tilde{F}_\theta(x)-F_\theta(x)=a_l(\tilde{F})f_l(x)-a_l(F)f_l(x)=a_l'f_l(x),$$

where $|a_l'f_l(x)|>\gamma$. Here, $a_l(F)$ denotes the lth coordinate of F. Then,

$$\gamma<|R(x)|\leq 6c_2<3\gamma/5$$

which is a contradiction for $c_2<\gamma/10$. Similar argument in the case $l=0$ gives a contradiction for $c_2<1/6$. Hence, $\sigma_{1,j}(F_\theta)\bigcap\sigma_{1,i}(\tilde{F}_\theta)=\emptyset$ and $\sum\limits_{F\in\mathcal{F}_{n,\mathbf{b}_0}(H)}\mu(\sigma_1(F_\theta))\ll|I|$. Together with (25) this gives

$$\sum_{F\in\mathcal{F}_{n,\mathbf{b}_0}(H)}\mu(\sigma(F_\theta))\ll|I|\Psi(H).$$

Summing this over all vectors $\mathbf{b}_0$ gives

$$\sum_{H=1}^{\infty}\sum_{\mathbf{b}_0}\sum_{F\in\mathcal{F}_{n,\mathbf{b}_0}(H)}\mu(\sigma(F_\theta))\ll\sum_{H=1}^{\infty}H^{n-1}\Psi(H)|I|<\infty.$$

The Borel–Cantelli lemma can now be used to complete the proof.

Case II. This time, for $F\in\mathcal{F}_n(H)$ use $\sigma(F_\theta)$ to denote the union of intervals $I_j(F,\theta)$ for which $1\leq|F'(\alpha_{j,F})|<c_1H^{1/2}$. Hence $\sigma(F_\theta)$ is the set of $x\in I$ which satisfy

$$|F_\theta(x)|<\Psi(H),$$

and x lies in some $I_j(F,\theta)$ for which

$$1\leq|F_\theta'(\alpha_{j,F})|<c_1H^{1/2}. \tag{26}$$

Now define expansion of $I_j(F,\theta)$ as follows:

$$\sigma_{2,j}(F_\theta):=\{x\in I:\,dist(x,I_j(F,\theta))<c_3H^{-1}|F'(\alpha_{j,F})|^{-1}\},\quad c_3>c(n).$$

Let $\sigma_2(F_\theta)=\cup_j\sigma_{2,j}(F_\theta)$. It is readily verified that

$$\mu(\sigma(F_\theta))\leq c_3^{-1}c(n)H\Psi(H)\mu(\sigma_2(F_\theta)). \tag{27}$$

Fix the vector

$$\mathbf{b}_1=(a_n,a_{n-1},\ldots,a_{k+1},H,a_{k-1},\ldots,a_{l+1},a_{l-1},\ldots,a_{m+1},a_{m-1},\ldots,a_1,a_0),$$

where $a_k=H$ with $l,m\neq k$ and $l>m$. Denote the subclass of $\mathcal{F}_n(H)$ of functions with the same vector $\mathbf{b}_1$ by $\mathcal{F}_{n,\mathbf{b}_1}(H)$. The number of different sets $\mathcal{F}_{n,\mathbf{b}_1}(H)$ is $\ll H^{n-2}$.

The intervals $\sigma_{2,j}(F_\theta)$ will be divided into two classes of essential and inessential intervals according to Sprindzuk's method, see [12] for more details. The interval $\sigma_{2,j}(F_\theta)$ will be essential if

$$\mu(\sigma_{2,j}(F_\theta)\cap\sigma_2(\tilde{F}_\theta))\leq\mu(\sigma_{2,j}(F_\theta))/2$$

for all $\tilde{F} \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ other than $F \in \mathcal{F}_{n,\mathbf{b}_1}(H)$. Otherwise it is called inessential.

First, the essential intervals are investigated. Summing the measure of essential intervals gives

$$\sum_{F \in \mathcal{F}_{n,\mathbf{b}_1}(H)} \sum_{\substack{j \\ \sigma_{2,j}(F_\theta) \text{ essential}}} \mu(\sigma_{2,j}(F_\theta)) \ll |I|.$$

From this, (27) and the fact that the number of vectors $\mathbf{b}_1$ is $\ll H^{n-2}$, we have

$$\sum_{\mathbf{b}_1} \sum_{F \in \mathcal{F}_{n,\mathbf{b}_1}(H)} \mu(\sigma(F_\theta)) \ll H^{n-1}\Psi(H)|I|.$$

Finally, we obtain

$$\sum_{H=1}^{\infty} \sum_{\mathbf{b}_1} \sum_{F \in \mathcal{F}_{n,\mathbf{b}_1}(H)} \mu(\sigma(F_\theta)) < \infty.$$

Thus, by the Borel–Cantelli Lemma, the set of points x which belong to infinitely many essential domains is of measure zero.

Now we consider the inessential intervals. Then, by definition, there is a $\tilde{F} \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ different from $F \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ such that

$$\mu(\sigma_{2,j}(F_\theta) \cap \sigma_2(\tilde{F}_\theta)) > \mu(\sigma_{2,j}(F_\theta))/2.$$

Let $x \in \sigma_{2,j}(F_\theta) \cap \sigma_{2,i}(\tilde{F}_\theta)$. Develop every function $F \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ for every j, where $\alpha_{j,F} \in \sigma(F_\theta)$, as a Taylor series on the interval $\sigma_{2,j}(F_\theta)$ so that

$$F_\theta(x) = F_\theta(\alpha_{j,F}) + F'_\theta(\alpha_{j,F})(x - \alpha_{j,F}) + F''_\theta(\xi_2)(x - \alpha_{j,F})^2/2,$$

where ξ_2 is between x and $\alpha_{j,F}$. Using the fact that $|x - \alpha_{j,F}| \ll H^{-1}$, we get

$$|F_\theta(x)| \ll \Psi(H) + H^{-1} + H \cdot H^{-2} \ll H^{-1} \tag{28}$$

for any $x \in \sigma_{2,j}(F_\theta)$ and $n \geq 2$. Furthermore, from the Mean Value Theorem, for $x \in \sigma_{2,j}(F_\theta)$ with $\alpha_{j,F} \in \sigma(F_\theta)$,

$$\begin{aligned} |F'_\theta(x)| &\leq |F'_\theta(\alpha_{j,F})| + |F''_\theta(\xi_3)(x - \alpha_{j,F})| \\ &\ll H^{1/2} + HH^{-1}|F'(\alpha_{j,F})|^{-1} \ll H^{1/2}. \end{aligned} \tag{29}$$

Consider the new function $R = \tilde{F}_\theta - F_\theta = a'_l f_l + a'_m f_m$, where both F and $\tilde{F}$ belong to $\mathcal{F}_{n,\mathbf{b}_1}(H)$. For these functions, conditions (28) and (29) hold on the set $\sigma_{2,j}(F_\theta) \bigcap \sigma_{2,i}(\tilde{F}_\theta)$. By (5), (28) and (29), we obtain

$$|R(x)| \ll H^{-1}, \quad |R'(x)| \ll H^{1/2}.$$

From (6) it is relatively straightforward to show that $|a'_i| \ll H^{1/2}$ for $i = l, m$ so that $H(R) \ll H^{1/2}$. Therefore, $|a'_l f_l(x) + a'_m f_m(x)| \ll H(R)^{-2}$. Divide by $f_l(x)$ and put $t = t_{ml} = f_m f_l^{-1}$. Then, $|a'_m t(x) + a'_l| \ll H(R)^{-2}$ which, by Khintchine's Theorem, holds infinitely often only on a set of measure zero. Finally, by Lemma 2, the set of points $x \in I$ which satisfy $|R(x)| \ll H^{-1}$ for infinitely many (a'_l, a'_m) also has zero measure.

Case III. This is very similar to the previous case. For $F \in \mathcal{F}_n(H)$ use $\sigma(F_\theta)$ to denote the union of intervals $I_j(F,\theta)$ for which $H^{-v} \leq |F'_\theta(\alpha_{j,F})| < 1$ with $0 < v < 1/4$. Hence $\sigma(F_\theta)$ is the set of $x \in I$ which satisfy

$$|F_\theta(x)| < \Psi(H),$$

and x lies in some $I_j(F,\theta)$ for which

$$H^{-v} \leq |F'_\theta(\alpha_{j,F})| < 1.$$

Fix the vector $\mathbf{b}_1$ as above and define the following expansions of $I_j(F,\theta)$:

$$\sigma_{3,j}(F_\theta) := \{x \in I : \, dist(x, I_j(F,\theta)) < c_4 H^{-1} |F'(\alpha_{j,F})|^{-1}\}, \quad c_4 > c(n),$$

$$\sigma'_{3,j}(F_\theta) := \{x \in I : \, dist(x, I_j(F,\theta)) < H^{-1+4v/3}\}.$$

From this,

$$\mu(\sigma(F)) \leq c_4^{-1} c(n) \mu(\sigma_3(F)) H \Psi(H), \tag{30}$$

where $\sigma_3(F_\theta) = \cup_j \sigma_{3,j}(F_\theta)$. It is clear that $\sigma_{3,j}(F_\theta) \subset \sigma'_{3,j}(F_\theta)$. Moreover, it is easy to see that

$$\sigma_{3,j}(F_\theta) \subset \sigma'_{3,i}(\tilde{F}_\theta) \tag{31}$$

for any $\tilde{F} \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ with $\sigma_{3,i}(\tilde{F}_\theta) \cap \sigma_{3,j}(F_\theta) \neq \emptyset$.

Redefine essential and inessential intervals $\sigma_{3,j}(F_\theta)$. The interval $\sigma_{3,j}(F_\theta)$ will be essential if for any $\tilde{F} \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ other than $F \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ we have $\sigma_{3,j}(F_\theta) \cap \sigma_3(\tilde{F}_\theta) = \emptyset$.

Summing the measures of the essential intervals $\sigma_3(F_\theta)$ gives

$$\sum_{F \in \mathcal{F}_{n,\mathbf{b}_1}(H)} \sum_{\substack{j \\ \sigma_{3,j}(F_\theta) \text{ essential}}} \mu(\sigma_{3,j}(F_\theta)) \ll |I|. \tag{32}$$

As $\#\mathbf{b}_1 \ll H^{n-2}$, from (30) and (32), we have

$$\sum_{H=1}^{\infty} \sum_{\mathbf{b}_1} \sum_{F \in \mathcal{F}_{n,\mathbf{b}_1}(H)} \mu(\sigma(F_\theta)) \ll \sum_{H=1}^{\infty} H^{n-1} \Psi(H) |I| < \infty.$$

By the Borel–Cantelli Lemma, the set of those x belonging to infinitely many essential intervals has zero measure.

Now let $\sigma_{3,j}(F_\theta)$ be an inessential interval. Then, by definition and (31), there is a $\tilde{F} \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ different from $F \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ such that

$$I_j(F,\theta) \subset \sigma_{3,j}(F_\theta) \subset (\sigma'_{3,j}(F_\theta) \cap \sigma'_{3,i}(\tilde{F}_\theta)).$$

Using Taylor's formula for F on $\sigma'_{3,j}(F_\theta)$, we obtain

$$|F_\theta(x)| \ll H^{-1+8v/3}. \tag{33}$$

By the Mean Value Theorem, for any $x \in \sigma'_{3,j}(F_\theta)$

$$|F'_\theta(x)| \leq |F'_\theta(\alpha_{j,F})| + |F''_\theta(\xi_5)(x - \alpha_{j,F})| \ll 1 + HH^{-1+4v/3} \ll H^{4v/3}. \tag{34}$$

Consider $R(x) = \tilde{F}_\theta(x) - F_\theta(x)$ with $F, \tilde{F} \in \mathcal{F}_{n,\mathbf{b}_1}(H)$, and $x \in \sigma'_{3,j}(F_\theta) \bigcap \sigma'_{3,i}(\tilde{F}_\theta)$. For R the inequalities $|R(x)| \ll H^{8v/3-1}$ and $|R'(x)| \ll H^{4v/3}$ hold; these follow from (33) and (34). As in Case II it is possible to show from (6) that $|a'_i| \ll H^{4v/3}$ $(i = l, m)$ so that $H(R) = \max\{|a'_l|, |a'_m|\} \ll H^{4v/3}$. Again, let $t = t_{ml} = f_m f_l^{-1}$. By (5) and (33),

$$|R(x)| = |a'_m t(x) + a'_l| \ll H^{8v/3-1} \ll H(R)^{(8v/3-1)/(4v/3)} \ll H(R)^{-1}$$

for $v < 1/4$. By Khintchine's theorem the last inequality holds infinitely often only for a set of measure zero. Hence, by Lemma 2, the measure of $\sigma'_3(F_\theta) \bigcap \sigma'_3(\tilde{F}_\theta)$ is zero and the measure of the set of x which belong to infinitely many inessential domains is also zero. The proof of the theorem is therefore complete. □

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АННОТАЦИЯ

В данной статье доказывается неоднородный аналог теоремы типа Грошева в случае сходимости для невырожденных кривых в евклидовом пространстве, когда функция аппроксимации является не обязательно монотонной. Наш результат естественно включает в себя и обобщает теорему для меры множества точек невырожденных кривых в однородном случае. В доказательстве используются неоднородный метод переноса и метод существенных и несущественных областей Спринджука.

Ключевые слова: *неоднородные диофантовы приближения, теорема Хинчина, невырожденная кривая.*