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Some properties of the resolvent kernels for integral equations with bi-Carleman kernels

We prove that, at regular values lying in a region of generalized strong convergence, the resolvent kernels corresponding to a continuous bi-Carleman kernel vanishing at infinity can be expressed as uniform limits of sequences of resolvent kernels associated with its approximating Hilbert-Schmidt-type subkernels.

Key words: *linear integral equation of the second kind, bounded integral linear operator, Fredholm resolvent, resolvent kernel, bi-Carleman kernel, Hilbert-Schmidt kernel, nuclear operator, regular value, characteristic set.*

1 Introduction

In the general theory of integral equations of the second kind in $L^2 = L^2(\mathbb{R})$, that is, equations of the form

$$f(s) - \lambda \int_{\mathbb{R}} \boldsymbol{T}(s,t) f(t) \, dt = g(s) \quad \text{for almost every } s \in \mathbb{R}, \tag{1}$$

it is customary to call an integral kernel $\boldsymbol{T}_{|\lambda}$ a *resolvent kernel for $\boldsymbol{T}$ at λ* if the integral operator it induces on L^2 is the Fredholm resolvent $T(I - \lambda T)^{-1}$ of the integral operator T, which is defined on L^2 by the kernel $\boldsymbol{T}$. Once the resolvent kernel $\boldsymbol{T}_{|\lambda}$ has been constructed, one can express the L^2-solution f to equation (1) in a direct and simple fashion as

$$f(s) = g(s) + \lambda \int_{\mathbb{R}} \boldsymbol{T}_{|\lambda}(s,t) g(t) \, dt \quad \text{for almost every } s \in \mathbb{R},$$

regardless of the particular choice of the function g of L^2. Here it should be noted that, in general, the property of being an integral operator is not shared by Fredholm resolvents of integral operators, and there is even an example, given in [1] (see also [2, Section 5, Theorem 8]), of an integral operator whose Fredholm resolvent at each non-zero regular value is not an integral operator. This phenomenon, however, can never

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occur for Carleman operators (the integral operators on L^2 whose kernels are Carleman, i.e., square integrable in the second variable for almost all values of the first) due to the fact that the right-multiple by a bounded operator of a Carleman operator is again a Carleman operator (see Proposition 2 below). Therefore, in the case when the kernel $\boldsymbol{T}$ is Carleman and λ is a regular value for T, the problem of solving equation (1) may be reduced to the problem of explicitly constructing in terms of $\boldsymbol{T}$ the resolvent kernel $\boldsymbol{T}_{|\lambda}$ which is a priori known to exist. For a precise formulation of this latter problem and for comments to the solution of some of its special cases we refer to the works by V. B. Korotkov [3], [2] (in both the references, see Problem 4 in §5). Here we only notice that in the case when a measurable kernel $\boldsymbol{T}$ of (1) is bi-Carleman (i.e., square integrable in each variable separately for almost all values of the other) but otherwise unrestricted, there seems to be as yet no analytic machinery for explicitly constructing its resolvent kernel $\boldsymbol{T}_{|\lambda}$ at every regular value λ. In order to approach such a problem, we will here confine our investigation to the case in which the bi-Carleman kernel $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{C}$ of T is a so-called K^0-kernel, i.e., it and its two Carleman functions $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s, \cdot)}$, $\boldsymbol{t}'(s) = \boldsymbol{T}(\cdot, s) : \mathbb{R} \to L^2$ are continuous and vanish at infinity (see Definition 3 below). Such conditions on $\boldsymbol{T}$ can always be achieved by means of a unitary equivalence transformation of T if the adjoint to T is also an integral operator (see Proposition 5 below). They therefore involve no loss of generality as far as the search of solutions to such integral equations as (1) is concerned.

One of the main technical advantages of dealing with the K^0-kernel is that its subkernels, such as the restrictions of it to compact squares in $\mathbb{R}^2$ centered at origin, are quite amenable to the methods of the classical theory of ordinary integral equations, and can be used to approximate the original kernel in suitable norms. This, for instance, can be used directly to establish an explicit theory of spectral functions for any Hermitian K^0-kernel by a development essentially the same as the one given by T. Carleman: for a symmetric Carleman kernel that is the pointwise limit of its symmetric Hilbert-Schmidt subkernels satisfying a mean square continuity condition, he constructed in [4, pp. 25-51] its spectral functions as pointwise limits of sequences of spectral functions for the subkernels. For further developments and applications of Carleman's spectral theory we refer to [5], [6], [7], [8, Appendix I], [9], [10], and [11].

As another application of the subkernel approach, we focus in present paper on the question whether and at what regular values λ the resolvent kernel $\boldsymbol{T}_{|\lambda}$ for a K^0-kernel $\boldsymbol{T}$ can be expressed as the limit of a sequence of resolvent kernels associated with the subkernels of $\boldsymbol{T}$. The main result of the paper is Theorem 10 describing such regular values λ in terms of generalized strong convergence, introduced by T. Kato in [12].

2 Notation, definitions, and auxiliary facts

2.1 Fredholm resolvents

Throughout this paper, the symbols $\mathbb{C}$ and $\mathbb{N}$ refer to the complex plane and the set of all positive integers, respectively, $\mathbb{R}$ is the real line equipped with the Lebesgue measure, and $L^2 = L^2(\mathbb{R})$ is the complex Hilbert space of (equivalence classes of) measurable complex-valued functions on $\mathbb{R}$ equipped with the inner product $\langle f, g \rangle = \int f(s)\overline{g(s)}\, ds$

and the norm $\|f\| = \langle f, f \rangle^{\frac{1}{2}}$. (Integrals with no indicated domain, such as the above, are always to be extended over $\mathbb{R}$.)

Let $\mathfrak{R}(L^2)$ denote the Banach algebra of all bounded linear operators acting on L^2; $\|\cdot\|$ will also denote the norm in $\mathfrak{R}(L^2)$. For an operator A of $\mathfrak{R}(L^2)$, A^* stands for the adjoint to A with respect to $\langle\cdot,\cdot\rangle$, and $\operatorname{Ran} A = \cup_{f \in L^2} Af$ for the range of A. An operator $U \in \mathfrak{R}(L^2)$ is said to be *unitary* if $\operatorname{Ran} U = L^2$ and $\langle Uf, Ug \rangle = \langle f, g \rangle$ for all f, $g \in L^2$. An operator $A \in \mathfrak{R}(L^2)$ is said to be *invertible* if it has an inverse which is also in $\mathfrak{R}(L^2)$, i.e., if there is an operator $B \in \mathfrak{R}(L^2)$ for which $BA = AB = I$, where I is the identity operator on L^2; B is denoted by A^{-1}. An operator $P \in \mathfrak{R}(L^2)$ is called a *projection* in L^2 if $P^2 = P$. A projection P in L^2 is said to be *orthogonal* if $P = P^*$. An operator $T \in \mathfrak{R}(L^2)$ is said to be *self-adjoint* if $T^* = T$. An operator $T \in \mathfrak{R}(L^2)$ is said to be *compact* if it transforms every bounded set in L^2 into a relatively compact set in L^2. (Recall that a set L in a normed space Y is said to be *relatively compact* in Y if each sequence of elements from L contains a subsequence converging in the norm of Y.) A (compact) operator $A \in \mathfrak{R}(L^2)$ is *nuclear* if $\sum_n |\langle Au_n, u_n \rangle| < \infty$ for any choice of an orthonormal basis $\{u_n\}$ of L^2.

Throughout the rest of this subsection, T denotes a bounded linear operator of $\mathfrak{R}(L^2)$. The set of *regular values* for T, denoted by $\Pi(T)$, is the set of complex numbers λ such that the operator $I - \lambda T$ is invertible, i.e., it has an inverse $R_\lambda(T) = (I - \lambda T)^{-1} \in \mathfrak{R}(L^2)$ that satisfies

$$(I - \lambda T) R_\lambda(T) = R_\lambda(T) (I - \lambda T) = I. \tag{2}$$

The operator

$$T_{|\lambda} := T R_\lambda(T) \ (= R_\lambda(T) T) \tag{3}$$

is then referred to as the *Fredholm resolvent* of T at λ. Remark that if λ is a regular value for T, then, for each fixed g in L^2, the (unique) solution f of L^2 to the second-kind equation $f - \lambda T f = g$ may be written as

$$f = g + \lambda T_{|\lambda} g$$

(follows from the formula

$$R_\lambda(T) = I + \lambda T_{|\lambda} \tag{4}$$

which is a rewrite of (2)). Recall that the inverse $R_\lambda(T)$ of $I - \lambda T$ as a function of T also satisfies the following identity, often referred to as the *second resolvent equation* (see, e.g., [13, Theorem 5.16.1]): for T, $A \in \mathfrak{R}(L^2)$,

$$\begin{aligned} R_\lambda(T) - R_\lambda(A) &= \lambda R_\lambda(T)(T - A)R_\lambda(A) \\ &= \lambda R_\lambda(A)(T - A)R_\lambda(T) \quad \text{for every } \lambda \in \Pi(T) \cap \Pi(A). \end{aligned} \tag{5}$$

(A slightly modified version of it is

$$\begin{aligned} T_{|\lambda} - A_{|\lambda} &= (I + \lambda T_{|\lambda})(T - A)(I + \lambda A_{|\lambda}) \\ &= (I + \lambda A_{|\lambda})(T - A)(I + \lambda T_{|\lambda}) \quad \text{for every } \lambda \in \Pi(T) \cap \Pi(A), \end{aligned} \tag{6}$$

which involves the Fredholm resolvents.) It should also be mentioned that the map $R_\lambda(T): \Pi(T) \to \mathfrak{R}(L^2)$ is continuous at every point λ of the open set $\Pi(T)$, in the

sense that $\|R_{\lambda_n}(T) - R_\lambda(T)\| \to 0$ when $\lambda_n \to \lambda$, $\lambda_n \in \Pi(T)$ (see, e.g., [14, Lemma 2 (XIII.4.3)]). Moreover, $R_\lambda(T)$ is given by an operator-norm convergent series ($T^0 = I$):

$$R_\lambda(T) = \sum_{n=0}^\infty \lambda^n T^n \quad \text{provided } |\lambda| < \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\|T^n\|}} \tag{7}$$

(see, e.g., [14, Theorem 1 (XIII.4.2)]). To simplify the formulae, we shall always write $R_\lambda^*(T)$ for the adjoint $(R_\lambda(T))^*$ to $R_\lambda(T)$.

The *characteristic set* $\Lambda(T)$ for T is defined to be the complementary set in $\mathbb{C}$ of $\Pi(T)$: $\Lambda(T) = \mathbb{C} \setminus \Pi(T)$; the name is adopted from [14, XIII.3.1].

Given a sequence $\{S_n\}_{n=1}^\infty$ of bounded operators on L^2, let $\nabla_\mathfrak{b}(\{S_n\})$ denote the set of all non-zero complex numbers ζ for which there exist positive constants $M(\zeta)$ and $N(\zeta)$ such that

$$\zeta \in \Pi(S_n) \text{ and } \|S_{n|\zeta}\| \leqslant M(\zeta) \quad \text{for } n > N(\zeta), \tag{8}$$

where, as in what follows, $S_{n|\zeta}$ stands for the Fredholm resolvent of the sequence term S_n at ζ, and let $\nabla_\mathfrak{s}(\{S_n\})$ denote the set of all non-zero complex numbers ζ ($\in \nabla_\mathfrak{b}(\{S_n\})$) for which the sequence $\{S_{n|\zeta}\}$ is convergent in the strong operator topology (i.e., the limit $\lim\limits_{n\to\infty} S_{n|\zeta}f$ exists in L^2 for every $f \in L^2$).

Remark 1. The sets $\nabla_\mathfrak{b}(\{S_n\})$ and $\nabla_\mathfrak{s}(\{S_n\})$ evidently remain unchanged if in their definition the Fredholm resolvents $S_{n|\zeta}$ are replaced by the operators $R_\zeta(S_n) = (I - \zeta S_n)^{-1} = I + \zeta S_{n|\zeta}$ (see (4)). So, if Δ_b (resp., Δ_s) is the *region of boundedness* (resp., *strong convergence*) for the resolvents $\{(\zeta I - S_n)^{-1}\}$, which was introduced and studied in [12, Section VIII-1.1], then the sets $\nabla_\mathfrak{b}(\{S_n\})$ and $\Delta_\mathrm{b} \setminus \{0\}$ (resp., $\nabla_\mathfrak{s}(\{S_n\})$ and $\Delta_\mathrm{s} \setminus \{0\}$) are mapped onto each other by the mapping $\zeta \to \zeta^{-1}$. In the course of the proof of Theorem 10 in Subsection 3.3, we keep this mapping in mind when referring to [12] for generalized strong convergence theory.

2.2 Integral operators

A linear operator $T : L^2 \to L^2$ is *integral* if there is a complex-valued measurable function $\boldsymbol{T}$ (*kernel*) on the Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ such that

$$(Tf)(s) = \int \boldsymbol{T}(s,t)f(t)\,dt$$

for every f in L^2 and almost every s in $\mathbb{R}$. Recall [15, Theorem 3.10] that integral operators are bounded from L^2 into itself, and need not be compact.

A measurable function $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{C}$ is said to be a *Carleman kernel* (resp., *Hilbert-Schmidt kernel*) if $\int |\boldsymbol{T}(s,t)|^2\,dt < \infty$ for almost every fixed s in $\mathbb{R}$ (resp., $\int \int |\boldsymbol{T}(s,t)|^2\,dt\,ds < \infty$). To each Carleman kernel $\boldsymbol{T}$ there corresponds a *Carleman function* $\boldsymbol{t} : \mathbb{R} \to L^2$ defined by $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$ for all s in $\mathbb{R}$ for which $\boldsymbol{T}(s,\cdot) \in L^2$. The Carleman kernel $\boldsymbol{T}$ is called *bi-Carleman* in case its conjugate transpose kernel $\boldsymbol{T}'$ ($\boldsymbol{T}'(s,t) = \overline{\boldsymbol{T}(t,s)}$) is also a Carleman kernel. Associated with the conjugate transpose $\boldsymbol{T}'$ of every bi-Carleman kernel $\boldsymbol{T}$ there is therefore a Carleman function $\boldsymbol{t}' : \mathbb{R} \to L^2$

defined by $\boldsymbol{t}'(s) = \overline{\boldsymbol{T}'(s, \cdot)}\,(= \boldsymbol{T}(\cdot, s))$ for all s in $\mathbb{R}$ for which $\boldsymbol{T}'(s, \cdot) \in L^2$. With each bi-Carleman kernel $\boldsymbol{T}$, we therefore associate the pair of Carleman functions $\boldsymbol{t}, \boldsymbol{t}' : \mathbb{R} \to L^2$, both defined, via $\boldsymbol{T}$, as above.

An integral operator whose kernel is Carleman (resp., bi-Carleman) is referred to as the *Carleman* (resp., *bi-Carleman*) operator. The integral operator T is called *bi-integral* if its adjoint T^* is also an integral operator; in that case if $\boldsymbol{T^*}$ is the kernel of T^* then, in the above notation, $\boldsymbol{T^*}(s, t) = \boldsymbol{T}'(s, t)$ for almost all $(s, t) \in \mathbb{R}^2$ (see, e.g., [15, Theorem 7.5]). A bi-Carleman operator is always a bi-integral operator, but not conversely. The bi-integral operators are generally involved in second-kind integral equations (like (1)) in L^2, as the adjoint equations to such equations are customarily required to be integral. The nuclear operators of $\mathfrak{R}(L^2)$ are examples of integral operators with Hilbert-Schmidt kernels (see, e.g., [16, Theorem VI.23, pp. 210–211]).

We shall employ the convention of referring to integral operators by italic caps and to the corresponding kernels (resp., Carleman functions) by the same letter, but written in upper case (resp., lower case) bold-face type. Thus, e.g., if T denotes, say, a bi-Carleman operator, then $\boldsymbol{T}$ and $\boldsymbol{t}$ and $\boldsymbol{t}'$ are to be used to denote its kernel and two Carleman functions, respectively.

We conclude this subsection by recalling an important algebraic property of Carleman operators which will be exploited frequently throughout the text, a property which is the content of the following so-called "Right-Multipilication Lemma" (see [17], [11, Corollary IV.2.8], or [15, Theorem 11.6]):

Proposition 2. *Let T be a Carleman operator, let $\boldsymbol{t}$ be the Carleman function associated with the inducing Carleman kernel of T, and let $A \in \mathfrak{R}(L^2)$ be arbitrary. Then the product operator TA is also a Carleman operator, and the composition function*

$$A^*(\boldsymbol{t}(\cdot)) : \mathbb{R} \to L^2 \tag{9}$$

is the Carleman function associated with its kernel.

2.3 K^0-kernels

If k is in $\mathbb{N}$ and B is a Banach space with norm $\|\cdot\|_B$, let $C(\mathbb{R}^k, B)$ denote the Banach space, with the norm $\|f\|_{C(\mathbb{R}^k, B)} = \sup_{x \in \mathbb{R}^k} \|f(x)\|_B$, of all continuous functions f from $\mathbb{R}^k$ into B such that $\lim_{|x| \to \infty} \|f(x)\|_B = 0$, where $|\cdot|$ is the euclidian norm in $\mathbb{R}^k$. Given an equivalence class $f \in L^2$ containing a function of $C(\mathbb{R}, \mathbb{C})$, the symbol $[f]$ is used to mean that function.

Among all possible bi-Carleman kernels on $\mathbb{R}^2$, the following definition distinguishes a special type: those which, together with their associated Carleman functions, are continuous and vanish at infinity.

Definition 3. A bi-Carleman kernel $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{C}$ is called a K^0-*kernel* if the following three conditions are satisfied:

(i) the function $\boldsymbol{T}$ is in $C\left(\mathbb{R}^2, \mathbb{C}\right)$,

(ii) the Carleman function $\boldsymbol{t}$ associated with $\boldsymbol{T}$, $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s, \cdot)}$, is in $C\left(\mathbb{R}, L^2\right)$,

(iii) the Carleman function $\boldsymbol{t}'$ associated with the conjugate transpose $\boldsymbol{T}'$ of $\boldsymbol{T}$, $\boldsymbol{t}'(s) = \overline{\boldsymbol{T}'(s, \cdot)} = \boldsymbol{T}(\cdot, s)$, is in $C\left(\mathbb{R}, L^2\right)$.

What follows is a brief discussion of some properties of K^0-kernels relevant for this paper. In the first place, note that the conditions figuring in Definition 3 do not depend on each other in general; it is therefore natural to discuss the role played by each of them separately. The more restrictive of these conditions is (i), in the sense that it rules out the possibility for any K^0-kernel (unless that kernel is identically zero) of being a function depending only on the sum, difference, or product of the variables; there are many other less trivial examples of inadmissible dependences. This circumstance may be of use in constructing examples of those bi-Carleman kernels that have both the properties (ii) and (iii), but do not enjoy (i); for another reason of existence of such type bi-Carleman kernels, we refer to the remark made in [18, p. 115] regarding boundedly supported kernels. Conversely it can be asserted, e.g., that if a function $\boldsymbol{T} \in C\left(\mathbb{R}^2, \mathbb{C}\right)$ additionally satisfies for all $(s, t) \in \mathbb{R}^2$ the inequality $|\boldsymbol{T}(s, t)| \leqslant p(s)q(t)$, with p, q being non-negative $C(\mathbb{R}, \mathbb{R})$-functions square integrable over $\mathbb{R}$, then $\boldsymbol{T}$ is a K^0-kernel, i.e., the Carleman functions, $\boldsymbol{t}$ and $\boldsymbol{t}'$, it induces are both in $C\left(\mathbb{R}, L^2\right)$. This assertion, obviously, pertains only to Hilbert-Schmidt kernels, and may be proved by an extension from the positive definite case with $p(s) \equiv q(s) \equiv (\boldsymbol{T}(s, s))^{\frac{1}{2}}$ to this general case of Buescu's argument in [19, pp. 247–249].

A few remarks are in order here concerning what can immediately be inferred from the $C\left(\mathbb{R}, L^2\right)$-behaviour of the Carleman functions, $\boldsymbol{t}$ and $\boldsymbol{t}'$, associated with a given K^0-kernel, $\boldsymbol{T}$ (thought of as a kernel of an integral operator $T \in \mathfrak{R}(L^2)$):

1) The images of $\mathbb{R}$ under $\boldsymbol{t}$ and $\boldsymbol{t}'$, i.e.,

$$\boldsymbol{t}(\mathbb{R}) := \bigcup_{s \in \mathbb{R}} \boldsymbol{t}(s) \quad \text{and} \quad \boldsymbol{t}'(\mathbb{R}) := \bigcup_{s \in \mathbb{R}} \boldsymbol{t}'(s), \tag{10}$$

are relatively compact sets in L^2.

2) The *Carleman norm-functions* $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$, defined on $\mathbb{R}$ by $\boldsymbol{\tau}(s) = \|\boldsymbol{t}(s)\|$ and $\boldsymbol{\tau}'(s) = \|\boldsymbol{t}'(s)\|$, respectively, are continuous and vanish at infinity, i.e.,

$$\boldsymbol{\tau}, \boldsymbol{\tau}' \in C(\mathbb{R}, \mathbb{R}). \tag{11}$$

3) The images Tf and T^*f of any $f \in L^2$ under T and T^*, respectively, have $C(\mathbb{R}, \mathbb{C})$-representatives in L^2, $[Tf]$ and $[T^*f]$, given pointwise by

$$[Tf](s) = \langle f, \boldsymbol{t}(s) \rangle, \quad [T^*f](s) = \langle f, \boldsymbol{t}'(s) \rangle \quad \text{at each } s \text{ in } \mathbb{R}. \tag{12}$$

4) The n-th iterant $\boldsymbol{T}^{[n]}$ ($n \geqslant 2$) of the K^0-kernel $\boldsymbol{T}$,

$$\boldsymbol{T}^{[n]}(s, t) :=$$

$$\underbrace{\int \ldots \int}_{n-1} \boldsymbol{T}(s, x_1) \ldots \boldsymbol{T}(x_{n-1}, t) \, dx_1 \ldots dx_{n-1} \left(= \langle T^{n-2}\left(\boldsymbol{t}'(t)\right), \boldsymbol{t}(s) \rangle \right), \tag{13}$$

is a K^0-kernel that defines the integral operator T^n. More generally, every two K^0-kernels $\boldsymbol{P}$ and $\boldsymbol{Q}$ might be said to be *multipliable* with each other, in the sense that their convolution

$$\boldsymbol{C}(s,t) := \int \boldsymbol{P}(s,x)\boldsymbol{Q}(x,t)\, dx\, (= \langle \boldsymbol{q}'(t), \boldsymbol{p}(s)\rangle)$$

exists at every point $(s,t) \in \mathbb{R}^2$, and forms a K^0-kernel that defines the product operator $C = PQ$. Indeed,

$$
\begin{aligned}
\int \langle \boldsymbol{q}'(t), \boldsymbol{p}(s)\rangle f(t)\, dt &= \int \left(\int \boldsymbol{P}(s,x)\boldsymbol{Q}(x,t)\, dx \right) f(t)\, dt = \langle f, Q^*(\boldsymbol{p}(s))\rangle \\
&= \langle Qf, \boldsymbol{p}(s)\rangle = \int \boldsymbol{P}(s,x) \left(\int \boldsymbol{Q}(x,t)f(t)\, dt \right) dx = [PQf](s)
\end{aligned}
\tag{14}
$$

for every f in L^2 and every s in $\mathbb{R}$. Since both $\boldsymbol{p}$ and $\boldsymbol{q}'$ are in $C(\mathbb{R}, L^2)$ and both P and Q are in $\mathfrak{R}(L^2)$, the fact that $\boldsymbol{C}$ satisfies Definition 3 may be derived from the joint continuity of the inner product in its two arguments, which helps in proving (i), and from Proposition 2, according to which

$$\boldsymbol{c}(s) = \overline{\boldsymbol{C}(s,\cdot)} = Q^*(\boldsymbol{p}(s)), \quad \boldsymbol{c}'(s) = \boldsymbol{C}(\cdot,s) = P(\boldsymbol{q}'(s)) \quad \text{for every } s \text{ in } \mathbb{R},$$

which helps in proving both (ii) and (iii).

2.4 Sub-$\boldsymbol{K}^0$-kernels

If $\boldsymbol{T}$ is a K^0-kernel of an integral operator T, then impose on $\boldsymbol{T}$ an extra condition of being of ad hoc parquet-like support:

(iv) there exist positive reals τ_n ($n \in \mathbb{N}$) strictly increasing with n to infinity, $\tau_n \uparrow \infty$ as $n \to \infty$, such that, for each fixed n, the *subkernels of $\boldsymbol{T}$*, $\boldsymbol{T}_n$ and $\widetilde{\boldsymbol{T}}_n$, defined on $\mathbb{R}^2$ by

$$\boldsymbol{T}_n(s,t) = \chi_n(s)\boldsymbol{T}(s,t) \quad \text{and} \quad \widetilde{\boldsymbol{T}}_n(s,t) = \boldsymbol{T}_n(s,t)\chi_n(t), \tag{15}$$

are K^0-kernels, and the integral operators,

$$T_n := P_n T \quad \text{and} \quad \widetilde{T}_n := P_n T P_n, \tag{16}$$

they induce on L^2 are nuclear.

Here in (iv), as in the rest of the paper, χ_n stands for the characteristic function of the open interval $\mathbb{I}_n = (-\tau_n, \tau_n)$, and P_n for an orthogonal projection in L^2, defined on each $f \in L^2$ by $P_n f = \chi_n f$; hence $(I - P_n)f = \widehat{\chi}_n f$ for each $f \in L^2$, with $\widehat{\chi}_n$ standing for the characteristic function of the set $\widehat{\mathbb{I}}_n = \mathbb{R} \setminus \mathbb{I}_n$.

Any K^0-kernel $\boldsymbol{T}$ which satisfies condition (iv) necessarily has to vanish everywhere on the straight lines $s = \pm\tau_n$ and $t = \pm\tau_n$, parallel to the axes of t and s, respectively.

The P_n ($n \in \mathbb{N}$) clearly form a sequence of orthogonal projections increasing to I with respect to the strong operator topology, so that, for every $f \in L^2$,

$$\|(P_n - I)f\| \searrow 0 \quad \text{as } n \to \infty. \tag{17}$$

So it follows immediately from (16) that, as $n \to \infty$,

$$
\begin{aligned}
\|(T_n - T)f\| &\to 0, & \|(\widetilde{T}_n - T)f\| &\to 0, \\
\|(T_n^* - T^*)f\| &\to 0, & \|(\widetilde{T}_n^* - T^*)f\| &\to 0.
\end{aligned}
\tag{18}
$$

Among the subkernels defined in (15), the $\boldsymbol{T}_n$'s have more in common with the original kernel $\boldsymbol{T}$, inasmuch as $[T_n f](s) = \int \boldsymbol{T}(s,t) f(t)\, dt$ for every f in L^2 and every s in $\mathbb{I}_n$, while the tilded subkernels $\widetilde{\boldsymbol{T}}_n$ are more suitable to deal with $\boldsymbol{T}$ being Hermitian, i.e., satisfying $\boldsymbol{T}(s,t) = \overline{\boldsymbol{T}(t,s)}$ for all $s, t \in \mathbb{R}$, because then they all are also Hermitian.

Now we list some basic properties of the subkernels defined in (15), most of which are obvious from the definition (compare [6], [9], [10]):

$$|\boldsymbol{T}_n(s,t)| \leqslant |\boldsymbol{T}(s,t)|, \quad |\widetilde{\boldsymbol{T}}_n(s,t)| \leqslant |\boldsymbol{T}(s,t)|, \quad \text{for all } s, t \in \mathbb{R},$$

$$\lim_{n \to \infty} \|\boldsymbol{T}_n - \boldsymbol{T}\|_{C(\mathbb{R}^2, \mathbb{C})} = 0, \quad \lim_{n \to \infty} \|\widetilde{\boldsymbol{T}}_n - \boldsymbol{T}\|_{C(\mathbb{R}^2, \mathbb{C})} = 0, \tag{19}$$

$$\int\int |\boldsymbol{T}_n(s,t)|^2\, dt\, ds < \infty, \quad \int\int |\widetilde{\boldsymbol{T}}_n(s,t)|^2\, dt\, ds < \infty,$$

$$
\begin{aligned}
\lim_{n \to \infty} \|\boldsymbol{t}_n - \boldsymbol{t}\|_{C(\mathbb{R}, L^2)} &= 0, & \lim_{n \to \infty} \|\boldsymbol{t}'_n - \boldsymbol{t}'\|_{C(\mathbb{R}, L^2)} &= 0, \\
\lim_{n \to \infty} \|\widetilde{\boldsymbol{t}}_n - \boldsymbol{t}\|_{C(\mathbb{R}, L^2)} &= 0, & \lim_{n \to \infty} \|\widetilde{\boldsymbol{t}}'_n - \boldsymbol{t}'\|_{C(\mathbb{R}, L^2)} &= 0,
\end{aligned}
\tag{20}
$$

where, for each n in $\mathbb{N}$, the Carleman functions $\boldsymbol{t}_n$, $\boldsymbol{t}'_n$ and $\widetilde{\boldsymbol{t}}_n$, $\widetilde{\boldsymbol{t}}'_n$, associated to the subkernels $\boldsymbol{T}_n$ and $\widetilde{\boldsymbol{T}}_n$, are defined, as usual, to be

$$
\begin{aligned}
\boldsymbol{t}_n(s) &= \overline{\boldsymbol{T}_n(s, \cdot)} \ (= \chi_n(s)\boldsymbol{t}(s)), & \boldsymbol{t}'_n(t) &= \boldsymbol{T}_n(\cdot, t) \ (= P_n(\boldsymbol{t}'(t))), \\
\widetilde{\boldsymbol{t}}_n(s) &= \overline{\widetilde{\boldsymbol{T}}_n(s, \cdot)} \ (= \chi_n(s)P_n(\boldsymbol{t}(s))), & \widetilde{\boldsymbol{t}}'_n(t) &= \widetilde{\boldsymbol{T}}_n(\cdot, t) \ (= \chi_n(t)P_n(\boldsymbol{t}'(t)))
\end{aligned}
\tag{21}
$$

for every $s, t \in \mathbb{R}$. The limits in (20) all hold due to (ii), (iii), (17), and a result from [12, Lemma 3.7, p. 151]. The result, just referred to, will be used in the text so often that it should be explicitly stated:

Lemma 4. *Let $B_n, B \in \mathfrak{R}(L^2)$, and suppose that for every $f \in L^2$, $\|B_n f - Bf\| \to 0$ as $n \to \infty$. Then for any relatively compact set U in L^2,*

$$\sup_{f \in U} \|B_n f - Bf\| \to 0 \quad \text{as } n \to \infty. \tag{22}$$

Applying this lemma to the sets $\boldsymbol{t}(\mathbb{R})$ and $\boldsymbol{t}'(\mathbb{R})$ of (10) immediately gives that, as $n \to \infty$,

$$\sup_{s \in \mathbb{R}} \|(B_n - B)(\boldsymbol{t}(s))\| \to 0 \quad \text{and} \quad \sup_{t \in \mathbb{R}} \|(B_n - B)(\boldsymbol{t}'(t))\| \to 0, \tag{23}$$

and putting $B_n = P_n$, $B = I$ here causes the three last limits in (20) to be zero.

We would like to close this section with a unitary equivalence result which is essentially contained in Theorem of [20], where it is proved for K^0-kernels supported on $[0, \infty)^2$ (i.e., for integral operators acting in $L^2[0, \infty)$) and with the explicit description of the real sequence $\{\tau_n\}$ (denoted there by "$\{t_n\}$") needed to enforce condition (iv). See also Theorem 1 in [21] and in [22].

Proposition 5. *Suppose that S is a bi-integral operator on L^2. Then there exists a unitary operator $U \colon L^2 \to L^2$ such that the operator $T = USU^{-1}$ is a bi-Carleman operator on L^2, whose kernel is a K^0-kernel satisfying condition* (iv).

By virtue of this result, one can confine one's attention (with no loss of generality) to second-kind integral equations (1) in which the kernel $\boldsymbol{T}$ possesses all the properties (i)-(iv). These four assumptions on $\boldsymbol{T}$ will remain in force for the rest of the paper, and the notations given in condition (iv) will be used frequently without warning.

3 Constructing resolvent $\boldsymbol{K}^0$-kernels

3.1 Resolvent kernels for $\boldsymbol{K}^0$-kernels

We start with the definition of the resolvent kernel for a K^0-kernel, which differs from the general one given at the beginning of the introduction at least in that it does not make explicit use of the notion of Fredholm resolvent.

Definition 6. Let $\boldsymbol{T}$ be a K^0-kernel, let λ be a complex number, and suppose that a K^0-kernel, to be denoted by $\boldsymbol{T}_{|\lambda}$, satisfies, for all s and t in $\mathbb{R}$, the two simultaneous integral equations

$$\boldsymbol{T}_{|\lambda}(s,t) - \lambda \int \boldsymbol{T}(s,x)\boldsymbol{T}_{|\lambda}(x,t)\,dx = \boldsymbol{T}(s,t), \tag{24}$$

$$\boldsymbol{T}_{|\lambda}(s,t) - \lambda \int \boldsymbol{T}_{|\lambda}(s,x)\boldsymbol{T}(x,t)\,dx = \boldsymbol{T}(s,t), \tag{25}$$

and the condition that for any f in L^2,

$$\int \left| \int \boldsymbol{T}_{|\lambda}(s,t)f(t)\,dt \right|^2 ds < \infty. \tag{26}$$

Then the K^0-kernel $\boldsymbol{T}_{|\lambda}$ will be called the *resolvent kernel* for $\boldsymbol{T}$ at λ, and the functions $\boldsymbol{t}_{|\lambda}$ and $\boldsymbol{t}'_{|\lambda}$ of $C(\mathbb{R}, L^2)$, defined via $\boldsymbol{T}_{|\lambda}$ by $\boldsymbol{t}_{|\lambda}(s) = \overline{\boldsymbol{T}_{|\lambda}(s, \cdot)}$ and $\boldsymbol{t}'_{|\lambda}(t) = \boldsymbol{T}_{|\lambda}(\cdot, t)$, will be called the *resolvent Carleman functions* for $\boldsymbol{T}$ at λ.

Theorem 7. *Let $T \in \mathfrak{R}(L^2)$ be an integral operator, with a kernel $\boldsymbol{T}$ that is a K^0-kernel, and let λ be a complex number. Then* (a) *if λ is a regular value for T, then the resolvent kernel for $\boldsymbol{T}$ exists at λ, and is a kernel of the Fredholm resolvent of T at λ, i.e., $\left(T_{|\lambda}f\right)(s) = \int \boldsymbol{T}_{|\lambda}(s,t)f(t)\,dt$ for every f in L^2 and almost every s in $\mathbb{R}$;* (b) *if the resolvent kernel for $\boldsymbol{T}$ exists at λ, then λ is a regular value for T.*

Proof. To prove statement (a), let λ be an arbitrary but fixed regular value for T ($\lambda \in \Pi(T)$), and define two functions $\boldsymbol{a}$, $\boldsymbol{a}' \colon \mathbb{R} \to L^2$ by writing

$$\boldsymbol{a}(s) = \left(\bar{\lambda}(T_{|\lambda})^* + I \right) (\boldsymbol{t}(s)), \quad \boldsymbol{a}'(s) = \left(\lambda T_{|\lambda} + I \right) (\boldsymbol{t}'(s)) \tag{27}$$

whenever $s \in \mathbb{R}$. So defined, $\boldsymbol{a}$ and $\boldsymbol{a}'$ then belong to the space $C\left(\mathbb{R}, L^2 \right)$, as $\boldsymbol{t}$ and $\boldsymbol{t}'$ (the Carleman functions associated to the K^0-kernel $\boldsymbol{T}$) are in $C\left(\mathbb{R}, L^2 \right)$, and $T_{|\lambda}$ (the Fredholm resolvent of T at λ) is in $\mathfrak{R}(L^2)$.

The functions $\boldsymbol{A}$, $\boldsymbol{A}' \colon \mathbb{R}^2 \to \mathbb{C}$, given by the formulae

$$\begin{aligned}
\boldsymbol{A}(s,t) &= \lambda \left\langle \boldsymbol{t}'(t), \boldsymbol{a}(s) \right\rangle + \boldsymbol{T}(s,t), \\
\boldsymbol{A}'(s,t) &= \bar{\lambda} \overline{\left\langle \boldsymbol{a}'(s), \boldsymbol{t}(t) \right\rangle} + \overline{\boldsymbol{T}(t,s)},
\end{aligned} \tag{28}$$

then belong to the space $C\left(\mathbb{R}^2, \mathbb{C} \right)$, due to the continuity of the inner product as a function from $L^2 \times L^2$ to $\mathbb{C}$. By using (27) it is also seen from (28) that these functions are conjugate transposes of each other, i.e., $\boldsymbol{A}'(s,t) = \overline{\boldsymbol{A}(t,s)}$ for all s, $t \in \mathbb{R}$. Simple manipulations involving formulae (28), (12), and (27) give rise to the following two strings of equations, satisfied at all points s in $\mathbb{R}$ by each function f in L^2:

$$\begin{aligned}
\int \boldsymbol{A}(s,t) f(t) \, dt &= \lambda \int \left\langle \boldsymbol{t}'(t), \boldsymbol{a}(s) \right\rangle f(t) \, dt + \int \boldsymbol{T}(s,t) f(t) \, dt \\
&= \left\langle f, \bar{\lambda} T^*(\boldsymbol{a}(s)) + \boldsymbol{t}(s) \right\rangle = \left\langle f, \boldsymbol{a}(s) \right\rangle, \\
\int \boldsymbol{A}'(s,t) f(t) \, dt &= \bar{\lambda} \int \overline{\left\langle \boldsymbol{a}'(s), \boldsymbol{t}(t) \right\rangle} f(t) \, dt + \int \overline{\boldsymbol{T}(t,s)} f(t) \, dt \\
&= \left\langle f, \lambda T(\boldsymbol{a}'(s)) + \boldsymbol{t}'(s) \right\rangle = \left\langle f, \boldsymbol{a}'(s) \right\rangle.
\end{aligned}$$

The equality of the extremes of each of these strings implies that $\overline{\boldsymbol{A}(s, \cdot)} \in \boldsymbol{a}(s)$ and $\boldsymbol{A}(\cdot, s) \in \boldsymbol{a}'(s)$ for every fixed s in $\mathbb{R}$. Furthermore, the following relations hold whenever f is in L^2:

$$\begin{aligned}
\int \boldsymbol{A}(\cdot, t) f(t) \, dt &= \left\langle f, \boldsymbol{a}(\cdot) \right\rangle = \left\langle \left(\lambda T_{|\lambda} + I \right) f, \boldsymbol{t}(\cdot) \right\rangle \\
&= \left\langle R_\lambda(T) f, \boldsymbol{t}(\cdot) \right\rangle = (T R_\lambda(T) f)(\cdot) = \left(T_{|\lambda} f \right)(\cdot) \in L^2,
\end{aligned} \tag{29}$$

showing that the Fredholm resolvent $T_{|\lambda}$ of T at λ is an integral operator on L^2, with the function $\boldsymbol{A}$ as its kernel (compare this with (26)).

The inner product when written in the integral form and the above observations about $\boldsymbol{A}$ allow the defining relationships for $\boldsymbol{A}$ and $\boldsymbol{A}'$ (see (28)) to be respectively written as the integral equations

$$\boldsymbol{A}(s,t) = \lambda \int \boldsymbol{A}(s,x) \boldsymbol{T}(x,t) \, dx + \boldsymbol{T}(s,t),$$

$$\boldsymbol{A}(s,t) = \lambda \int \boldsymbol{T}(s,x) \boldsymbol{A}(x,t) \, dx + \boldsymbol{T}(s,t),$$

holding for all s, $t \in \mathbb{R}$. Together with (29), these imply that the K^0-kernel $\boldsymbol{A}$ is a resolvent kernel for $\boldsymbol{T}$ at λ (in the sense of Definition 6).

To prove statement (b), let there exist a K^0-kernel $\boldsymbol{T}_{|\lambda}$ satisfying (24) through (26). It is to be proved that λ belongs to $\Pi(T)$, i.e., that the operator $I - \lambda T$ is invertible. To this effect, therefore, remark first that the integral operator A given by $(Af)(s) = \int \boldsymbol{T}_{|\lambda}(s,t)f(t)\,dt$ is bounded from L^2 into L^2, owing to condition (26) and to the Banach Theorem (see [15, p. 14]). Then, due to the multipliability property of K^0-kernels (see (14)), the kernel-function equations (24) and (25) give rise to the operator equalities $(I - \lambda T)A = T$ and $A(I - \lambda T) = T$, respectively. The latter are easily seen to be equivalent respectively to the following ones $(I - \lambda T)(I + \lambda A) = I$ and $(I + \lambda A)(I - \lambda T) = I$, which together imply that the operator $I - \lambda T$ is invertible with inverse $I + \lambda A$. The theorem is proved.   □

Remark 8. The proof just given establishes that resolvent kernels in the sense of Definition 6 are in one-to-one correspondence with Fredholm resolvents. In view of this correspondence: (1) $\Pi(T)$ might as well be defined as the set of all those $\lambda \in \mathbb{C}$ at which the resolvent kernel in the sense of Definition 6 exists (thus, whenever $\boldsymbol{T}_{|\lambda}$, $\boldsymbol{t}_{|\lambda}$, or $\boldsymbol{t}'_{|\lambda}$ appear in what follows, it may and will always be understood that λ belongs to $\Pi(T)$); (2) the resolvent kernel $\boldsymbol{T}_{|\lambda}$ for the K^0-kernel $\boldsymbol{T}$ at λ might as well be defined as that K^0-kernel which induces $T_{|\lambda}$, the Fredholm resolvent at λ of that integral operator T whose kernel is $\boldsymbol{T}$. Using (3) and (9), the values of the resolvent Carleman functions for $\boldsymbol{T}$ at each fixed regular value $\lambda \in \Pi(T)$ can therefore be ascertained by writing

$$\boldsymbol{t}_{|\lambda}(\cdot) = R^*_\lambda(T)(\boldsymbol{t}(\cdot)), \quad \boldsymbol{t}'_{|\lambda}(\cdot) = R_\lambda(T)\,(\boldsymbol{t}'(\cdot)), \tag{30}$$

where $\boldsymbol{t}$ and $\boldsymbol{t}'$ are Carleman functions corresponding to $\boldsymbol{T}$ (compare with (27) via (4)). The resolvent kernel $\boldsymbol{T}_{|\lambda}$ for $\boldsymbol{T}$, in its turn, can be exactly recovered from the knowledge of the resolvent Carleman functions $\boldsymbol{t}_{|\lambda}$ and $\boldsymbol{t}'_{|\lambda}$ by the formulae

$$\begin{aligned}
\overline{\boldsymbol{T}_{|\lambda}(s,t)} &= \bar\lambda\left\langle \boldsymbol{t}_{|\lambda}(s), \boldsymbol{t}'(t)\right\rangle + \overline{\boldsymbol{T}(s,t)}, \\
\boldsymbol{T}_{|\lambda}(s,t) &= \lambda\left\langle \boldsymbol{t}'_{|\lambda}(t), \boldsymbol{t}(s)\right\rangle + \boldsymbol{T}(s,t),
\end{aligned} \tag{31}$$

respectively (compare with (28)). Formulae (30) and (31) will be of use in the subsequent analysis.

3.2   Resolvent kernels for sub-K^0-kernels

Here, as subsequently, we shall denote the resolvent kernels at λ for the subkernel $\boldsymbol{T}_n$ (resp., $\widetilde{\boldsymbol{T}}_n$) by $\boldsymbol{T}_{n|\lambda}$ (resp., $\widetilde{\boldsymbol{T}}_{n|\lambda}$), and the resolvent Carleman functions for these subkernels at λ by $\boldsymbol{t}_{n|\lambda}$ and $\boldsymbol{t}'_{n|\lambda}$ (resp., $\widetilde{\boldsymbol{t}}_{n|\lambda}$ and $\widetilde{\boldsymbol{t}}'_{n|\lambda}$). Then the following formulae are none other than valid versions of (30) and (31) for $\boldsymbol{t}_{n|\lambda_n}$, $\boldsymbol{t}'_{n|\lambda_n}$, and $\boldsymbol{T}_{n|\lambda_n}$; all are developed making use of (21):

$$\boldsymbol{t}_{n|\lambda_n}(s) = \overline{\boldsymbol{T}_{n|\lambda_n}(s,\cdot)} = R^*_{\lambda_n}(T_n)\,(\boldsymbol{t}_n(s)) = \chi_n(s)R^*_{\lambda_n}(T_n)\,(\boldsymbol{t}(s)), \tag{32}$$

$$\boldsymbol{t}'_{n|\lambda_n}(t) = \boldsymbol{T}_{n|\lambda_n}(\cdot,t) = R_{\lambda_n}(T_n)\,(\boldsymbol{t}'_n(t)) = R_{\lambda_n}(T_n)\,P_n\,(\boldsymbol{t}'(t)), \tag{33}$$

$$\begin{aligned}
\overline{\boldsymbol{T}_{n|\lambda_n}(s,t)} &= \bar\lambda_n\left\langle \boldsymbol{t}_{n|\lambda_n}(s), P_n\,(\boldsymbol{t}'(t))\right\rangle + \overline{\boldsymbol{T}_n(s,t)}, \\
\boldsymbol{T}_{n|\lambda_n}(s,t) &= \lambda_n\chi_n(s)\left\langle \boldsymbol{t}'_{n|\lambda_n}(t), \boldsymbol{t}(s)\right\rangle + \boldsymbol{T}_n(s,t),
\end{aligned} \tag{34}$$

where s, $t \in \mathbb{R}$ are arbitrary. It is readily seen from (15) and (34) that each K^0-kernel $\boldsymbol{T}_{n|\lambda}(s,t)$ has bounded s-support (namely, lying in $[-\tau_n, \tau_n]$), so the condition (26) of Definition 6 is automatically satisfied with $\boldsymbol{T}_{n|\lambda}$ in the role of $\boldsymbol{T}_{|\lambda}$. Thus, in this role, $\boldsymbol{T}_{n|\lambda}$ is the only solution of the simultaneous integral equations (24) and (25) (with $\boldsymbol{T}$ replaced by $\boldsymbol{T}_n$) which is a K^0-kernel. The problem of explicitly finding that solution in terms of $\boldsymbol{T}_n$ is completely solved via the Fredholm-determinant method, as follows. For $\boldsymbol{T}_n$ a subkernel of $\boldsymbol{T}$, consider its Fredholm determinant $D_{\boldsymbol{T}_n}(\lambda)$ defined by the series

$$D_{\boldsymbol{T}_n}(\lambda) := 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \ldots \int \boldsymbol{T}_n \begin{pmatrix} x_1 & \cdots & x_m \\ x_1 & \cdots & x_m \end{pmatrix} dx_1 \ldots dx_m, \qquad (35)$$

for every $\lambda \in \mathbb{C}$, and its first Fredholm minor $D_{\boldsymbol{T}_n}(s,t \mid \lambda)$ defined by the series

$$D_{\boldsymbol{T}_n}(s,t \mid \lambda) = \boldsymbol{T}_n(s,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \ldots \int \boldsymbol{T}_n \begin{pmatrix} s & x_1 & \cdots & x_m \\ t & x_1 & \cdots & x_m \end{pmatrix} dx_1 \ldots dx_m, \quad (36)$$

for all points s, $t \in \mathbb{R}$ and for every $\lambda \in \mathbb{C}$, where

$$\boldsymbol{T}_n \begin{pmatrix} x_1 & \cdots & x_\nu \\ y_1 & \cdots & y_\nu \end{pmatrix} := \det \begin{pmatrix} \boldsymbol{T}_n(x_1, y_1) & \cdots & \boldsymbol{T}_n(x_1, y_\nu) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \boldsymbol{T}_n(x_\nu, y_1) & \cdots & \boldsymbol{T}_n(x_\nu, y_\nu) \end{pmatrix}.$$

The next proposition can reliably be inferred from results of the Carleman-Mikhlin-Smithies theory of the Fredholm determinant and the first Fredholm minor for the Hilbert-Schmidt kernels with possibly unbounded support (see [23], [24], and [25]).

Proposition 9. *Let $\lambda \in \mathbb{C}$ be arbitrary but fixed. Then*

1) the series of (35) is absolutely convergent in $\mathbb{C}$, and the series of (36) is absolutely convergent in $C(\mathbb{R}^2, \mathbb{C})$ and in $L^2(\mathbb{R}^2)$;

2) if $D_{\boldsymbol{T}_n}(\lambda) \neq 0$ then resolvent kernel for $\boldsymbol{T}_n$ at λ exists and is the quotient of the first Fredholm minor and the Fredholm determinant:

$$\boldsymbol{T}_{n|\lambda}(s,t) \equiv \frac{D_{\boldsymbol{T}_n}(s,t \mid \lambda)}{D_{\boldsymbol{T}_n}(\lambda)}; \qquad (37)$$

3) if $D_{\boldsymbol{T}_n}(\lambda) = 0$, then the resolvent kernel for $\boldsymbol{T}_n$ does not exist at λ.

For each $n \in \mathbb{N}$, therefore, the characteristic set $\Lambda(T_n)$ is composed of all the zeros of the entire function $D_{\boldsymbol{T}_n}(\lambda)$, and is at most a denumerable set of complex numbers clustering at ∞. For $\widetilde{\boldsymbol{T}}_{n|\lambda}$, a formula like (37) is built up in the same way but replacing $\boldsymbol{T}_n$ by $\widetilde{\boldsymbol{T}}_n$. Since $\widetilde{T}_n^m = T_n^m P_n$ for $m \in \mathbb{N}$, the m-th iterants of $\widetilde{\boldsymbol{T}}_n$ and $\boldsymbol{T}_n$ (see (13)) stand therefore in a similar relation to each other, namely: $\widetilde{\boldsymbol{T}}_n^{[m]}(s,t) = \chi_n(t)\boldsymbol{T}_n^{[m]}(s,t)$ for all s, $t \in \mathbb{R}$. Then it follows from the rules for calculating the coefficients of powers of λ in the Fredholm series (35) and (36) that $D_{\widetilde{\boldsymbol{T}}_n}(\lambda) \equiv D_{\boldsymbol{T}_n}(\lambda)$ and $D_{\widetilde{\boldsymbol{T}}_n}(s,t \mid \lambda) \equiv \chi_n(t)D_{\boldsymbol{T}_n}(s,t \mid \lambda)$. Hence, for each n,

$$\Lambda(T_n) = \Lambda(\widetilde{T}_n), \quad \Pi(T_n) = \Pi(\widetilde{T}_n), \qquad (38)$$

$$\widetilde{\boldsymbol{T}}_{n|\lambda}(s,t) = \frac{D_{\widetilde{\boldsymbol{T}}_n}(s,t \mid \lambda)}{D_{\widetilde{\boldsymbol{T}}_n}(\lambda)} = \chi_n(t)\boldsymbol{T}_{n|\lambda}(s,t), \qquad (39)$$

$$\widetilde{\boldsymbol{t}}_{n|\lambda}(s) = P_n\left(\boldsymbol{t}_{n|\lambda}(s)\right), \quad \widetilde{\boldsymbol{t}}'_{n|\lambda}(t) = \chi_n(t)\boldsymbol{t}'_{n|\lambda}(t). \qquad (40)$$

3.3   The main result

Given an arbitrary sequence $\{\lambda_n\}_{n=1}^\infty$ of complex numbers satisfying $\lambda_n \in \Pi(T_n)$ for each n and converging to some $\lambda \in \mathbb{C}$, the $C(\mathbb{R}^2, \mathbb{C})$-valued sequence of the resolvent kernels

$$\left\{ \boldsymbol{T}_{n|\lambda_n} \right\}_{n=1}^\infty, \tag{41}$$

all of whose terms are known explicitly in terms of the original K^0-kernel $\boldsymbol{T}$ via the Fredholm formulae (35)-(37), and the $C(\mathbb{R}, L^2)$-valued sequences of the respective Carleman functions

$$\left\{ \boldsymbol{t}_{n|\lambda_n} \right\}_{n=1}^\infty, \quad \left\{ \boldsymbol{t}'_{n|\lambda_n} \right\}_{n=1}^\infty \tag{42}$$

are not known to converge in general. If they did converge, relevant questions would be, e.g.: if the sequence (41) converges in $C(\mathbb{R}^2, \mathbb{C})$, possibly up to the extraction of a subsequence, to a function $\boldsymbol{A}$ say, whether λ is necessarily a regular value for T, and if λ does turn out to belong to $\Pi(T)$, whether $\boldsymbol{A} = \boldsymbol{T}_{|\lambda}$. Similar questions can be asked concerning the sequences of (42), but we postpone them all to a later paper. The (in a sense converse) question we deal with in this paper is: given that the above λ is a (non-zero) regular value for T, what further connections between $\{\lambda_n\}$ and λ guarantee the existence, in suitable senses, of the limit-relations

$$\boldsymbol{t}_{|\lambda} = \lim_{n\to\infty} \boldsymbol{t}_{n|\lambda_n}, \quad \boldsymbol{t}'_{|\lambda} = \lim_{n\to\infty} \boldsymbol{t}'_{n|\lambda_n}, \quad \boldsymbol{T}_{|\lambda} = \lim_{n\to\infty} \boldsymbol{T}_{n|\lambda_n}?$$

In the theorem which follows, we characterize one such connection by means of sets like $\nabla_\mathfrak{s}$, defined at the end of Subsection 2.1.

Theorem 10.   *Let $\{\beta_n\}_{n=1}^\infty$ be an arbitrary sequence of complex numbers satisfying*

$$\lim_{n\to\infty} \beta_n = 0, \tag{43}$$

and define $\lambda_n(\lambda) := \lambda(1-\beta_n\lambda)^{-1}$, so that one can consider that $\lambda_n(\lambda) \to \lambda$ when $n \to \infty$ for each fixed $\lambda \in \mathbb{C}$. Then $\varnothing \neq \nabla_\mathfrak{s}(\{\beta_n I + T_n\}) \subseteq \nabla_\mathfrak{s}(\{\beta_n I + \widetilde{T}_n\}) \subseteq \Pi(T)$ and the following limits hold:

$$\boldsymbol{t}'_{|\lambda}(t) = \lim_{n\to\infty} \boldsymbol{t}'_{n|\lambda_n(\lambda)}(t) \quad (\lambda \in \nabla_\mathfrak{s}(\{\beta_n I + \widetilde{T}_n\}),\ t \in \mathbb{R}), \tag{44}$$

$$\boldsymbol{t}_{|\lambda}(s) = \lim_{n\to\infty} \boldsymbol{t}_{n|\lambda_n(\lambda)}(s) \quad (\lambda \in \nabla_\mathfrak{s}(\{\beta_n I + T_n\}),\ s \in \mathbb{R}), \tag{45}$$

$$\boldsymbol{T}_{|\lambda}(s,t) = \lim_{n\to\infty} \boldsymbol{T}_{n|\lambda_n(\lambda)}(s,t) \quad (\lambda \in \nabla_\mathfrak{s}(\{\beta_n I + \widetilde{T}_n\}),\ (s,t) \in \mathbb{R}^2), \tag{46}$$

where:

   (a) *the convergence in (44) is in the $C(\mathbb{R}, L^2)$ norm for each fixed $\lambda \in \nabla_\mathfrak{s}(\{\beta_n I + \widetilde{T}_n\})$ (see (63)), and is uniform in λ on every compact subset $\widetilde{\mathfrak{K}}$ of $\nabla_\mathfrak{s}(\{\beta_n I + \widetilde{T}_n\})$ for each fixed $t \in \mathbb{R}$ (see (64));*

   (b) *the convergence in (45) is in the $C(\mathbb{R}, L^2)$ norm for each fixed $\lambda \in \nabla_\mathfrak{s}(\{\beta_n I + T_n\})$ (see (65)), and is uniform in λ on every compact subset $\mathfrak{K}$ of $\nabla_\mathfrak{s}(\{\beta_n I + T_n\})$ for each fixed $s \in \mathbb{R}$ (see (66)); and*

   (c) *the convergence in (46) is in the $C(\mathbb{R}^2, \mathbb{C})$ norm for each fixed $\lambda \in \nabla_\mathfrak{s}(\{\beta_n I + \widetilde{T}_n\})$ (see (67)), and is uniform in λ on every compact subset $\widetilde{\mathfrak{K}}$ of $\nabla_\mathfrak{s}(\{\beta_n I + \widetilde{T}_n\})$ for each fixed $(s,t) \in \mathbb{R}^2$ (see (68)).*

Proof. Let us begin by collecting (mainly from [12]) some preparatory results, to be numbered below from (48) to (58). To simplify the notation, write $A_n := \beta_n I + T_n$, $\widetilde{A}_n := \beta_n I + \widetilde{T}_n$. Choose a (non-zero) regular value $\zeta \in \Pi(T)$ satisfying $|\zeta|\,\|T\| < 1$, and hence satisfying for some $N(\zeta) > 0$ the inequality

$$|\zeta|\,\|A_n\| \leqslant |\zeta| \left(\max_{n>N(\zeta)} |\beta_n| + \|T\| \right) < 1 \quad \text{for all } n > N(\zeta). \tag{47}$$

Then ζ does belong to $\nabla_\flat(\{A_n\})$, because

$$\|A_{n|\zeta}\| \leqslant \frac{\|A_n\|}{1 - |\zeta|\,\|A_n\|} \leqslant M(\zeta) = \frac{\displaystyle\max_{n>N(\zeta)} \|A_n\|}{1 - |\zeta| \left(\displaystyle\max_{n>N(\zeta)} |\beta_n| + \|T\| \right)} \quad \text{for all } n > N(\zeta)$$

(compare (8)). The result is that the intersection of $\nabla_\flat(\{A_n\})$ and $\Pi(T)$ is non-void. Similarly it can be shown that $\nabla_\flat(\{\widetilde{A}_n\}) \cap \Pi(T) \neq \varnothing$. Therefore, since, because of (43) and (18), the sequences $\{A_n\}$ and $\{\widetilde{A}_n\}$ both converge to T in the strong operator topology, it follows by the criterion for generalized strong convergence (see [12, Theorem VIII-1.5]) that

$$\nabla_\mathfrak{s}(\{A_n\}) = \nabla_\flat(\{A_n\}) \cap \Pi(T), \quad \nabla_\mathfrak{s}(\{\widetilde{A}_n\}) = \nabla_\flat(\{\widetilde{A}_n\}) \cap \Pi(T), \tag{48}$$

$$\lim_{n\to\infty} \left\| \left(A_{n|\lambda} - T_{|\lambda} \right) f \right\| = 0 \quad \text{for all } \lambda \in \nabla_\mathfrak{s}(\{A_n\}) \text{ and } f \in L^2,$$

$$\lim_{n\to\infty} \|(\widetilde{A}_{n|\lambda} - T_{|\lambda})f\| = 0 \quad \text{for all } \lambda \in \nabla_\mathfrak{s}(\{\widetilde{A}_n\}) \text{ and } f \in L^2. \tag{49}$$

Further, given a $\lambda \in \nabla_\flat(\{A_n\}) \cup \nabla_\flat(\{\widetilde{A}_n\})$, the following formulae hold for sufficiently large n:

$$R_\lambda(A_n) = \frac{1}{1 - \beta_n\lambda} R_{\lambda_n(\lambda)}(T_n), \quad R_\lambda(\widetilde{A}_n) = \frac{1}{1 - \beta_n\lambda} R_{\lambda_n(\lambda)}(\widetilde{T}_n),$$

$$R_{\lambda_n(\lambda)}(T_n) = (1 - \beta_n\lambda)\left(I + \lambda A_{n|\lambda} \right) = I + \lambda\beta_n I + \lambda A_{n|\lambda} + \lambda^2\beta_n A_{n|\lambda},$$

$$A_{n|\lambda} = \left(\frac{1}{1 - \beta_n\lambda} \right)^2 T_{n|\lambda_n(\lambda)} + \frac{\beta_n}{1 - \beta_n\lambda} I, \tag{50}$$

$$\widetilde{A}_{n|\lambda} = \left(\frac{1}{1 - \beta_n\lambda} \right)^2 \widetilde{T}_{n|\lambda_n(\lambda)} + \frac{\beta_n}{1 - \beta_n\lambda} I.$$

These are obtained by a purely formal calculation, and use that fact that $\Pi(\widetilde{T}_n) = \Pi(T_n)$ for each fixed $n \in \mathbb{N}$ (see (38)). The equations in the last two lines combine to give, using (39),

$$A_{n|\lambda} P_n = \widetilde{A}_{n|\lambda} + \frac{\beta_n}{1 - \beta_n\lambda}(I - P_n). \tag{51}$$

This implies in particular that $\|\widetilde{A}_{n|\lambda}\| \leqslant \|A_{n|\lambda}\| + |\beta_n||1 - \beta_n\lambda|^{-1}$, whence (43) leads to the inclusion relation $\nabla_\flat(\{A_n\}) \subseteq \nabla_\flat(\{\widetilde{A}_n\})$, from which it follows via (48) that $\varnothing \neq \nabla_\mathfrak{s}(\{A_n\}) \subseteq \nabla_\mathfrak{s}(\{\widetilde{A}_n\}) \subseteq \Pi(T)$, as asserted.

In what follows, let $\widetilde{\mathfrak{K}}$ denote a compact (closed and bounded) subset of $\nabla_{\mathfrak{s}}(\{\widetilde{A}_n\})$. Then, according to Theorem VIII-1.1 in [12] there exists a positive constant $M(\widetilde{\mathfrak{K}})$ such that

$$\sup_{\lambda \in \widetilde{\mathfrak{K}}} \|\widetilde{A}_{n|\lambda}\| \leqslant M(\widetilde{\mathfrak{K}}) \quad \text{for all sufficiently large } n, \tag{52}$$

and, according to Theorem VIII-1.2 therein, the convergence in (49) is uniform over $\widetilde{\mathfrak{K}}$:

$$\lim_{n \to \infty} \sup_{\lambda \in \widetilde{\mathfrak{K}}} \|(\widetilde{A}_{n|\lambda} - T_{|\lambda})f\| = 0 \quad \text{for each fixed } f \in L^2. \tag{53}$$

Now use (52), (53), and the observation from (43) that

$$\sup_{\lambda \in \widetilde{\mathfrak{K}}} \left| \frac{\beta_n}{1 - \beta_n \lambda} \right| \leqslant \frac{|\beta_n|}{1 - |\beta_n| \sup_{\lambda \in \widetilde{\mathfrak{K}}} |\lambda|} \to 0 \quad \text{as } n \to \infty, \tag{54}$$

to infer, via the connecting formula (51), that

$$\lim_{n \to \infty} \sup_{\lambda \in \widetilde{\mathfrak{K}}} \left\| \left(A_{n|\lambda} P_n - T_{|\lambda} \right) f \right\| = 0 \quad \text{for each fixed } f \in L^2, \tag{55}$$

$$\sup_{\lambda \in \widetilde{\mathfrak{K}}} \left\| A_{n|\lambda} P_n \right\| < M(\widetilde{\mathfrak{K}}) + 1 \quad \text{for all sufficiently large } n. \tag{56}$$

Throughout what follows let $\mathfrak{K}$ denote a compact subset of $\nabla_{\mathfrak{s}}(\{A_n\})$. Then Theorem VIII-1.1 in [12], this time applied to the operator sequence $\{A_n\}$, yields the conclusion that there exists a positive constant $M(\mathfrak{K})$ such that

$$\sup_{\lambda \in \mathfrak{K}} \left\| A_{n|\lambda} \right\| \leqslant M(\mathfrak{K}) \quad \text{for all sufficiently large } n, \tag{57}$$

and hence there holds

$$\lim_{n \to \infty} \sup_{\lambda \in \mathfrak{K}} \left\| \left(A_{n|\lambda} - T_{|\lambda} \right)^* f \right\| = 0 \quad \text{for each fixed } f \in L^2. \tag{58}$$

Indeed, given any $f \in L^2$, the following relations hold:

$$\lim_{n \to \infty} \sup_{\lambda \in \mathfrak{K}} \left\| \left(A_{n|\lambda} - T_{|\lambda} \right)^* f \right\|$$
$$= \lim_{n \to \infty} \sup_{\lambda \in \mathfrak{K}} \left\| \left(I + \bar{\lambda} \left(A_{n|\lambda} \right)^* \right) (T - A_n)^* R_\lambda^*(T) f \right\| \qquad \text{by (6)}$$
$$\leqslant \sup_{\lambda \in \mathfrak{K}} (1 + |\lambda| M(\mathfrak{K})) \lim_{n \to \infty} \sup_{\lambda \in \mathfrak{K}} \|(T - A_n)^* R_\lambda^*(T) f\| \qquad \text{by (57)}$$
$$= 0 \qquad \text{by Lemma 4,}$$

in as much as $(A_n)^* \to T^*$ strongly as $n \to \infty$ (see (18), (43)) and the set $\bigcup_{\lambda \in \mathfrak{K}} R_\lambda^*(T)f$ is relatively compact in L^2 (being the image under the continuous map $R_\lambda^*(T)f \colon \Pi(T) \to L^2$ (see Subsection 2.1) of the compact subset $\mathfrak{K}$ of $\Pi(T)$). Similarly, it can be proved that

$$\lim_{n \to \infty} \sup_{\lambda \in \widetilde{\mathfrak{K}}} \left\| P_n \left(A_{n|\lambda} P_n - T_{|\lambda} \right)^* f \right\| = 0 \quad \text{for each fixed } f \in L^2. \tag{59}$$

With these preparations, we are ready to establish that the limit formulae (44)-(46) all hold, each uniformly in two senses, exactly as stated in the enunciation of the theorem. For that purpose, use formulae (33), (32), (30), (50), and then the triangle inequality to formally write

$$
\begin{aligned}
\sup \left\| \boldsymbol{t}'_{n|\lambda_n(\lambda)}(t) - \boldsymbol{t}'_{|\lambda}(t) \right\| & \\
= \sup \Big\| & \left(P_n - I + \lambda \beta_n P_n + \lambda (A_{n|\lambda} P_n - T_{|\lambda}) + \lambda^2 \beta_n A_{n|\lambda} P_n \right) (\boldsymbol{t}'(t)) \Big\| \\
& \leqslant \sup \left(\left\| (P_n - I) (\boldsymbol{t}'(t)) \right\| \right) + \sup \left(|\beta_n| \, |\lambda| \, \| P_n (\boldsymbol{t}'(t)) \| \right) \\
& + \sup \left(|\lambda| \left\| (A_{n|\lambda} P_n - T_{|\lambda}) (\boldsymbol{t}'(t)) \right\| \right) + \sup \left(|\beta_n| \, |\lambda|^2 \left\| A_{n|\lambda} P_n \right\| \boldsymbol{\tau}'(t) \right), \quad (60)
\end{aligned}
$$

$$
\begin{aligned}
\sup \left\| \boldsymbol{t}_{n|\lambda_n(\lambda)}(s) - \boldsymbol{t}_{|\lambda}(s) \right\| & \\
= \sup \Big\| & \left(\chi_n(s) \left(I + \lambda \beta_n I + \lambda A_{n|\lambda} + \lambda^2 \beta_n A_{n|\lambda} \right) - I - \lambda T_{|\lambda} \right)^* (\boldsymbol{t}(s)) \Big\| \\
& \leqslant \sup \left(|\beta_n| \, \chi_n(s) \, |\lambda| \, \boldsymbol{\tau}(s) \right) + \sup \left(\widehat{\chi}_n(s) \boldsymbol{\tau}(s) \right) \\
& + \sup \left(\chi_n(s) \, |\lambda| \left\| (A_{n|\lambda} - T_{|\lambda})^* (\boldsymbol{t}(s)) \right\| \right) + \sup \left(\widehat{\chi}_n(s) \, |\lambda| \, \| T_{|\lambda} \| \, \boldsymbol{\tau}(s) \right) \\
& + \sup \left(|\beta_n| \, \chi_n(s) \, |\lambda|^2 \, \| A_{n|\lambda} \| \, \boldsymbol{\tau}(s) \right) \quad (61)
\end{aligned}
$$

("formally" because we have not specified the domain over which the suprema are being taken). Now use equations (31), (34), and the triangle and the Cauchy-Schwarz inequality to also formally write

$$
\begin{aligned}
\sup \left| \boldsymbol{T}_{n|\lambda_n(\lambda)}(s,t) - \boldsymbol{T}_{|\lambda}(s,t) \right| & \\
= \sup \Big| & \lambda_n(\lambda) \chi_n(s) \left\langle \boldsymbol{t}'_{n|\lambda_n(\lambda)}(t), \boldsymbol{t}(s) \right\rangle - \lambda \left\langle \boldsymbol{t}'_{|\lambda}(t), \boldsymbol{t}(s) \right\rangle + \boldsymbol{T}_n(s,t) - \boldsymbol{T}(s,t) \Big| \\
& \leqslant \sup \left(\chi_n(s) \, |\lambda| \left| \left\langle \boldsymbol{t}'_{n|\lambda_n(\lambda)}(t) - \boldsymbol{t}'_{|\lambda}(t), \boldsymbol{t}(s) \right\rangle \right| \right) \\
& + \sup \left(\chi_n(s) \, |\lambda_n(\lambda) - \lambda| \left| \left\langle \boldsymbol{t}'_{n|\lambda_n(\lambda)}(t), \boldsymbol{t}(s) \right\rangle \right| \right) \\
& + \sup \left(\widehat{\chi}_n(s) \, |\lambda| \left| \left\langle \boldsymbol{t}'_{|\lambda}(t), \boldsymbol{t}(s) \right\rangle \right| \right) + \sup \left| \boldsymbol{T}_n(s,t) - \boldsymbol{T}(s,t) \right| \\
& \leqslant \sup \left(\chi_n(s) \, |\lambda| \left\| \boldsymbol{t}'_{n|\lambda_n(\lambda)}(t) - \boldsymbol{t}'_{|\lambda}(t) \right\| \boldsymbol{\tau}(s) \right) \\
& + \sup \left(\chi_n(s) \, |\lambda|^2 \left| \frac{\beta_n}{1 - \beta_n \lambda} \right| \left\| \boldsymbol{t}'_{n|\lambda_n(\lambda)}(t) \right\| \boldsymbol{\tau}(s) \right) \\
& + \sup \left(\widehat{\chi}_n(s) \, |\lambda| \, \| R_\lambda(T) \| \, \boldsymbol{\tau}'(t) \boldsymbol{\tau}(s) \right) + \sup \left| \boldsymbol{T}_n(s,t) - \boldsymbol{T}(s,t) \right|. \quad (62)
\end{aligned}
$$

(a) For a fixed $\lambda \in \nabla_{\mathfrak{s}}(\{\widetilde{A}_n\})$ take the suprema in (60) over all $t \in \mathbb{R}$. Then each summand on the right-hand side of (60) becomes an n-th term of a null sequence of $C(\mathbb{R}, L^2)$-norm values, by means of (17), (55), (43), and (23). This proves (44) in the following uniform version:

$$
\lim_{n \to \infty} \left\| \boldsymbol{t}'_{n|\lambda_n(\lambda)} - \boldsymbol{t}'_{|\lambda} \right\|_{C(\mathbb{R}, L^2)} = 0 \quad \text{for each fixed } \lambda \in \nabla_{\mathfrak{s}}(\{\beta_n I + \widetilde{T}_n\}). \quad (63)
$$

Next, because of (17), (55), (56), (43), and of the boundedness of the set $\widetilde{\mathfrak{K}}$, the suprema at the right-hand side of (60), all taken, this time, over all $\lambda \in \widetilde{\mathfrak{K}}$, tend as $n \to \infty$ to zero, which proves that the limit (44) holds in the sense that

$$
\lim_{n \to \infty} \sup_{\lambda \in \widetilde{\mathfrak{K}}} \left\| \boldsymbol{t}'_{n|\lambda_n(\lambda)}(t) - \boldsymbol{t}'_{|\lambda}(t) \right\| = 0 \quad \text{for each fixed } t \in \mathbb{R}. \quad (64)
$$

(b) As for the convergence in (45), its uniformity with respect to s,

$$\lim_{n\to\infty} \left\| \boldsymbol{t}_{n|\lambda_n(\lambda)} - \boldsymbol{t}_{|\lambda} \right\|_{C(\mathbb{R},L^2)} = 0 \quad \text{for each fixed } \lambda \in \nabla_{\mathfrak{s}}(\{\beta_n I + T_n\}), \qquad (65)$$

may be proved similarly to (63), first taking the suprema in (61) to be over $\mathbb{R}$ with respect to s and then taking account of (58), (11), (43), and (23).

To see that formula (45) also holds in its asserted form

$$\lim_{n\to\infty} \sup_{\lambda\in\mathfrak{K}} \left\| \boldsymbol{t}_{n|\lambda_n(\lambda)}(s) - \boldsymbol{t}_{|\lambda}(s) \right\| = 0 \quad \text{for each fixed } s \in \mathbb{R}, \qquad (66)$$

extend the suprema in (61) over $\mathfrak{K}$ with respect to λ and then apply (58), (57), (11), and (43) to the right-hand-side terms there.

(c) Two uniform versions claimed in the theorem for the limit (46) are written as

$$\lim_{n\to\infty} \left\| \boldsymbol{T}_{n|\lambda_n(\lambda)} - \boldsymbol{T}_{|\lambda} \right\|_{C(\mathbb{R}^2,\mathbb{C})} = 0 \quad \text{for each fixed } \lambda \in \nabla_{\mathfrak{s}}(\{\beta_n I + \widetilde{T}_n\}), \qquad (67)$$

$$\lim_{n\to\infty} \sup_{\lambda\in\widetilde{\mathfrak{K}}} \left| \boldsymbol{T}_{n|\lambda_n(\lambda)}(s,t) - \boldsymbol{T}_{|\lambda}(s,t) \right| = 0 \quad \text{for each fixed } (s,t) \in \mathbb{R}^2 \qquad (68)$$

and will be proved by directly invoking (62). If the suprema involved therein are taken over all points $(s,t) \in \mathbb{R}^2$, then the above-established relations (19), (63), (54), and (11) together imply that all four terms on the extreme right side of (62) converge to 0 as $n \to \infty$, which proves (67). Similarly, the validity of (68) can be deduced from (64), (19), (11), and (54) upon taking the suprema in (62) (with s and t kept fixed) over all $\lambda \in \mathbb{C}$ belonging to the bounded set $\widetilde{\mathfrak{K}}$. The theorem is proved. $\qquad\square$

Remark 11. Because of the observation at the beginning of the above proof the punctured disk $D_{\|T\|} = \{\lambda \in \mathbb{C} \mid 0 < |\lambda| < \frac{1}{\|T\|}\}$ has the property that

$$D_{\|T\|} \subset \nabla_{\mathfrak{s}}(\{\beta_n I + T_n\}) \subset \nabla_{\mathfrak{s}}(\{\beta_n I + \widetilde{T}_n\})$$

for each complex null sequence $\{\beta_n\}$. It therefore follows that if $\lambda \in D_{\|T\|}$, the sequence $\{\lambda_n(\lambda)\}$ figuring in formulae (63), (65), and (67) can be replaced by any sequence approaching λ, while retaining the uniform convergences. In particular, one can simply take each $\lambda_n(\lambda)$ equal to λ. Meanwhile there is another, more practical, expression for $\boldsymbol{T}_{|\lambda}$ at $\lambda \in D_{\|T\|}$, which may be obtained as follows:

$$
\begin{aligned}
\boldsymbol{T}_{|\lambda}(s,t) &= \boldsymbol{T}(s,t) + \lambda\langle R_\lambda(T)(\boldsymbol{t}'(t)), \boldsymbol{t}(s)\rangle &&\text{by (31) and (30)} \\
&= \boldsymbol{T}(s,t) + \lambda\langle \left(\sum_{n=0}^{\infty} \lambda^n T^n\right)(\boldsymbol{t}'(t)), \boldsymbol{t}(s)\rangle &&\text{by (7)} \\
&= \boldsymbol{T}(s,t) + \sum_{n=0}^{\infty} \langle \lambda^{n+1} T^n(\boldsymbol{t}'(t)), \boldsymbol{t}(s)\rangle &&\text{by (7)} \\
&= \sum_{n=1}^{\infty} \lambda^{n-1} \boldsymbol{T}^{[n]}(s,t). &&\text{by (13)}
\end{aligned}
$$

The series in the last line is the Neumann series for $\boldsymbol{T}_{|\lambda}$; it is convergent to $\boldsymbol{T}_{|\lambda}$ in $C(\mathbb{R}^2, \mathbb{C})$ for λ satisfying the inequality in (7), as

$$\|\boldsymbol{T}^{[n]}\|_{C(\mathbb{R}^2,\mathbb{C})} = \sup_{(s,t)\in\mathbb{R}^2} \left|\langle T^{n-2}\boldsymbol{t}'(t), \boldsymbol{t}(s)\rangle\right| \leqslant \|\boldsymbol{\tau}'\|_{C(\mathbb{R},\mathbb{R})} \|\boldsymbol{\tau}\|_{C(\mathbb{R},\mathbb{R})} \|T^{n-2}\|.$$

Remark 12. Applying the respective results of Theorem 10 in conjunction with the inequalities

$$\|\widetilde{\boldsymbol{t}}'_{n|\lambda_n(\lambda)}(t) - \boldsymbol{t}'_{|\lambda}(t)\| \leqslant \chi_n(t) \left\|\boldsymbol{t}'_{n|\lambda_n(\lambda)}(t) - \boldsymbol{t}'_{|\lambda}(t)\right\| + \widehat{\chi}_n(t) \left\|\boldsymbol{t}'_{|\lambda}(t)\right\|,$$

$$|\widetilde{\boldsymbol{T}}_{n|\lambda_n(\lambda)}(s,t) - \boldsymbol{T}_{|\lambda}(s,t)| \leqslant \chi_n(t) \left|\boldsymbol{T}_{n|\lambda_n(\lambda)}(s,t) - \boldsymbol{T}_{|\lambda}(s,t)\right| + \widehat{\chi}_n(t) \left|\boldsymbol{T}_{|\lambda}(s,t)\right|$$

(see (39) and (40)) yields that the limits (63), (64), (67), and (68) all remain valid upon replacing $\boldsymbol{t}'_{n|\lambda_n(\lambda)}$ and $\boldsymbol{T}_{n|\lambda_n(\lambda)}$ by $\widetilde{\boldsymbol{t}}'_{n|\lambda_n(\lambda)}$ and $\widetilde{\boldsymbol{T}}_{n|\lambda_n(\lambda)}$, respectively. In turn, the limits (65) and (66) continue to hold with $\boldsymbol{t}_{n|\lambda_n(\lambda)}$ and $\nabla_{\mathfrak{s}}(\{\beta_n I + T_n\})$ replaced respectively by $\widetilde{\boldsymbol{t}}_{n|\lambda_n(\lambda)}$ and $\nabla_{\mathfrak{s}}(\{\beta_n I + \widetilde{T}_n\})$, and to prove this use can be made of the inequalities

$$\|\widetilde{\boldsymbol{t}}_{n|\lambda_n(\lambda)}(s) - \boldsymbol{t}_{|\lambda}(s)\| \leqslant \left\|P_n(\boldsymbol{t}_{n|\lambda_n(\lambda)}(s) - \boldsymbol{t}_{|\lambda}(s))\right\| + \left\|(I - P_n)\left(\boldsymbol{t}_{|\lambda}(s)\right)\right\|,$$

$$\left\|P_n(\boldsymbol{t}_{n|\lambda_n(\lambda)}(s) - \boldsymbol{t}_{|\lambda}(s))\right\|$$

$$= \left\|P_n \left(\chi_n(s) \left(I + \lambda\beta_n I + \lambda A_{n|\lambda} + \lambda^2\beta_n A_{n|\lambda}\right) - I - \lambda T_{|\lambda}\right)^* (\boldsymbol{t}(s))\right\|$$

$$\leqslant |\beta_n| \chi_n(s) |\lambda| \boldsymbol{\tau}(s) + \widehat{\chi}_n(s)\boldsymbol{\tau}(s)$$

$$+ \chi_n(s) |\lambda| \left\|P_n \left(A_{n|\lambda} - T_{|\lambda}\right)^* (\boldsymbol{t}(s))\right\| + \widehat{\chi}_n(s) |\lambda| \left\|T_{|\lambda}\right\| \boldsymbol{\tau}(s)$$

$$+ |\beta_n| \chi_n(s) |\lambda|^2 \left\|A_{n|\lambda}P_n\right\| \boldsymbol{\tau}(s)$$

(compare (61)) and of the properties (59) and (56).

In connection with Theorem 10 the following natural question can be asked: in what cases are the sets $\overset{\circ}{\Pi}(T) := \Pi(T) \setminus \{0\}$ and $\nabla_{\mathfrak{s}}(\{\frac{\lambda_n-\lambda}{\lambda\lambda_n}I + T_n\})$ coincident? One answer to this question is given in the following theorem.

Theorem 13. *Suppose that*

$$\|(T - T_n)T_n^m\| \to 0 \quad as \ n \to \infty, \tag{69}$$

for some m in $\mathbb{N}$. Then

$$\nabla_{\mathfrak{s}}(\{\beta_n I + T_n\}) = \overset{\circ}{\Pi}(T) \subset \nabla_{\mathfrak{b}}(\{\beta_n I + T_n\}) \tag{70}$$

for each complex sequence $\{\beta_n\}$ converging to 0.

Proof. Continue to denote $A_n := \beta_n I + T_n$ as in the previous proof. Let λ be a fixed non-zero regular value for T. A straightforward calculation yields the equation

$$\left((I - \lambda T) \sum_{k=0}^{m-1} \lambda^k A_n^k + \lambda^m A_n^m\right) (I - \lambda A_n)$$

$$= (I - \lambda T)\left(I + \lambda^{m+1} R_\lambda(T)(T - A_n)A_n^m\right). \tag{71}$$

Expanding binomially $(\beta_n I + T_n)^m$ and utilizing conditions (43) and (69) gives

$$
\begin{aligned}
\|(T - A_n)A_n^m\| = \|(T - \beta_n I - T_n)\,(\beta_n I + T_n)^m\| \\
\leqslant \|(T - T_n)T_n^m\| + \|\beta_n T_n^m\| \\
+ \|(T - \beta_n I - T_n)\| \sum_{k=1}^{m} \binom{m}{k} |\beta_n^k|\, \|T_n^{m-k}\| \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

so $|\lambda|^{m+1}\, \|R_\lambda(T)(T - A_n)A_n^m\| < \frac{1}{2}$ for all n sufficiently large. Note that, for such n, the right-hand side of equation (71) does represent an invertible operator on L^2. This makes the last factor

$$
I - \lambda A_n = (1 - \beta_n\lambda)\left(I - \tfrac{\lambda}{1-\beta_n\lambda}T_n\right) \tag{72}
$$

on the left-hand side one-to-one and so invertible, as T_n is compact. Hence, for such n, $\frac{\lambda}{1-\beta_n\lambda} \in \Pi(T_n)$, $\lambda \in \Pi(A_n)$, and

$$
\left\|A_{n|\lambda}\right\| = \frac{1}{|\lambda|}\,\|R_\lambda(A_n) - I\|
$$

$$
= \frac{1}{|\lambda|}\left\|\left(I + \lambda^{m+1}R_\lambda(T)(T - A_n)A_n^m\right)^{-1} R_\lambda(T)\left((1 - \lambda T)\sum_{k=0}^{m-1}(\lambda A_n)^k + (\lambda A_n)^m\right) - I\right\|
$$

$$
\leqslant \frac{1}{|\lambda|}\,\frac{\|R_\lambda(T)\|\,(1 + |\lambda|\,\|T\|)\sum_{k=0}^{m}|\lambda|^k\,\|A_n\|^k}{1 - |\lambda|^{m+1}\|R_\lambda(T)(T - A_n)A_n^m\|} + \frac{1}{|\lambda|}
$$

$$
\leqslant M(\lambda) := \frac{2}{|\lambda|}\,\|R_\lambda(T)\|\,(1 + |\lambda|\,\|T\|)\sum_{k=0}^{m}|\lambda|^k\left(\max_{n\in\mathbb{N}}|\beta_n| + \|T\|\right)^k + \frac{1}{|\lambda|},
$$

where in the second equality use has been made of equation (71). Thus (see (8)), $\lambda \in \nabla_{\mathfrak{b}}(\{A_n\})$, and (70) now follows by (48). The theorem is proved. $\qquad\square$

Remark 14. Observe by (16) that (69) implies

$$
\|(T - \widetilde{T}_n)\widetilde{T}_n^m\| = \|(T - T_n)T_n^m P_n\| \to 0 \quad \text{as } n \to \infty. \tag{73}
$$

The same result as in the above theorem is obtained, similarly, with the sequence $\{\widetilde{T}_n\}$ satisfying (73) and it reads as follows:

$$
\nabla_{\mathfrak{s}}(\{\beta_n I + \widetilde{T}_n\}) = \overset{\circ}{\Pi}(T) \subset \nabla_{\mathfrak{b}}(\{\beta_n I + \widetilde{T}_n\})
$$

for each complex null sequence $\{\beta_n\}$. Consequently, under condition (69),

$$
\nabla_{\mathfrak{s}}(\{\beta_n I + T_n\}) = \nabla_{\mathfrak{s}}(\{\beta_n I + \widetilde{T}_n\}) = \overset{\circ}{\Pi}(T) \tag{74}
$$

for each complex null sequence $\{\beta_n\}$.

The following two corollaries may be of interest for further applications.

Corollary 15. *If* $\lambda \in \overset{\circ}{\Pi}(T)$ *and condition* (69) *holds, then* λ *is not the limit of any sequence* $\{\xi_n\}$ *satisfying* $\xi_n \in \Lambda(T_n)$ *at each* n.

Proof. Assume, on the contrary, that there exists a sequence $\{\xi_n\}$ with $\xi_n \in \Lambda(T_n)$ such that $\xi_n \to \lambda \in \overset{\circ}{\Pi}(T)$ as $n \to \infty$. Theorem 13 says that $\lambda \in \Pi(\beta_n I + T_n)$ for all n sufficiently large, where $\beta_n = \frac{\xi_n - \lambda}{\lambda \xi_n}$. This implies via (72) that $\frac{\lambda}{1 - \beta_n \lambda} = \xi_n \in \Pi(T_n)$ for all sufficiently large n, which is a contradiction. The corollary is proved. $\square$

Corollary 16. *If an operator* T *with condition* (69) *is self-adjoint, then*

$$\nabla_{\flat}(\{T_n\}) = \nabla_{\flat}(\{\widetilde{T}_n\}) = \nabla_{\mathfrak{s}}(\{\widetilde{T}_n\}) = \nabla_{\mathfrak{s}}(\{T_n\}) = \overset{\circ}{\Pi}(T). \tag{75}$$

Proof. From the facts proved above it follows that $\overset{\circ}{\Pi}(T) \subset \nabla_{\flat}(\{T_n\}) \subset \nabla_{\flat}(\{\widetilde{T}_n\})$. This and (74) together show that to prove (75) it is enough to prove that $\nabla_{\flat}(\{\widetilde{T}_n\}) \subset \overset{\circ}{\Pi}(T)$. Suppose $\lambda \in \nabla_{\flat}(\{\widetilde{T}_n\})$, so there is a positive constant M such that

$$\|\widetilde{T}_{n|\lambda}\| \leqslant M \quad \text{for all sufficiently large } n, \tag{76}$$

but suppose, contrary to $\lambda \in \overset{\circ}{\Pi}(T)$, that $\lambda \in \Lambda(T)$. Then, by Theorem VIII.24 of [16, p. 290], there exists a sequence $\lambda_n \in \Lambda(\widetilde{T}_n)$ $(n \in \mathbb{N})$ such that $\lambda_n \to \lambda$ as $n \to \infty$. Consequently,

$$|\lambda| \, \|\widetilde{T}_{n|\lambda}\| + 1 \geqslant \|R_\lambda(\widetilde{T}_n)\| \geqslant \frac{|\lambda_n|}{|\lambda_n - \lambda|} \to \infty,$$

which, however, is incompatible with (76). The corollary is proved. $\square$

Remark 17. In terms of kernels, condition (69) (resp., (73)) means that nuclear operators, induced on L^2 by the (explicit) kernels

$$\boldsymbol{J}_n(s, t) = \widehat{\chi}_n(s) \int_{\mathbb{I}_n} \boldsymbol{T}(s, x) \boldsymbol{T}_n^{[m]}(x, t) \, dx$$

$$(\text{resp., } \widetilde{\boldsymbol{J}}_n(s, t) = \widehat{\chi}_n(s) \int_{\mathbb{I}_n} \boldsymbol{T}(s, x) \widetilde{\boldsymbol{T}}_n^{[m]}(x, t) \, dx),$$

have their operator norm going to 0 as n goes to infinity. In particular, if the nuclear operators $(I - P_n) T P_n$, with kernels $\widehat{\chi}_n(s) \boldsymbol{T}(s, t) \chi_n(t)$, converge to zero operator in the operator norm as $n \to \infty$, then both conditions (69) and (73) automatically hold with any fixed m in $\mathbb{N}$, and this may happen even if T is not a compact operator. Note, incidentally, that for the latter there is a stronger conclusion about the uniform-on-compacta convergence on all of $\overset{\circ}{\Pi}(T)$:

Theorem 18. *If* T *is a compact operator, then the following limits hold:*

$$\lim_{n \to \infty} \sup_{\lambda \in \mathfrak{K}} \left\| \boldsymbol{t}'_{n|\lambda} - \boldsymbol{t}'_{|\lambda} \right\|_{C(\mathbb{R}, L^2)} = 0, \quad \lim_{n \to \infty} \sup_{\lambda \in \mathfrak{K}} \left\| \boldsymbol{t}_{n|\lambda} - \boldsymbol{t}_{|\lambda} \right\|_{C(\mathbb{R}, L^2)} = 0,$$

$$\lim_{n \to \infty} \sup_{\lambda \in \mathfrak{K}} \left\| \boldsymbol{T}_{n|\lambda} - \boldsymbol{T}_{|\lambda} \right\|_{C(\mathbb{R}^2, \mathbb{C})} = 0 \tag{77}$$

for each compact subset $\mathfrak{K}$ *of* $\overset{\circ}{\Pi}(T)$ (*compare with* (63)-(68)).

Proof. Let $\mathfrak{K}$ be a compact subset of $\overset{\circ}{\Pi}(T)$. Since under the stated hypotheses on T

$$\lim_{n\to\infty} \|T_n - T\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\widetilde{T}_n - T\| = 0, \tag{78}$$

it follows from (52), (57), and Theorem 13 (all applied with β_n all taken equal to zero) that for some positive constant M

$$\sup_{\lambda\in\mathfrak{K}} \|\widetilde{T}_{n|\lambda}\| + \sup_{\lambda\in\mathfrak{K}} \|T_{n|\lambda}\| \leqslant M \quad \text{for all sufficiently large } n. \tag{79}$$

Transforming the Fredholm resolvent differences $\widetilde{T}_{n|\lambda} - T_{|\lambda}$ and $T_{n|\lambda} - T_{|\lambda}$ into products of operators via the second resolvent equation (6) and subsequently using (78) and (79) then leads to the limit-relations:

$$\lim_{n\to\infty} \sup_{\lambda\in\mathfrak{K}} \|\widetilde{T}_{n|\lambda} - T_{|\lambda}\| = 0, \quad \lim_{n\to\infty} \sup_{\lambda\in\mathfrak{K}} \|T_{n|\lambda} - T_{|\lambda}\| = 0. \tag{80}$$

For $\beta_n = 0$, proceeding the inequalities (60)-(62) yields, respectively, the following estimates

$$\sup_{\lambda\in\mathfrak{K}} \|\boldsymbol{t}'_{n|\lambda} - \boldsymbol{t}'_{|\lambda}\|_{C(\mathbb{R},L^2)} \leqslant \sup_{t\in\mathbb{R}} \|(P_n - I)(\boldsymbol{t}'(t))\|$$
$$+ \|\boldsymbol{\tau}'\|_{C(\mathbb{R},\mathbb{R})} \sup_{\lambda\in\mathfrak{K}} \left(|\lambda| \|\widetilde{T}_{n|\lambda} - T_{|\lambda}\|\right),$$
$$\sup_{\lambda\in\mathfrak{K}} \|\boldsymbol{t}_{n|\lambda} - \boldsymbol{t}_{|\lambda}\|_{C(\mathbb{R},L^2)} \leqslant \|\widehat{\chi}_n\boldsymbol{\tau}\|_{C(\mathbb{R},\mathbb{R})} \sup_{\lambda\in\mathfrak{K}} \left(1 + |\lambda| \|T_{|\lambda}\|\right)$$
$$+ \|\boldsymbol{\tau}\|_{C(\mathbb{R},\mathbb{R})} \sup_{\lambda\in\mathfrak{K}} \left(|\lambda| \|T_{n|\lambda} - T_{|\lambda}\|\right),$$
$$\sup_{\lambda\in\mathfrak{K}} \|\boldsymbol{T}_{n|\lambda} - \boldsymbol{T}_{|\lambda}\|_{C(\mathbb{R}^2,\mathbb{C})} \leqslant \|\boldsymbol{\tau}\|_{C(\mathbb{R},\mathbb{R})} \sup_{\lambda\in\mathfrak{K}} \|\boldsymbol{t}'_{n|\lambda} - \boldsymbol{t}'_{|\lambda}\|_{C(\mathbb{R},L^2)}$$
$$+ \|\widehat{\chi}_n\boldsymbol{\tau}\|_{C(\mathbb{R},\mathbb{R})} \|\boldsymbol{\tau}'\|_{C(\mathbb{R},\mathbb{R})} \sup_{\lambda\in\mathfrak{K}} \left(|\lambda| \|R_\lambda(T)\|\right) + \|\boldsymbol{T}_n - \boldsymbol{T}\|_{C(\mathbb{R}^2,\mathbb{C})},$$

whence the limits in (77) all hold by virtue of (11), (23), (19), and (80). The theorem is proved. $\qquad\square$

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АННОТАЦИЯ

Доказывается, что в регулярных точках из области обобщенной сильной сходимости разрешающие ядра для непрерывного бикарлемановского ядра, исчезающего на бесконечности, могут быть выражены как равномерные пределы последовательностей разрешающих ядер для аппроксимирующих его подъядер типа Гильберта – Шмидта.

Ключевые слова: *линейное интегральное уравнение 2-го рода, ограниченный линейный интегральный оператор, резольвента Фредгольма, разрешающее ядро, бикарлемановское ядро, ядро Гильберта – Шмидта, ядерный оператор, регулярное значение, характеристическое множество.*