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© N. Budarina, V. Bernik, F. Götze[1]

Effective estimations of the measure of the sets of real numbers in which integer polynomials take small value

In this paper we obtain the effective estimates in the terms of n and Q for the measure of the sets of real numbers with the given approximation property by algebraic numbers of degree n and height bounded by $Q \in \mathbb{N}$.

Key words: *integer polynomials, Lebesgue measure, approximation by algebraic numbers.*

1. Introduction

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an integer polynomial of degree $\deg P = n$ and height $H = H(P) = \max_{0\leqslant j\leqslant n} |a_j|$. Denote by $\mathcal{P}_{\le n}$ the class of integer polynomials P of degree at most n and $\mathcal{P}_n$ the class of integer polynomials P of degree n. Denote by $J \subset \mathbb{R}$ some interval. For convenience, let $J = [-1/2, 1/2]$. Using Minkowski theorem on linear forms or Dirichlet box principle it is easy to prove that for any $x \in J$ and $Q \in \mathbb{N}_{>1}$ there exists a polynomial $P \in \mathcal{P}_{\le n}$, $H(P) \le Q$ satisfying

$$|P(x)| < 2^{-2}Q^{-n}. \tag{1}$$

Then it is not difficult to show that for all $x \in J$ the inequality

$$|P(x)| < 2^{-2}H(P)^{-n} \tag{2}$$

has infinitely many solutions in polynomials $P \in \mathcal{P}_{\le n}$. The inequalities (1) and (2) can not be improved in principal. It is not difficult to show that for $\gamma = \sqrt[n+1]{2}$ the inequality $|P(\gamma)| > c(\gamma)H(P)^{-n}$ holds for any $P \in \mathcal{P}_{\le n}$. However, Khintchine [1] has shown that for any $\epsilon > 0$ and almost all $x \in \mathbb{R}$ (in the sence of Lebesgue measure) the inequality

$$|P(x)| < \epsilon H(P)^{-n}$$

[1]Institute for Applied Mathematics, Khabarovsk Division, Far-Eastern Branch of the Russian Academy of Sciences, 680000 Khabarovsk, Russia, Dzerzhinsky st., 54; Institute of Mathematics, Belarus Academy of Sciences, 220072 Minsk, Belarus, Surganova st., 11; Faculty of Mathematics, University of Bielefeld, P. O. Box 10 01 31, 33501 Bielefeld, Germany . E-mail: buda77@mail.ru, bernik@im.bas-net.by, goetze@math.uni-bielefeld.de

holds for infinitely many polynomials $P \in \mathcal{P}_{\le n}$.

Let $\Psi(x)$ be a monotonic decreasing function of $x \in \mathbb{R}_+$. Denote by $\mathcal{L}_n(\Psi)$ the set of real numbers $x \in J$ such that the inequality

$$|P(x)| < \Psi(H(P)) \tag{3}$$

has infinitely many solutions in polynomials $P \in \mathcal{P}_{\le n}$. In [2, 3] it was proved that

$$\mu(\mathcal{L}_n(\Psi)) = \begin{cases} 0 & \text{if} \quad \sum\limits_{H=1}^{\infty} H^{n-1}\Psi(H) < \infty, \\ \mu(J) & \text{if} \quad \sum\limits_{H=1}^{\infty} H^{n-1}\Psi(H) = \infty. \end{cases} \tag{4}$$

Note that for $n = 1$ the result (4) coincides with Khintchine theorem on approximation of real numbers by rational numbers [4]. If $\Psi(H) = H^{-n-\epsilon}$ then the series in (4) converges and it is known as a Mahler hypothesis [5] which was been proven by Sprindzuk [6]. Recently the result in (4) was generalized to the non-degenerate curves [7, 8] and to simultaneous approximation in the fields of real, complex and p-adic numbers [9, 10].

For the investigation of the distribution of real algebraic integers the following effective estimate of the measure was used (which is based on the results from [2]).

Proposition 1. *[11] Let $n \geq 2$ be an integer. Denote by $\mathcal{M}_n(Q)$ the set of $x \in J$ for which the inequalities*

$$|P(x)| \leq n^{-1}2^{-n-5}Q^{-n} \qquad \textit{and} \qquad H(P) \leq Q \tag{5}$$

have a solution in $P \in \mathcal{P}_{\le n}$. Then there exists a sufficiently large $Q_0 = Q_0(n, J)$ such that for all $Q > Q_0$ we have $\mu(\mathcal{M}_n(Q)) \leq 2^{-2}\mu(J)$.

In Proposition 1 the two variables n and Q have different nature: the variable $n = \deg P$ is fixed and the variable Q must be chosen to be sufficiently large (note that Q is not defined explicitly).

The effective versions of metric theorems have begun to develop in recent years. This led to estimates for the number of integer polynomials with given distance between the roots [12, 13, 14, 15], to estimates for discriminants [16] and resultants [17].

Let $\mu(A)$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and $|I|$ be the length of an interval $I \subset \mathbb{R}$. In what follows $c_0 = c_0(n), c_1, c_2, \ldots$ denote positive constants which depend only on n. Let $\#S$ will denote the number of elements in a finite set S. We will also use the Vinogradov symbol $f \ll g$ which means that there exists a constant $c > 0$ such that $f \leq cg$. The notation $f \asymp g$ means that $f \ll g \ll f$.

We introduce two classes of polynomials

$$\mathcal{P}_{\le n}(Q) = \{P_n \in \mathcal{P}_{\le n} : H(P) \leqslant Q\} \quad \text{and} \quad \mathcal{P}_n(Q) = \{P_n \in \mathcal{P}_n : H(P) \leqslant Q\}. \tag{6}$$

Let $I = [0, 1]$ and $\delta = \delta(n, Q)$. Denote by $M_n(Q, I, \delta)$ the set of $x \in I$ for which the inequality

$$|P(x)| < \delta \tag{7}$$

has a solution in polynomials $P \in \mathcal{P}_{\leq n}(Q)$. In this paper we are interested on the metric properties of the set $M_n(Q, I, \delta)$. We are going to prove a few theorems with estimations of the form

$$\mu(M_n(Q, I, \delta)) < s(n)Q^l \delta \mu(I), \tag{8}$$

where $s(n)$ is a function of n. Then we can choose $\delta = \delta_1(s(n))^{-1}Q^{-l}$ and conclude that the inequality (7) holds for $P \in \mathcal{P}_{\leq n}(Q)$ only on the set of $x \in I$ with the measure at most $\delta_1\mu(I)$, $0 < \delta_1 \leq 1$, for the rest of x from the set of the measure at least $(1 - \delta_1)\mu(I)$ the inequality opposite to (7) holds.

First, we are going to prove the theorem which is based on the known inequalities in the theory of transcendental numbers [18].

Theorem 1. *For any δ satisfying*

$$2^{-3n^2+2}Q^{-2n-3} \leq \delta \leq 2^{-6n^2}Q^{-2n-1} \tag{9}$$

we have

$$\mu(M_n(Q, I, \delta)) < 2^{6n^2}Q^{2n+1}\delta\mu(I).$$

Note that (9) holds for $Q \geq 2^{3n^2/2+1}$. The proof of Theorem 1 is based on Lemma 1. But if instead we will use Lemma 2 then we obtain that

$$\mu(M_n(Q, I, \delta)) < e^{10n \log_2 n}Q^{3n-1}\delta\mu(I)$$

for sufficiently large Q and sufficiently small δ. In the following theorem we are proving the best possible result for the measure of the set $M_n(Q, I, \delta)$ in the terms of the height of the polynomials. For this result we require that $Q \geq 2^{44n^4}$ and instead of 2^{6n^2} in Theorem 1 we will have $2^{3n^3+n^2}n^2$.

Let $\beta_n > 0$ be a constant depending on n and $\epsilon \in \mathbb{R}_+$. Denote by $L_n(Q, I)$ the set of $x \in I$ for which the inequality

$$|P(x)| < \beta_n Q^{-n} \tag{10}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$.

Theorem 2. *For any real number ϵ, satisfying $2^{-n^2} \leq \epsilon \leq 1$, and $\beta_n = 2^{-3n^3-n^2}n^{-2}\epsilon$, for $Q \geq 2^{44n^4}$ we have that*

$$\mu(L_n(Q, I)) < \epsilon\mu(I).$$

2. Auxiliary statements

By translation and taking the reciprocals (if necessary) each polynomial $P \in \mathcal{P}_n$ can be transformed into a polynomial R satisfying

$$|a_n(R)| \geq H(R)/n, \tag{11}$$

and $H(R) \asymp H(P)$. Consider the polynomials $P \in \mathcal{P}_n$ satisfying (11). Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the polynomial P in $\mathbb{C}$. Define the sets

$$S_P(\alpha_i) = \{x \in \mathbb{R} : \ |x - \alpha_i| = \min_{1 \leqslant m \leqslant n} |x - \alpha_m|\}, \quad 1 \leq i \leq n.$$

Further assume without loss of generality that $i = 1$. Reorder the other roots of P so that

$$|\alpha_1 - \alpha_2| \ \leq \ |\alpha_1 - \alpha_3| \ \leq \ldots \leq \ |\alpha_1 - \alpha_n|.$$

For the polynomial P define the real numbers ρ_j by

$$|\alpha_1 - \alpha_j| \ = \ H(P)^{-\rho_j}, \quad 2 \leq j \leq n, \quad \rho_2 \geq \rho_3 \geq \ldots \geq \rho_n.$$

Let $\epsilon > 0$ be sufficiently small, $d > 0$ be a large fixed number, $\epsilon_1 = \epsilon/d$ and $T = [\epsilon_1^{-1}]+1$. Also, define the integers l_j, $2 \leq j \leq n$, by the relations

$$\frac{l_j - 1}{T} \leq \rho_{1j} < \frac{l_j}{T}, \qquad l_2 \geq l_3 \geq \ldots \geq l_n \geq 0.$$

Finally, define the numbers q_i by $q_i = \frac{l_{i+1}+\ldots+l_n}{T}$, $(1 \leq i \leq n-1)$. All irreducible polynomials $P \in \mathcal{P}_n(Q)$ satisfying (11) and corresponding to the same vector $\mathbf{l} = (l_2, \ldots, l_n)$ are grouped together into a class $\mathcal{P}_n(Q, \mathbf{l})$, and the number of such classes is finite and depends only on n and ϵ_1, i.e. is at most $C(n, \epsilon_1)$, see [6]. Also, we define the class $\mathcal{P}_n(\mathbf{l})$ to consist of all irreducible polynomials $\mathcal{P}_n$ satisfying (11) and corresponding to a vector $\mathbf{l}$. In 4.2.4 we fix the vector $\mathbf{l}$ and will continue the proof for this fixed vector.

A number of lemmas for later use are now given.

Lemma 1. *[19] Suppose $P(z) \in \mathbb{Z}[z]$ is a polynomial without multiple roots of degree n and height H, and let $\alpha_1, \ldots, \alpha_n$ be its roots. Then for any number θ,*

$$\min_{i=1,\ldots,n} |\theta - \alpha_i| \cdot e^{-n^2} H^{-n} \leq |P(\theta)| \leq \min_{i=1,\ldots,n} |\theta - \alpha_i| \cdot n^2 H (H+2)^{n-1} \max\{1, |\theta|^n\}.$$

Denote by $L(P)$ the length of the polynomial $P(t) = \sum_{i=0}^n a_i t^i$, i.e. $L(P) = \sum_{i=0}^n |a_i|$. Obviously, $H(P) \leq L(P) \leq (n+1)H(P)$.

Lemma 2. *[18] Suppose $P(z) \in \mathbb{Z}[z]$ is an irreducible polynomial of degree n, height $H = H(P)$, length $L = L(P)$, and let $\alpha_1, \ldots, \alpha_n$ be its roots. Then for any θ,*

$$\min_{i=1,\ldots,n} |\theta - \alpha_i| \cdot n^{-3n} 2^{-3n} H^{-2n+2} \leq |P(\theta)|$$

or

$$\min_{i=1,\ldots,n} |\theta - \alpha_i| \cdot 2^{-n+1} n^{-n+1} L(P)^{-2n+2} \leq |P(\theta)|.$$

In Lemma 2 in contrast of Lemma 1, H^{-2n+2} occurs instead of H^{-n}. But instead of e^{-n^2} we have $n^{-3n} 2^{-3n} = e^{-3n \ln n - 3n \ln 2}$.

Lemma 3. *[6, 20] Let $x \in S_P(\alpha_1)$. Then*

$$|x - \alpha_1| \leqslant n \frac{|P(x)|}{|P'(x)|} \quad \textit{for} \ \ P'(x) \neq 0,$$

$$|x - \alpha_1| \leqslant 2^{n-1} |P(x)| |P'(\alpha_1)|^{-1} \quad \textit{for} \ \ P'(\alpha_1) \neq 0, \tag{12}$$

and

$$|x - \alpha_1| \leqslant \min_{2 \leqslant j \leqslant n} (2^{n-j} |P(x)| |P'(\alpha_1)|^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_k|)^{\frac{1}{j}} \quad \textit{for} \ \ P'(\alpha_1) \neq 0.$$

The next lemma is proved in [20].

Lemma 4. *Let $P \in \mathcal{P}_n(\mathbf{l})$ satisfying (11). Then*

$$|P'(\alpha_1)| \geq nH(P)^{1-q_1} \quad and \quad |P^{(l)}(\alpha_1)| \leq H(P)^{1-q_l+(n-l)\epsilon_1}, \quad 1 \leq l \leq n-1.$$

Lemma 5. *[21] Let J be an interval in $\mathbb{R}$ and let B be a measurable subset of J with $\mu(B) \geq |J|/v$, where $v \in \mathbb{N}$. Suppose that $|P(x)| < H(P)^{-u}$ for all $x \in B$, where $u > 0$ and $\deg P \leq n$. Then $|P(x)| < (3v)^n(n+1)^{n+1}H(P)^{-u}$ for all $x \in J$.*

Lemma 6. *[21] Let $\delta_0 > 0$ and $Q > Q_0(\delta_0)$. Further, let P_1 and P_2 be two integer polynomials of degree at most n with no common roots and $\max(H(P_1), H(P_2)) \leq Q$. Let $J \subset \mathbb{R}$ be an interval with $\mu(J) = Q^{-\eta}$, $\eta > 0$. If there exists $\tau > 0$ such that for all $x \in J$*

$$|P_j(x)| < Q^{-\tau},$$

for $j = 1, 2$, then

$$\tau + 1 + 2\max(\tau + 1 - \eta, 0) < 2n + \delta_0. \tag{13}$$

Lemma 7. *[6] Let $P(x) = a_nx^n + \ldots a_0$. If $|a_n| \gg H(P)$ then for any i, $1 \leqslant i \leqslant n$ there exists a constant $c > 0$ such that $|\alpha_i| < c$.*

Lemma 8. *[22] Let $P_1(x), \ldots, P_r(x)$ be non-zero complex polynomials of degree $n_1, \ldots, n_r$, respectively, and set $n = n_1 + \ldots + n_r$. We then have*

$$2^{-n}H(P_1)\ldots H(P_r) \leq H(P_1 \ldots P_r) \leq 2^nH(P_1)\ldots H(P_r).$$

3. Proof of Theorem 1

For polynomials P of degree $\deg P \geq 2$ we proceed by mathematical induction with inductive hypothesis being that for $1 \leq s \leq n-1$ we have

$$\mu\left(x \in I : \exists P \in \mathcal{P}_{\leq s}(Q) \ s.t. \ |P(x)| < \delta_s \ \right) < 2^{6s^2}Q^{2s+1}\delta_s\mu(I).$$

Note that the base case for $s = 1$ follows from (17).

3.1. Polynomials without common roots

Define two classes of polynomials without common roots, so that

$$P_{\leq n}(j,T) = \{P \in \mathcal{P}_{\leq n}, \ |a_j| = H(P) = T\}, \ \ 0 \leq j \leq n,$$
$$P_{\leq n}(T) = \cup_{j=0}^n \mathcal{P}_{\leq n}(j,T).$$

Obviously, the following estimates

$$\begin{aligned} \#P_{\leq n}(j,T) &\leq (2T+1)^n, \\ \#P_{\leq n}(T) &\leq (n+1)(2T+1)^n \end{aligned} \tag{14}$$

hold.

Denote by $\bar\sigma(P)$ the set of solutions $x \in I$ of the inequality $|P(x)| < \delta$ for a fixed polynomial $P \in P_{\le n}(j,T)$. Let $x_i \in \bar\sigma(P) \cap S_P(\alpha_i)$. Then, by Lemma 1, we have

$$|x_i - \alpha_i| \le e^{n^2}\delta T^n.$$

Summing the last estimate over all roots of polynomial P, we have

$$\mu(\bar\sigma(P)) \le 2ne^{n^2}\delta T^n. \tag{15}$$

Then sum the estimate (15) over all polynomials P in $P_{\le n}(T)$ and obtain

$$\begin{gathered}\sum_{j=0}^{n}\sum_{P\in\mathcal{P}_{\le n}(j,T)}\mu(\bar\sigma(P)) \le \\ \le 2n(n+1)e^{n^2}T^n(2T+1)^n\delta\mu(I) \le 2^{1-n}n(n+1)e^{n^2}(2T+1)^{2n}\delta\mu(I).\end{gathered} \tag{16}$$

To find an estimate of the measure for all polynomials P in $\mathcal{P}_n(Q)$ we sum the estimate (16) over T from 1 to Q and approximate the sum over T by the integral on T from 1 to $Q+1$. Thus,

$$\begin{gathered}\sum_{P\in\mathcal{P}_{\le n}(Q)}\mu(\bar\sigma(P)) \le 2^{1-n}n(n+1)e^{n^2}\delta\mu(I)\sum_{T=1}^{Q}(2T+1)^{2n} < \\ < 2^{-n}n(n+1)(2n+1)^{-1}(2Q+3)^{2n+1}e^{n^2}\delta\mu(I) = \\ = 2^{-n}n(n+1)(2n+1)^{-1}(2Q)^{2n+1}(1+3/(2Q))^{2n+1}e^{n^2}\delta\mu(I) < \\ < 2^{n+1}nQ^{2n+1}e^{n^2+3(2n+1)/(2Q)}\delta\mu(I) \le 2^{n+1}ne^{n^2+(2n+1)/2}Q^{2n+1}\delta\mu(I)\end{gathered} \tag{17}$$

for $n \ge 1$ and $Q \ge 3$.

3.2. Reducible polynomials

Let $P \in \mathcal{P}_{\le n}(Q)$ be an reducible polynomial of the form

$$P(x) = P_1(x)P_2(x),\ \ \deg P_1 = n_1,\ \ \deg P_2 = n - n_1,\ \ 1 \le n_1 \le n-1,$$

and the inequality $|P(x)| < \delta$ holds for $x \in I$. Let $2^{-1}Q^\lambda < H(P_1) \le Q^\lambda$ where $0 \le \lambda \le 1$. Then by Lemma 8, $H(P_2) < 2^{n+1}Q^{1-\lambda}$. By the continuity of P there exists θ such that

$$\mu\left(x \in \bar\sigma(P):\ |P_1(x)| < \theta\right) = \mu(\bar\sigma(P))/2. \tag{18}$$

Then for the complement to (18) we have

$$\mu\left(x \in \bar\sigma(P):\ |P_1(x)| \ge \theta\right) = \mu(\bar\sigma(P))/2 \tag{19}$$

or

$$\mu\left(x \in \bar\sigma(P):\ |P_2(x)| < \delta\theta^{-1}\right) = \mu(\bar\sigma(P))/2. \tag{20}$$

Then, according to Lemma 5 and by (18), (20), for all $x \in \bar\sigma(P)$ we have

$$|P_1(x)| < 2^{n_1}3^{n_1}(n_1+1)^{n_1+1}\theta,\quad |P_2(x)| < 2^{n-n_1}3^{n-n_1}(n-n_1+1)^{n-n_1+1}\delta\theta^{-1}. \tag{21}$$

For $\theta \leq 2^{6n^2-2-6n_1^2-n_1}3^{-n_1}(n_1+1)^{-n_1-1}Q^{2n+1-\lambda(2n_1+1)}\delta$ we apply inductive hypothesis to polynomials P_1 and obtain that the measure of $x \in I$ for which there is the polynomial $P(x) = P_1(x)P_2(x)$ with P_1 satisfying (21) does not exceed $2^{6n^2-2}Q^{2n+1}\delta\mu(I)$.

For $\theta > 2^{6n^2-2-6n_1^2-n_1}3^{-n_1}(n_1+1)^{-n_1-1}Q^{2n+1-\lambda(2n_1+1)}\delta$ we apply inductive hypothesis to polynomials P_2 and obtain that the measure of $x \in I$ for which there is the polynomial $P(x) = P_1(x)P_2(x)$ with P_2 satisfying (21) does not exceed

$$\begin{aligned}2^{2n^2+4n+12n_1^2-14nn_1-2n_1+3}3^n(n_1+1)^{n_1+1}(n-n_1+1)^{n-n_1+1}Q^{4n_1\lambda-2n_1-2n\lambda}\mu(I) \leq \\ \leq 2^{2n^2+4n+12n_1^2-14nn_1-2n_1+5}3^n n^n Q^{4n_1\lambda-2n_1-2n\lambda}\mu(I)\end{aligned} \tag{22}$$

since $(n_1+1)^{n_1+1}(n-n_1+1)^{n-n_1+1} \leq 2^2 n^n$. Define two functions

$$f_1(n_1) = 2n^2+4n+12n_1^2-14nn_1-2n_1+5, \qquad f_2(\lambda, n_1) = 4n_1\lambda - 2n_1 - 2n\lambda.$$

It is readily verified that the function $f_1(n_1)$ attains its maximum value of $2n^2-10n+15$ on the domain $D_1 = \{1 \leq n_1 \leq n-1\}$ for $n \geq 2$, and the function $f_2(\lambda, n_1)$ attains its maximum value of -2 on the domain $D_2 = \{(\lambda, n_1) : 0 \leq \lambda \leq 1 \text{ and } 1 \leq n_1 \leq n-1\}$.

Therefore, the estimate in (22) does not exceed

$$2^{2n^2-10n+15}3^n n^n Q^{-2}\mu(I) = 2^{2n^2-10n+15+n\log_2 3+n\log_2 n}Q^{-2}\mu(I) < 2^{3n^2}Q^{-2}\mu(I) \tag{23}$$

for $n \geq 2$. The last estimate does not exceed $2^{6n^2-2}Q^{2n+1}\delta\mu(I)$ for $\delta \geq 2^{-3n^2+2}Q^{-2n-3}$. Then, using a trivial estimate for the measure of the set $M_n(Q, I, \delta)$, we obtain (9).

4. Proof of Theorem 2

Let $0 < t < 1$ be a sufficiently small number which we will specify later.

First, consider linear polynomials. Let $P(x) = ax + b \in \mathcal{P}_1(Q)$ satisfying $|P(x)| < \beta_1 Q^{-1}$. Then $|x + b/a| < \beta_1 Q^{-1}|a|^{-1}$ and the measure does not exceed

$$\begin{aligned}2\sum_{|a|\leq Q}\beta_1 Q^{-1}|a|^{-1}|a|\mu(I) = 2\sum_{|a|\leq Q}\beta_1 Q^{-1}\mu(I) \leq \\ \leq 2(2Q+1)\beta_1 Q^{-1}\mu(I) < 2^3\beta_1\mu(I) \leq t\mu(I)\end{aligned}$$

for $\beta_1 \leq 2^{-3}t$. For polynomials P of degree $\deg P \geq 2$ we proceed by mathematical induction with the inductive hypothesis being that for $1 \leq m \leq n-1$ we have

$$\mu\left(x \in I : \exists P \in \mathcal{P}_m(Q) s.t. \begin{array}{c}|P(x)| < \beta_m Q^{-m}, \\ \beta_m = 2^{-3m^3}m^{-2}t\end{array}\right) < f(m)t\mu(I)$$

for $Q \geq 2^{44m^4}$, where $f(m) = 2^{m^2}$. The base case for $m = 1$ follows from the estimate for the linear polynomials from above.

From now on we assume that $n \geq 2$.

4.1. Connection between the derivative at the root and the derivative at a point closest to the root

4.1.1. Case 1: $|P'(x)| > 2n^{5/2}\beta_n^{1/2}Q^{-\frac{n-1}{2}}$.

Denote by $\sigma_*(P)$ the set of solutions $x \in I$ of the system

$$|P(x)| < \beta_n Q^{-n},\ |P'(x)| > n^{5/2}\beta_n^{1/2}Q^{-\frac{n-1}{2}}$$

for a fixed polynomial $P \in \mathcal{P}_n(Q)$. Let $P \in \mathcal{P}_n(Q)$ and $x \in \sigma_*(P) \cap S_P(\alpha_1)$. By Lemma 3

$$|x-\alpha_1| \le n\beta_n Q^{-n}|P'(x)|^{-1} < 2^{-1}n^{-3/2}\beta_n^{1/2}Q^{-(n+1)/2}.$$

By the Taylor's formula

$$P'(x) = \sum_{j=1}^{n}((j-1)!)^{-1}P^{(j)}(\alpha_1)(x-\alpha_1)^{j-1}.$$

Estimating each terms for $2 \le j \le n$ and using the fact that $x \in [0,1]$ and trivial estimate $|P^{(j)}(x)| < n^{j+1}Q$, gives

$$|P^{(j)}(\alpha_1)(x-\alpha_1)^{j-1}| < n^{j+1}Q(2^{-1}n^{-3/2}\beta_n^{1/2}Q^{-(n+1)/2})^{j-1} \le 2^{-1}n^{3/2}\beta_n^{1/2}Q^{-\frac{n-1}{2}}$$

since $\beta_n < 1$, which implies

$$\sum_{j=2}^{n}|((j-1)!)^{-1}P^{(j)}(\alpha_1)(x-\alpha_1)^{j-1}| \le 2^{-1}n^{3/2}\beta_n^{1/2}Q^{-\frac{n-1}{2}}\sum_{j=2}^{n}|((j-1)!)^{-1} <$$
$$< 2^{-1}n^{3/2}\beta_n^{1/2}(e-1)Q^{-\frac{n-1}{2}} < n^{5/2}\beta_n^{1/2}Q^{-\frac{n-1}{2}}$$

and

$$|P'(x)|/2 < |P'(\alpha_1)| < 2|P'(x)|.$$

Therefore, the set $\sigma_*(P) \cap S_P(\alpha_1)$ is contained in $\sigma(P,\alpha_1)$ which is defined by

$$|x-\alpha_1| < 2n\beta_n Q^{-n}|P'(\alpha_1)|^{-1}. \tag{24}$$

4.1.2. Case 2: $|P'(x)| \le 2n^{5/2}\beta_n^{1/2}Q^{-\frac{n-1}{2}}$.

Denote by $\sigma^*(P)$ the set of solutions $x \in I$ of the system

$$|P(x)| < \beta_n Q^{-n},\ |P'(x)| \le 2n^{5/2}\beta_n^{1/2}Q^{-\frac{n-1}{2}}$$

for a fixed polynomial $P \in \mathcal{P}_n(Q)$. Let $P \in \mathcal{P}_n(Q)$ and $x \in \sigma^*(P) \cap S_P(\alpha_1)$. Show that the value of the derivative of P at the α_1, $P(\alpha_1) = 0$, satisfies

$$|P'(\alpha_1)| < 2^n n^2\beta_n^{1/2}Q^{-\frac{n-1}{2}}. \tag{25}$$

To show this, assume the opposite of (25). Then develop P' as a Taylor series in the neighborhood of α_1 and use the estimate $|x-\alpha_1| \le 2^{-1}n^{-2}\beta_n^{1/2}Q^{-\frac{n+1}{2}}$ from Lemma 3. It follows that $|P'(\alpha_1)| \le |P'(x)| + \sum\limits_{j=2}^{n}|((j-1)!)^{-1}P^{(j)}(\alpha_1)(x-\alpha_1)^{j-1}| < 2^n n^2\beta_n^{1/2}Q^{-\frac{n-1}{2}}$ for $n \ge 2$, which contradicts to the condition that $|P'(\alpha_1)| \ge 2^n n^2\beta_n^{1/2}Q^{-\frac{n-1}{2}}$.

4.1.3. Partitioning the roots

Each of the roots of a polynomial $P \in \mathcal{P}_n(Q)$ will lie in one of the following sets:

$$\begin{aligned} &T_k: && d_{k+1}Q^{\frac{-(k-1)}{2}} <|P'(\alpha_1)| \le d_k Q^{\frac{-(k-2)}{2}}, \qquad 0 \le k \le n,\\ &T_{n+1}: && |P'(\alpha_1)| \le d_{n+1}Q^{\frac{-(n-1)}{2}}, \end{aligned}$$

where $d_0 = 2n^2$, $d_{n+1} = n^{5/2}\beta_n^{1/2}$, $d_i = 2^{-n/2+1}$, $1 \le i \le n$. These inequalities partitioned the roots of $P \in \mathcal{P}_n(Q)$ and are labelled $A_0^{(k)}(P)$, $k = 0, 1, \dots, n+1$, respectively. Note that in $\bigcup\limits_{k=0}^{n} T_k$ from the Subsection 4.1.1, $|P'(\alpha_1)| \asymp |P'(x)|$ for $x \in S_P(\alpha_1)$.

The proof of theorem is divided into two cases: the irreducible and reducible polynomials. The proof in the case of irreducible polynomials relies upon the division of $|P'(\alpha_1)|$ which is given above. Note that the Propositions 2–3 hold for all polynomials $P \in \mathcal{P}_n(Q)$ not only the irreducible.

4.2. Irreducible polynomials

4.2.1. Establishing Case A: $|P'(\alpha_1)| \in T_0$.

Define the set $L_{n,0}(Q, I)$ of $x \in I \cap S_P(\alpha_1)$ with $\alpha_1 \in A_0^{(0)}(P)$ for which the system

$$|P(x)| < \beta_n Q^{-n}, \qquad d_1 Q^{\frac{1}{2}} < |P'(\alpha_1)| \le 2n^2 Q, \tag{26}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$.

Proposition 2. *For $Q \ge \frac{n}{3\ln 2}$ we have*

$$\mu(L_{n,0}(Q, I)) < 2t\mu(I).$$

Proof. For a polynomial $P \in \mathcal{P}_n(Q)$ and $\alpha_1 \in A_0^{(0)}(P)$ define the interval

$$\sigma_0(P, \alpha_1) := \{x \in I : |x - \alpha_1| < c_0|P'(\alpha_1)|^{-1}\}, \qquad c_0 < 1, \tag{27}$$

and

$$\sigma_0(P) = \cup_{\alpha_1 \in A_0^{(0)}(P)} \sigma_0(P, \alpha_1).$$

From (24), (26) and (27) it follows that $\mu(\sigma_0(P, \alpha_1)) < \mu(I)$ for $2c_0 d_1^{-1} Q^{-1/2} < \mu(I)$ and $\sigma(P, \alpha_1) \subseteq \sigma_0(P, \alpha_1)$ for $2n\beta_n Q^{-n} \le c_0$. By (24) and (27) we obtain

$$\mu(\sigma(P)) < 2n\beta_n c_0^{-1} Q^{-n} \mu(\sigma_0(P)). \tag{28}$$

Fix the vector $\mathbf{b_1} = (a_n, \dots, a_1)$ which consists of the coefficients of the polynomial $P(t) = \sum\limits_{j=0}^{n} a_j t^j \in \mathcal{P}_n(Q)$. Let the subclass of polynomials $P \in \mathcal{P}_n(Q)$ with the same vector $\mathbf{b_1}$ be denoted by $\mathcal{P}_n(Q, \mathbf{b_1})$. The intervals $\sigma_0(P, \alpha_1)$ divide into two classes using

Sprindzuk's method of essential and inessential domains [6]. The interval $\sigma_0(P,\alpha_1)$ is called *inessential* if there is a polynomial $\tilde{P}\in\mathcal{P}_n(Q,\mathbf{b_1})$ (with $\tilde{P}\neq P$), such that

$$\mu(\sigma_0(P,\alpha_1)\cap\sigma_0(\tilde{P}))\geq 1/2\mu(\sigma_0(P,\alpha_1)), \tag{29}$$

and *essential* otherwise.

First, the essential intervals $\sigma_0(P,\alpha_1)$ are investigated. By definition

$$\sum_{P\in\mathcal{P}_n(Q,\mathbf{b_1})}\ \sum_{\alpha_1\in A_0^{(0)}(P)}\sigma_0(P,\alpha_1)<2n\mu(I).$$

Using the last estimate, (29) and the fact that the number of different vectors $\mathbf{b_1}$ does not exceed $(2Q+1)^n$ gives

$$\sum_{\mathbf{b_1}}\ \sum_{P\in\mathcal{P}_n(Q,\mathbf{b_1})}\ \sum_{\alpha_1\in A_0^{(0)}(P)}\sigma(P,\alpha_1)<$$
$$<2^{n+1}n^2e^{n/(2Q)}\beta_nc_0^{-1}\mu(I)\leq t\mu(I)\leq 2^{n+3}n^2\beta_nc_0^{-1}\mu(I)\leq t\mu(I)$$

for $2^{n+3}n^2\beta_nc_0^{-1}\leq t$ and $Q\geq\frac{n}{3\ln 2}$.

Second, we consider the inessential intervals $\sigma_0(P,\alpha_1)$. Decompose the polynomial P into Taylor series on the interval $\sigma_0(P,\alpha_1)$ so that

$$P(x)=\sum_{j=1}^{n}(j!)^{-1}P^{(j)}(\alpha_1)(x-\alpha_1)^j.$$

Using (26) and (27), estimate each term of the decomposition

$$|P^{(j)}(\alpha_1)(x-\alpha_1)^j|<c_0/n\ \text{ for }\ n^{j+2}d_1^{-j}Q^{1-j/2}\leq c_0^{1-j},\ 2\leq j\leq n,$$

to obtain

$$|P(x)|<2c_0,\quad x\in\sigma_0(P,\alpha_1). \tag{30}$$

Let $\sigma_0(P,\tilde{P})=\sigma_0(P,\alpha_1)\cap\sigma_0(\tilde{P})$, where $P,\tilde{P}\in\mathcal{P}_n(Q,\mathbf{b_1})$ and $P\neq\tilde{P}$. Then on the set $\sigma_0(P,\tilde{P})$ with the measure at least $1/2\mu(\sigma_0(P,\alpha_1))$ for the polynomials P and $\tilde{P}$ the inequality (30) holds. Now consider the new polynomial $R(x)=P(x)-\bar{P}(x)=a_0'$ since the polynomials P and $\tilde{P}$ have the same coefficients $a_n,a_{n-1},\ldots,a_1$. Thus, by (30) we have

$$|R(x)|<4c_0. \tag{31}$$

Therefore, $1\leq|a_0'|<4c_0$ which gives a contradiction for $c_0\leq 2^{-2}$.

Since $d_1=2^{-n/2+1}$ and $\beta_n=2^{-3n^3}n^{-2}t$ then the choice of $c_0=n^{-4}2^{-n+2}$ satisfies all the conditions for c_0 from above with $n\geq 2$ and $t\leq 1$. □

4.2.2. Establishing Case B: $|P'(\alpha_1)|\in T_k$, $1\leq k\leq n-1$.

For fixed k, $1\leq k\leq n-1$ define the set $L_{n,k}(Q,I)$ of $x\in I\cap S_P(\alpha_1)$ with $\alpha_1\in A_0^{(k)}(P)$ for which the system

$$|P(x)|<\beta_nQ^{-n},\quad |P'(\alpha_1)|\in T_k, \tag{32}$$

has a solution in polynomials $P\in\mathcal{P}_n(Q)$.

Proposition 3. *For* $Q \ge \max\left(\frac{n-k}{3\ln 2}, 2^{44k^4-1}\right)$ *we have*

$$\mu(L_{n,k}(Q,I) < (f(k)+1)t\mu(I).$$

Proof. For a polynomial $P \in \mathcal{P}_n(Q)$ and $\alpha_1 \in A_0^{(k)}(P)$ define the interval

$$\sigma_k(P,\alpha_1) := \{x \in I : |x-\alpha_1| < c_k Q^{-k}|P'(\alpha_1)|^{-1}\}, \;\; c_k < 1.$$

Let $\sigma_k(P) = \bigcup_{\alpha_1 \in A_0^{(k)}(P)} \sigma_k(P,\alpha_1)$. From (24), (32) and definition of $\sigma_k(P,\alpha_1)$, $1 \le k \le n-1$, it follows that $\mu(\sigma_k(P,\alpha_1)) < \mu(I)$ for $2c_k d_{k+1}^{-1} Q^{-(k+1)/2} < \mu(I)$ and $\sigma(P,\alpha_1) \subseteq \sigma_k(P,\alpha_1)$ for $2n\beta_n c_k^{-1} Q^{-n+k} \le 1$. For $1 \le k \le n-1$ fix the vector $\mathbf{b_{k+1}} = (a_n,\dots,a_{k+1})$. Let the subclass of polynomials $P \in \mathcal{P}_n(Q)$ with the same vector $\mathbf{b_{k+1}}$ be denoted by $\mathcal{P}_n(Q,\mathbf{b_{k+1}})$. The intervals $\sigma_k(P,\alpha_1)$ divide into two classes of essential and inessential domains.

First, the essential intervals $\sigma_k(P,\alpha_1)$ are investigated. By definition

$$\sum_{P \in \mathcal{P}_n(Q,\mathbf{b_{k+1}})} \sum_{\alpha_1 \in A_0^{(k)}(P)} \mu(\sigma_k(P,\alpha_1)) < 2n\mu(I).$$

Using the last estimate and the fact that the number of different vectors $\mathbf{b_{k+1}}$ does not exceed $(2Q+1)^{n-k}$, it follows that

$$\sum_{\mathbf{b_{k+1}}} \sum_{P \in \mathcal{P}_n(Q,\mathbf{b_{k+1}})} \mu(\sigma(P)) < 2^{n-k+3} n^2 \beta_n c_k^{-1} \mu(I) \le t\mu(I) \tag{33}$$

for $2^{n-k+3} n^2 \beta_n c_k^{-1} \le t$ and $Q \ge \frac{n-k}{3\ln 2}$.

Second, we consider the inessential intervals $\sigma_k(P,\alpha_1)$. Let $\sigma_k(P,\bar{P}) = \sigma_k(P,\alpha_1) \cap \sigma_k(\bar{P})$, where $P,\bar{P} \in \mathcal{P}_n(Q,\mathbf{b_{k+1}})$ and $P \ne \bar{P}$. Develop P and P' into Taylor series on $\sigma_k(P,\alpha_1)$ and $\sigma_k(\bar{P},\bar{\alpha}_1)$ respectively. Then on the set $\sigma_k(P,\bar{P})$ with the measure at least $1/2\mu(\sigma_k(P,\alpha_1))$ for the polynomials P and $\bar{P}$ the following inequality

$$\max(|P(x)|,|\bar{P}(x)|) < 2c_k Q^{-k}, \tag{34}$$

holds for $n^{j+2} d_{k+1}^{-j} Q^{1+k-j(k+1)/2} \le c_k^{1-j}$, $2 \le j \le n$.

By (34) and Lemma 5, the new polynomials $R(t) = P(t) - \bar{P}(t)$ of $\deg R \le k$ with $H(R) \le 2Q$ satisfy

$$|R(x)| < 2^{k+2} 3^k (k+1)^{k+1} c_k Q^{-k} = 2^{2k+2} 3^k (k+1)^{k+1} c_k Q_1^{-k}, \quad H(R) \le 2Q = Q_1 \tag{35}$$

for any $x \in \sigma_k(P,\alpha_1)$. Applying inductive hypothesis to polynomials R satisfying (35), we obtain that the measure of the set of x belonging to inessential intervals does not exceed $f(k)t\mu(I)$ for $Q \ge 2^{44k^4-1}$ and $2^{2k+2}3^k(k+1)^{k+1}c_k < \beta_k$. Since $d_{k+1} = 2^{-n/2+1}$ and $\beta_m = 2^{-3m^3} m^{-2} t$, $1 \le m \le n$, then the choice of $c_k = 2^{2n^2+n-k+3} n^2 \beta_n$ satisfies all the conditions for c_k from above with $n \ge 2$ and $t \ge 2^{-2n^2}$. Combining the estimates of measure for essential and inessential intervals, we obtain $\mu(L_{n,k}(Q,I)) < (f(k)+1)t\mu(I)$ for $Q \ge \max\left(\frac{n-k}{3\ln 2}, 2^{44k^4-1}\right)$. $\square$

Therefore, for the union $\bigcup_{k=1}^{n-1} L_{n,k}(Q,I)$ we have

$$\mu(\bigcup_{k=1}^{n-1} L_{n,k}(Q,I)) \leq \sum_{k=1}^{n-1} \mu(L_{n,k}(Q,I)) < \sum_{k=1}^{n-1}(1+f(k))t\mu(I)$$

with $Q \geq 2^{44(n-1)^4-1}$.

4.2.3. Establishing Case C: $|P'(\alpha_1)| \in T_n$.

Define the set $L_{n,n}(Q,I)$ of $x \in I \cap S_P(\alpha_1)$ with $\alpha_1 \in A_0^{(n)}(P)$ for which the system

$$|P(x)| < \beta_n Q^{-n}, \;\; n^5 \beta_n^{1/2} Q^{-(n-1)/2} < |P'(\alpha_1)| \leq d_n Q^{-(n-2)/2} \tag{36}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$.

Proposition 4. *For $Q \geq 2^{6n}$ we have $\mu(L_{n,n}(Q,I)) < 2t\mu(I)$.*

Proof. Divide the interval I into smaller intervals I_i with the lengths Q^{-u_1} where $u_1 > 0$ since $\mu(I_i) < \mu(I)$. We say the polynomial P belongs to the interval I_i if there exists $x \in I_i$ such that (36) and the corresponding estimates for $P(x)$ hold. If there is at most one irreducible polynomial $P \in \mathcal{P}_n(Q)$ that belongs to every I_i then by (24) the measure of those x, that satisfy (36), does not exceed

$$2^2 n^{-3} \beta_n^{1/2} Q^{u_1-(n+1)/2} \mu(I) \leq t\mu(I) \tag{37}$$

for $u_1 = (n+1)/2$ and $2^2 n^{-3} \beta_n^{1/2} \leq t$.

If at least two irreducible polynomials $P_i \in \mathcal{P}_n(Q)$ of the form $P_i(x) = k_i P(x)$ for the same irreducible polynomial $P \in \mathcal{P}_n(Q)$, $k_i \in \mathbb{Z}$, belong to the interval I_i then the measure in this case coincides with the measure in (37).

The assumption that at least two irreducible polynomials belong to the interval I_i will lead to a contradiction. To show this, suppose that P_1 and P_2 belong to I_i. Develop P_1 as a Taylor series in the neighbourhood I_i of α_1 to obtain

$$|P_1(x)| \leq (n-1)n^3 Q^{-n} + 2^{-n/2+1} Q^{-n+1/2} \leq 2^{-n/2+1} n Q^{-n+1/2}, \quad x \in I_i,$$

for $Q \geq 2^{n-2} n^6$ and $d_n = 2^{-n/2+1}$. Here each term in the Taylor series for $2 \leq j \leq n$ has the form

$$|(j!)^{-1} P_1^{(j)}(\alpha_1)||x-\alpha_1|^j < n^{j+1} Q^{1-j(n+1)/2} \leq n^3 Q^{-n} \qquad \text{for} \quad Q \geq n^{2/(n+1)}.$$

Obviously, the same estimate holds for P_2 on I_i. Thus, for $j = 1, 2$ we have

$$|P_j(x)| < \begin{cases} Q^{-n+1/2} & \text{for } n \geq 8, \\ 3Q^{-n+1/2} & \text{for } n < 8. \end{cases}$$

For $Q \geq 2^{6n}$ and $n < 8$ we have $|P_j(x)| < Q^{-n+1/2+7/(24n)}$, $j = 1, 2$. Apply Lemma 6 to polynomials P_1 and P_2 with $\tau = n - 1/2 - \mu$ and $\eta = (n+1)/2$, where $\mu = 0$ for $n \geq 8$ and $\mu = 7/(24n)$ for $n < 8$. Therefore, the left hand side in (13) has the form

$$\tau + 1 + 2\max(\tau + 1 - \eta, 0) = 2n - 3\mu + 1/2.$$

Choose $\delta_0 < 1/(8n)$. Then we will have a contradiction in (13) for $n \geq 2$. $\square$

4.2.4. Establishing Case D: $|P'(\alpha_1)| \in T_{n+1}$.

Define the set $L_{n,n+1}(Q,I)$ of $x \in I \cap S_P(\alpha_1)$ with $\alpha_1 \in A_0^{(n+1)}(P)$ for which the system

$$|P(x)| < \beta_n Q^{-n}, \qquad |P'(\alpha_1)| \le n^5 \beta_n^{1/2} Q^{-(n-1)/2} \tag{38}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$.

Proposition 5. *For $Q \ge 2^{44n^4}$ we have $\mu(L_{n,n+1}(Q,I)) < t\mu(I)$.*

Proof. Define by $\sigma_*(P)$ the set of solutions $x \in I$ of (38) for a fixed polynomial $P \in \mathcal{P}_n(Q)$. Note that the set $L_{n,n+1}(Q,I)$ can be written as

$$L_{n,n+1}(Q,I) = \ L_{\le} \cup L_{>}$$

where $L_{\le} = \bigcup\limits_{P \in \mathcal{P}_n\left(Q^{\frac{1}{n+1}}\right)} \sigma_*(P)$ and $L_{>} = \bigcup\limits_{P \in \mathcal{P}_n(Q) \setminus \mathcal{P}_n\left(Q^{\frac{1}{n+1}}\right)} \sigma_*(P)$.

Next, we are going to establish the following two separate cases.

Case 1: $\mu(L_{\le}) < t\mu(I)/2$.

Let $x \in \sigma_*(P) \cap S_P(\alpha_1)$ for some $P \in \mathcal{P}_n\left(Q^{\frac{1}{n+1}}\right)$. Then by (38) and Lemma 3 (for $j = n$), we have

$$|x - \alpha_1| \le (\beta_n Q^{-n} |a_n|^{-1})^{1/n} \le (\beta_n Q^{-n})^{1/n} \tag{39}$$

since $|a_n| \ge 1$. Summing the estimate (39) over all polynomials $P \in \mathcal{P}_n\left(Q^{\frac{1}{n+1}}\right)$, we obtain

$$\mu(L_{\le}) \le 2n\beta_n^{1/n}(2Q^{\frac{1}{n+1}} + 1)^{n+1} Q^{-1} \le n\beta_n^{1/n} 2^{2n+3} \le t\mu(I)/2$$

for $t \ge 2^{2n+4} n \beta_n^{1/n}$. Note that for $\beta_n = 2^{-3n^3} n^{-2} t$ we get that

$$t \ge 2^{-2n^2} \ge n^{(n-2)/(n-1)} 2^{(2n^2+4n-3n^3)/(n-1)} \qquad \text{for} \quad n \ge 2.$$

Case 2: $\mu(L_{>}) < t\mu(I)/2$.

For $k \in \mathbb{N}$, let $\mathcal{P}_{\mathbf{l}}^k$ denote the subclass of $\mathcal{P}_n(\mathbf{l})$ given by

$$\mathcal{P}_{\mathbf{l}}^k = \{P \in \mathcal{P}_n(\mathbf{l}) : \ 2^k \le H(P) < 2^{k+1}\}.$$

Let $k_0 = [\frac{1}{n+1} \log_2 Q]$. Then we have

$$\mathcal{P}_n(Q) \setminus \mathcal{P}_n\left(Q^{\frac{1}{n+1}}\right) = \bigcup_{\mathbf{l}} \bigcup_{k=k_0}^{[\log_2 Q]} \mathcal{P}_{\mathbf{l}}^k.$$

Now divide the interval I into smaller intervals J_i' with $\mu(J_i') = 2^{k(u'+\gamma)}$ where

$$u' = \min_{1 \le j \le n} \{(-1 - n + q_j)/j\}, \ q_n = 0.$$

First show that the assumption that at least two irreducible polynomials from $\mathcal{P}_{\mathbf{l}}^k$ without common roots belong to the interval J_i' will lead to a contradiction. To show this, suppose that P_1 and P_2 belong to J_i'. Let $|a_n| \geq H(P)/n$. Let $q_1 \geq 1 + 1/n$ then by Lemma 4 and (38) we have $n^{-1}H(P)^{1-q_1} < |P'(\alpha_1)| \leq 2^{-1}n^5\beta_n^{1/2}Q^{-(n-1)/2}$, which implies that $q_1 \geq (n+1)/2 - 1/n$ for $H(P) \leq Q$ and $Q \geq 2^{n^4}$. Assumption that $q_1 < 1 + 1/n$ gives a contradiction. Develop P_1 as a Taylor series in the neighbourhood J_i' of α_1 to obtain

$$|P_1(x)| \leq n2^{k(-n+(n+1)\gamma)}, \quad x \in J_i'$$

for $\epsilon_1 \leq \gamma/n$ and sufficiently large k, where

$$\begin{aligned} |(j!)^{-1}P_1^{(j)}(\alpha_1)||x-\alpha_1|^j &\ll 2^{k(1-q_j+(n-j)\epsilon_1)}2^{k(j\gamma+j(\frac{-1-n+q_j}{j}))} = \\ &= 2^{k(-n+j\gamma+(n-j)\epsilon_1)}, \qquad 1 \leq j \leq n. \end{aligned}$$

Obviously, the same estimate holds for P_2 on J_i'. Note that for $Q \geq 2^{n^4}$ and $k \leq [\log_2 Q]$ we have that $|P_s(x)| < 2^{k(-n+(n+1)\gamma)+1/(n^3)}$, $s = 1, 2$. Apply Lemma 6 to polynomials P_1 and P_2 with $\tau = n - n\gamma - 1/(n^3)$ and $\eta = -u' - \gamma$. Therefore, the left hand side in (13) has the form

$$\tau + 1 + 2\max(\tau + 1 - \eta, 0) \;\; = \;\; 3n + 3 + 2(-1 - n + q_j)/j - \gamma(3n-2) - 3/(n^3)$$

Since $q_1 \geq (n+1)/2 - 1/n$, it is readily seen that $\tau + 1 + 2\max(\tau + 1 - \eta, 0) > 2n + 2 - \gamma(3n-2) - 2/n - 3/(n^3)$ for $j = 1$ and $\tau + 1 + 2\max(\tau + 1 - \eta, 0) > 2n + 2 - \gamma(3n-2) - 3/(n^3)$ for $2 \leq j \leq n$. Let $\delta_0 \leq 1/(n^3)$. We will have a contradiction in (13) for $\gamma \leq (2n^3 - 2n^2 - 4)/(n^3(3n-2))$. We can choose $\gamma = 1/(4n)$.

Therefore, there is at most one irreducible polynomial $P \in \mathcal{P}_1^k$ that belongs to J_i'. For $P \in \mathcal{P}_1^k$ denote by $\nu(P, \alpha_1)$ the set of $x \in S_P(\alpha_1)$ satisfying (38). According to Lemma 3 we have that

$$\mu(\nu(P, \alpha_1)) < 2^{n+ku'}.$$

Using the inclusion $\sigma_*(P) \subseteq \bigcup_{\alpha_1 \in A_0^{(n+1)}(P)} \nu(P, \alpha_1)$ for any polynomial P and the fact that the number of polynomials $P \in \mathcal{P}_1^k$ does not exceed the number of intervals J', we obtain

$$\begin{aligned} \mu(L_>) &\leq \sum_{\mathbf{1}} \sum_{k=k_0}^{[\log_2 Q]} n2^{n+ku'}2^{k(-u'-\gamma)}\mu(I) < \\ &< 2^n nC(n,\epsilon_1)\mu(I)(Q^{-\gamma/(n+1)} - Q^{-\gamma})(1-2^{-\gamma})^{-1} \leq \\ &\leq 2^{n^2+n}n^2C(n,\epsilon_1)Q^{-\gamma/(n+1)}\mu(I) \end{aligned} \tag{40}$$

since $1 - 2^{-1/(4n)} > n^{-1}2^{-n^2}$ for $n \geq 2$. Using the fact that $C(n, \epsilon_1) < (2nT)^n = (2n(\epsilon_1)^{-1})^n = (2n^2/\gamma)^n = 2^{3n}n^{3n}$ and $\epsilon_1 = \gamma/n$, we get

$$\mu(L_>) < n^{3n+2}2^{n^2+4n}Q^{-1/(4n(n+1))}\mu(I) \leq t\mu(I)/2$$

for $t \geq 2^{n^2+4n+1}n^{3n+2}Q^{-1/(4n(n+1))}$. For $n \geq 2$, $t \geq 2^{-2n^2}$ and $Q \geq 2^{44n^4}$ we complete the proof in the Case 2. □

4.3. Reducible polynomials

Let $P \in \mathcal{P}_n(Q)$ be an reducible polynomial of the form

$$P(f) = P_1(f)P_2(f), \quad \deg P_1 = n_1, \quad \deg P_2 = n - n_1, \quad 1 \le n_1 \le n-1$$

and the inequality $|P(x)| < \beta_n Q^{-n}$ holds for $x \in I$.

For fixed P by $\lambda(P)$ denote the set of $x \in I$ satisfying $|P(x)| < \beta_n Q^{-n}$. Now reducible polynomials $P \in \mathcal{P}_n(Q)$ we split into two classes $\mathcal{P}_n'(Q)$ and $\mathcal{P}_n''(Q)$. Reducible polynomial P belongs to $\mathcal{P}_n'(Q)$ if it has a factor P_1 satisfying the following property: let $\mu(\lambda(P)) < 2\mu(\lambda_1(P_1))$ where $\mu(\lambda_1(P_1)) = \{x \in \mu(\lambda(P)) : |P_1(x)| < \beta_{n_1} Q^{-n_1}\}$. Then by inductive hypothesis

$$\sum_{P \in \mathcal{P}_n'(Q)} \mu(\lambda(P)) < 2 \sum_{n_1=1}^{n-1} f(n_1) t \mu(I)$$

for $Q \ge 2^{44(n-1)^4}$.

If $\mu(\lambda(P)) \ge 2\mu(\lambda_1(P_1))$ then on the set $\lambda(P) \setminus \lambda_1(P_1)$ we have

$$|P_2(x)| < \beta_n (\beta_{n_1})^{-1} Q^{-n+n_1}$$

and the measure of the last set over all $P_2 \in \mathcal{P}_{n-n_1}(Q)$ does not exceed $f(n-n_1)t\mu(I)$ for $\beta_n \le \beta_{n_1}\beta_{n-n_1}$ by inductive hypothesis. In this case $\mu(\lambda(P)) \le 2\mu(\lambda(P) \setminus \lambda_1(P_1))$ and

$$\sum_{P \in \mathcal{P}_n''(Q)} \mu(\lambda(P)) < 2 \sum_{n_1=1}^{n-1} f(n-n_1) t \mu(I)$$

for $Q \ge 2^{44(n-1)^4}$. Note that the inequality $\beta_n \le \beta_{n_1}\beta_{n-n_1}$ holds for $\beta_m = 2^{-3m^3} m^{-2} t$ $(1 \le m \le n)$, $t \ge 2^{-2n^2}$ and $n \ge 2$.

4.4. End of the proof of Theorem 2

Combining the results of the Subsections 4.1–4.4, we get

$$\mu(L_n(Q, I)) < f(n) t \mu(I)$$

for $Q \ge 2^{44n^4}$, where for $n \ge 2$ we have

$$f(n) \ge 5 + \sum_{k=1}^{n-1}(1 + f(k)) + 2\sum_{k=1}^{n-1} f(k) + 2\sum_{k=1}^{n-1} f(n-k) = n + 4 + 5\sum_{k=1}^{n-1} f(k). \quad (41)$$

Since $f(k)$ is an increasing function, then, clearly, the function $f(k) = 2^{k^2}$ will satisfy (41) for $n \ge 2$.

Choose $\epsilon = tf(n)$. Since $t \ge 2^{-2n^2}$ and $f(n) = 2^{n^2}$ then $\epsilon = tf(n) \ge 2^{-n^2}$. This concludes the proof of Theorem 2.

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АННОТАЦИЯ

В данной статье получены эффективные оценки в терминах n и Q для меры множеств действительных чисел с заданным свойством аппроксимации алгебраическими числами степени n и высоты, ограниченной $Q \in \mathbb{N}$.

Ключевые слова: *целочисленные многочлены, мера Лебега, приближения алгебраическими числами.*