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A new subclass of meromorphic function with positive coefficients defined by Hurwitz-Lerch Zeta functions

In this paper, we introduce and study a new subclass of meromorphic univalent functions defined by Hurwitz-Lerch Zeta function. We obtain coefficient inequalities, extreme points, radius of starlikeness and convexity. Finally we obtain partial sums and neighborhood properties for the class $\sigma^*(\gamma, k, \lambda, b, s)$.

Key words: *meromorphic function, extreme point, partial sums, neighborhood.*

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1 Introduction

Let S be denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Denote by $S^*(\gamma)$ and $K^*(\gamma)$, $0 \le \gamma < 1$ the subclasses of functions in S that are starlike and convex functions of order γ respectively. Analytically $f \in S^*(\gamma)$ if and only if f is of the form (1) and satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \quad z \in U.$$

Similarly, $f \in K^*(\gamma)$ if and only if f is of the form (1) and satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma, \quad z \in U.$$

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Also denote by T the subclasses of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0$$

introduced and studied by Silverman [1], let $T^*(\gamma) = T \cap S^*(\gamma), CV(\gamma) = T \cap K^*(\gamma)$. The classes $T^*(\gamma)$ and $K^*(\gamma)$ posses some interesting properties and have been extensively studied by Silverman [1] and others. In 1991, Goodman [2, 3] introduced an interesting subclass uniformly convex (uniformly starlike) of the class CV of convex functions (ST starlike functions) denoted by UCV (UST). A function $f(z)$ is uniformly convex (uniformly starlike) in U if $f(z)$ in CV (ST) has the property that for every circular arc γ contained in U with center ξ also in U, the arc $f(\gamma)$ is a convex arc (starlike arc) with respect to $f(\xi)$.

Motivated by Goodman [2, 3], Ronning [4, 5] introduced and studied the following subclasses of S. A function $f \in S$ is said to be in the class $S_p(\gamma, k)$ uniformly $k-$starlike functions if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad 0 \leq \gamma < 1, \quad k \geq 0, \quad z \in U \tag{2}$$

and is said to be in the class UCV (γ, k), uniformly $k-$convex functions if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad 0 \leq \gamma < 1, \quad k \geq 0, \quad z \in U. \tag{3}$$

Indeed it follows from (2) and (3) that

$$f \in UCV(\gamma, k) \quad \Leftrightarrow \quad zf' \in S_p(\gamma, k).$$

Further Ahuja et al. [6], Bharathi et al. [7], Murugusundaramoorthy et al. [8] and others have studied and investigated interesting properties for the classes $S_p(\gamma, k)$ and $UCV(\gamma, k)$.

Let σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0 \tag{4}$$

which are analytic in the punctured open disk $U^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = U \setminus \{0\}$.

Let $f, g \in \sigma$, where f is given by (4) and g is defined by

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \quad b_n \geq 0.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z).$$

Let $\sigma_s, \sigma^*(\gamma)$ and $\sigma_k(\gamma), 0 \leq \gamma < 1$ denote the subclasses of σ that are meromorphic univalent, meromorphically starlike functions of order γ and meromorphically convex functions of order γ respectively. Analytically, $f \in \sigma^*(\gamma)$ if and only if f is of the form (4) and satisfies

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in U.$$

Similarly, $f \in \sigma_k(\gamma)$ if and only if f is of the form (4) and satisfies

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in U$$

and similar other classes of meromorphically univalent functions have been extensively studied by Altintas et al. [9], Aouf [10] and Mogra et al. [11].

The following we recall a general Hurwitz-Lerch Zeta function $\phi(z, s, a)$ defined by (see [12], p. 121)

$$\phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

for $a \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$, where $\mathbb{Z}_0^- = \mathbb{Z} \setminus \mathbb{N},\ \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\},\ \mathbb{N} = \{1, 2, 3, \dots\}$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\phi(z, s, a)$ can be found in the recent investigation by ,for example, Choi and Srivastava [13], Ferreira and Lopez [14], Garat et al. [15], Lin and Srivastava [16], Luo and Srivastava [17], Srivastava et al. [18], Ghanim [19] and others.

By making use of Hurwitz-Lerch Zeta function $\phi(z, s, a)$, Srivastava and Attiya [20] recently introduced and investigated the integral operator

$$\mathcal{J}_{b,s}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s c_n z^n, \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}, z \in U.$$

Motivated essentially by the above mentioned Srivastava-Attiya operator $\mathcal{J}_{b,s}$, we now introduce the linear operator

$$\mathcal{W}_{b,s} : \sigma \to \sigma$$

defined in terms of the Hadamard product (or convolution), by

$$\mathcal{W}_{b,s}f(z) = \Theta_{b,s}(z) * f(z), \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^- \cup \{1\}, s \in \mathbb{C}, f \in \sigma, z \in U^*, \tag{5}$$

where for convenience,

$$\Theta_{b,s}(z) = (b-1)^s \left[\phi(z, b, s) - b^{-s} + \frac{1}{z(b-1)^s}\right], \quad z \in U^*.$$

It can be easily be seen from (5) that

$$\mathcal{W}_{b,s}f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} L(n, b, s) a_n z^n, \tag{6}$$

$$\text{where} \quad L(n, b, s) = \left(\frac{b-1}{b+n}\right)^s.$$

Indeed, the operator $\mathcal{W}_{b,s}$ can be defined for $b \in \mathbb{C} \setminus \mathbb{Z}_0^- \cup \{1\}$, where

$$\mathcal{W}_{0,s} f(z) = \lim_{b \to 0} \{\mathcal{W}_{b,s} f(z)\}.$$

We observe that

$$\mathcal{W}_{b,0} f(z) = f(z)$$

and

$$\mathcal{W}_{1,\gamma} = \frac{\gamma - 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt, \qquad (\Re(\gamma) > 1.$$

Furthermore, from the definition (6), we find that

$$\mathcal{W}_{s+1,b} f(z) = \frac{b-1}{z^b} \int_0^z t^{b-1} \mathcal{W}_{b,s} f(t) dt, \qquad \Re(b) > 1. \tag{7}$$

Differentiating both sides of (7) with respect to z, we get the following useful relationship:

$$z \left(\mathcal{W}_{s+1,b} f\right)'(z) = (b-1) \mathcal{W}_{b,s} f(z) - b \mathcal{W}_{s+1,b} f(z).$$

In order to prove our results wee need the following lemmas.

Lemma 1. *If γ is a real number and $\omega = -(u + iv)$ is a complex number then*

$$\Re(\omega) \geq \gamma \Leftrightarrow |\omega + (1-\gamma)| - |\omega - (1-\gamma)| \geq 0.$$

Lemma 2. *If $\omega = u + iv$ is a complex number and γ is a real number then*

$$-\Re(\omega) \geq k|\omega + 1| + \gamma \Leftrightarrow -\Re\left(\omega(1 + ke^{i\theta}) + ke^{i\theta}\right) \geq \gamma, \quad -\pi \leq \theta \leq \pi.$$

Motivated by Sivaprasad Kumar et al. [21] and Atshan and Kulkarni [22], now we define a new subclass $\sigma^*(\gamma, k, \lambda, b, s)$ of σ.

Definition 1. For $0 \leq \gamma < 1, k \geq 0$ and $0 \leq \lambda < \frac{1}{2}$, we let $\sigma^*(\gamma, k, \lambda, b, s)$ be the subclass of σ_s consisting of functions of the form (4) and satisfying the analytic criterion

$$-\Re\left(\frac{z(\mathcal{W}_{b,s} f(z))'}{\mathcal{W}_{b,s} f(z)} + \lambda z^2 \frac{(\mathcal{W}_{b,s} f(z))''}{\mathcal{W}_{b,s} f(z)} + \gamma\right) > k\left|\frac{z(\mathcal{W}_{b,s} f(z))'}{\mathcal{W}_{b,s} f(z)} + \lambda z^2 \frac{(\mathcal{W}_{b,s} f(z))''}{\mathcal{W}_{b,s} f(z)} + 1\right|.$$

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bounds, extreme points, radii of meromorphic starlikeness and convexity for the class $\sigma^*(\gamma, k, \lambda, b, s)$. Further, we obtain partial sums and neighborhood properties for the class also.

2 Coefficient estimates

In this section we obtain necessary and sufficient condition for a function f to be in the class $\sigma^*(\gamma, k, \lambda, b, s)$.

Theorem 1. *Let $f \in \sigma$ be given by (4). Then $f \in \sigma^*(\gamma, k, \lambda, b, s)$ if and only if*

$$\sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]L(n,b,s)a_n \leq (1-\gamma)-2\lambda(1+k). \tag{8}$$

Proof. Let $f \in \sigma^*(\gamma, k, \lambda, b, s)$. Then by definition and using Lemma 1, it is enough to show that

$$-\Re\left\{\left(\frac{z(\mathcal{W}_{b,s}f(z))'}{\mathcal{W}_{b,s}f(z)}+\lambda z^2\frac{(\mathcal{W}_{b,s}f(z))''}{\mathcal{W}_{b,s}f(z)}\right)(1+ke^{i\theta})+ke^{i\theta}\right\} > \gamma, \quad -\pi \leq \theta \leq \pi. \tag{9}$$

For convenience

$$\begin{aligned}
C(z) &= -\left[z(\mathcal{W}_{b,s}f(z))'+\lambda z^2(\mathcal{W}_{b,s}f(z))''\right](1+ke^{i\theta})-ke^{i\theta}\mathcal{W}_{b,s}f(z),\\
D(z) &= \mathcal{W}_{b,s}f(z).
\end{aligned}$$

That is, the equation (9) is equivalent to

$$-\Re\left(\frac{C(z)}{D(z)}\right) \geq \gamma.$$

In view of Lemma 1, we only need to prove that

$$|C(z)+(1-\gamma)D(z)|-|C(z)-(1-\gamma)D(z)| \geq 0.$$

Therefore

$$|C(z)+(1-\gamma)D(z)| \geq (2-\gamma-2\lambda(k+1))\frac{1}{|z|}-$$
$$-\sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda)+(k+\gamma-1)]L(n,b,s)a_n|z|^n$$

and

$$|C(z)-(1-\gamma)D(z)| \leq (\gamma+2\lambda(k+1)))\frac{1}{|z|}+$$
$$+\sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda)+(k+\gamma+1)]L(n,b,s)a_n|z|^n.$$

It is to show that

$$|C(z)+(1-\gamma)D(z)|-|C(z)-(1+\gamma)D(z)| \geq$$
$$\geq (2(1-\gamma)-4\lambda(k+1))\frac{1}{|z|}-2\sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]L(n,b,s)a_n|z|^n \geq 0,$$

by the given condition (8). Conversely suppose $f \in \sigma^*(\gamma, k, \lambda, b, s)$. Then by Lemma 1, we have (9).

Choosing the values of z on the positive real axis, the inequality (9) reduces to

$$\Re\left\{\frac{\left[1-\gamma-2\lambda\left(1+ke^{i\theta}\right)\right]\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}\left[n(1+(n-1)\lambda)\left(1+ke^{i\theta}\right)+\left(\gamma+ke^{i\theta}\right)\right]L(n,b,s)z^{n-1}}{\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}L(n,b,s)a_nz^{n-1}}\right\}\geq 0.$$

Since $\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$\Re\left\{\frac{[1-\gamma-2\lambda(1+k)]\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]L(n,b,s)a_nr^{n-1}}{\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}L(n,b,s)r^{n-1}}\right\}\geq 0.$$

Letting $r \to 1^-$ and by the mean value theorem, we have obtained the inequality (8). $\square$

Corollary 1.1. *If $f \in \sigma^*(\gamma, k, \lambda, b, s)$ then*

$$a_n \leq \frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]L(n,b,s)}.$$

By taking $\lambda = 0$ in Theorem 1, we get the following corollary.

Corollary 1.2. *If $f \in \sigma^*(\gamma, k, \lambda, b, s)$ then*

$$a_n \leq \frac{1-\gamma}{[n(1+k)+(\gamma+k)]L(n,b,s)}. \tag{10}$$

Theorem 2. *If $f \in \sigma^*(\gamma, k, \lambda, b, s)$ then for $0 < |z| = r < 1$,*

$$\frac{1}{r}-\frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)}\,r \leq |f(z)| \leq \frac{1}{r}+\frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)}r.$$

This result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)}\,z, \text{ at } z=r, ir. \tag{11}$$

P r o o f. Since $f(z) = \frac{1}{z} + \sum\limits_{n=2}^{\infty} a_n z^n$, we have

$$|f(z)| = \frac{1}{r}+\sum_{n=1}^{\infty}a_nr^n \leq \frac{1}{r}+r\sum_{n=2}^{\infty}a_n. \tag{12}$$

Since $n \geq 1$, $(2k+\gamma+1) \leq n(k+1)(k+\gamma)L(n,b,s)$, using Theorem 1, we have

$$(2k+\gamma+1)\sum_{n=1}^{\infty}a_n \leq \sum_{n=1}^{\infty}n(k+1)(k+\gamma)L(n,b,s) \leq (1-\gamma)-2\lambda(k+1)$$

$$\Rightarrow \qquad \sum_{n=1}^{\infty}a_n \leq \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)}.$$

Using the above inequality in (12), we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)}r$$

and

$$|f(z)| \geq \frac{1}{r} - \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)}r.$$

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)}z.$ □

Corollary 2.1. *If $f \in \sigma^*(\gamma,k,\lambda,b,s)$ then*

$$\frac{1}{r^2} - \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)L(1,b,s)}.$$

The result is sharp for the function given by (11).

3 Extreme points

Theorem 3. *Let $f_0(z) = \frac{1}{z}$ and*

$$f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(\lambda+k)]L(n,b,s)}z^n, \quad n \geq 1. \tag{13}$$

Then $f \in \sigma^*(\gamma,k,\lambda,b,s)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} u_n f_n(z), u_n \geq 0 \qquad \textit{and} \qquad \sum_{n=1}^{\infty} u_n = 1. \tag{14}$$

P r o o f. Suppose $f(z)$ can be expressed as in (14). Then

$$f(z) = \sum_{n=0}^{\infty} u_n f_n(z) = u_0 f_0(z) + \sum_{n=1}^{\infty} u_n f_n(z) =$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} u_n \frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(\lambda+k)]L(n,b,s)}z^n.$$

Therefore

$$\sum_{n=1}^{\infty} \left(u_n \frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(\lambda+k)]L(n,b,s)} \times \right.$$
$$\left. \times \frac{[n(1+k)(1+(n-1)\lambda)+(\lambda+k)]L(n,b,s)}{(1-\gamma)-2\lambda(k+1)} z^n \right) = \sum_{n=1}^{\infty} u_n = 1 - u_0 \leq 1.$$

So by Theorem (1), $f \in \sigma^*(\gamma,k,\lambda,b,s)$.

Conversely suppose that $f \in \sigma^*(\gamma,k,\lambda,b,s)$. Since

$$a_n \leq \frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(\lambda+k)]L(n,b,s)} \; n \geq 1.$$

We set
$$u_n = \frac{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]L(n,b,s)}{(1-\gamma)-2\lambda(k+1)}a_n, \quad n \geq 1$$
and
$$u_0 = 1 - \sum_{n=1}^{\infty} u_n.$$
Then we have
$$f(z) = \sum_{n=0}^{\infty} u_n f_n(z) = u_0 f_0(z) + \sum_{n=1}^{\infty} u_n f_n(z).$$
Hence the result follows. □

4 Radii of meromorphically starlike and convexity

Theorem 4. *Let $f \in \sigma^*(\gamma, k, \lambda, b, s)$. Then f is meromorphically starlike of order δ, $(0 \leq \delta \leq 1)$ in the disc $|z| < r_1$, where*
$$r_1 = \inf_n \left[\frac{(1-\delta)}{(n+2-\delta)} \frac{[n(1+k)(1+(n-1)\lambda)+(\lambda+k)]L(n,b,s)}{(1-\gamma)-2\lambda(k+1)} \right]^{\frac{1}{n+1}}, \quad n \geq 1.$$
The result is sharp for the extremal function $f(z)$ given by (13).

P r o o f. The function $f \in \sigma^*(\gamma, k, \lambda, b, s)$ of the form (4) is meromorphically starlike of order δ is the disc $|z| < r_1$ if and only if it satisfies the condition
$$\left| \frac{zf'(z)}{f(z)} + 1 \right| < (1-\delta). \tag{15}$$
Since
$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq \left| \frac{\sum\limits_{n=1}^{\infty} (n+1) a_n z^{n+1}}{1 + \sum\limits_{n=1}^{\infty} a_n z^{n+1}} \right| \leq \frac{\sum\limits_{n=1}^{\infty} (n+1)|a_n||z|^{n+1}}{1 - \sum\limits_{n=1}^{\infty} |a_n||z|^{n+1}}.$$
The above expression is less than $(1-\delta)$ if $\sum\limits_{n=1}^{\infty} \frac{(n+2-\delta)}{(1-\delta)} a_n |z|^{n+1} < 1$.

Using the fact that $f(z) \in \sigma^*(\gamma, k, \lambda, b, s)$ if and only if
$$\sum_{n=1}^{\infty} \frac{[n(1+k)(1+(n-1)\lambda)+(\lambda+k)]L(n,b,s)}{(1-\gamma)-2\lambda(k+1)} a_n \leq 1.$$
Thus, (15) will be true if
$$\frac{(n+2-\delta)}{(1-\delta)}|z|^{n+1} < \frac{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]L(n,b,s)}{(1-\gamma)-2\lambda(k+1)}$$
or equivalently
$$|z|^{n+1} < \frac{(1-\delta)}{(n+2-\delta)} \frac{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]L(n,b,s)}{(1-\gamma)-2\lambda(k+1)}$$

which yields the starlikeness of the family. □

The proof of the following theorem is analogous to that of Theorem 4, and so we omit the proof.

Theorem 5. *Let $f \in \sigma^*(\gamma, k, \lambda, b, s)$. Then f is meromorphically convex of order δ, $(0 \leq \delta \leq 1)$ in the disc $|z| < r_2$, where*

$$r_2 = \inf_n \left[\frac{(1-\delta)}{n(n+2-\delta)} \frac{[n(1+k)(1+(n-1)\lambda) + (\gamma+k)]L(n,b,s)}{(1-\gamma) - 2\lambda(k+1)} \right]^{\frac{1}{n+1}}, \quad n \geq 1.$$

The result is sharp for the extremal function $f(z)$ given by (13).

5 Partial Sums

Let $f \in \sigma$ be a function of the form (4). Motivated by Silverman [23] and Silvia [24] and also see [25], we define the partial sums f_m defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n, \quad m \in \mathbb{N}.$$

In this section we consider partial sums of function from the class $\sigma^*(\gamma, k, \lambda, b, s)$ and obtain sharp lower bounds for the real part of the ratios of f to f_m and f' to f'_m.

Theorem 6. *Let $f \in \sigma^*(\gamma, k, \lambda, b, s)$ be given by (4) and define the partial sums $f_1(z)$ and $f_m(z)$ by*

$$f_1(z) = \frac{1}{z} \text{ and } f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} |a_n| z^n, \quad m \in \mathbb{N} \setminus \{1\}.$$

Suppose also that $\sum\limits_{n=1}^{\infty} d_n |a_n| \leq 1$, where

$$d_n \geq \begin{cases} 1, & \text{if} \quad n = 1, 2, \ldots, m \\ \frac{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]L(n,b,s)}{(1-\gamma)-2\lambda(k+1)}, & \text{if} \quad n = m+1, m+2, \ldots. \end{cases} \tag{16}$$

Then $f \in \sigma^*(\gamma, k, \lambda, b, s)$. Furthermore

$$\Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}} \tag{17}$$

$$\text{and } \Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1 + d_{m+1}}. \tag{18}$$

P r o o f. For the coefficient d_n given by (16) it is not difficult to verify that

$$d_{m+1} > d_m > 1.$$

Therefore we have

$$\sum_{n=1}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |a_n| d_m \leq 1 \tag{19}$$

by using the hypothesis (16). By setting

$$g_1(z) = d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right) = 1 + \frac{d_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=1}^{\infty} |a_n| z^{n-1}}$$

then it sufficient to show that

$$\Re\big(g_1(z)\big) \geq 0 \quad \text{or} \quad \left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq 1, \quad z \in U$$

and applying (19), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{m+1}\sum\limits_{n=m+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{m} |a_n| - d_{m+1}\sum\limits_{n=m+1}^{\infty} |a_n|} \leq 1,$$

which ready yields the assertion (17) of Theorem 6. In order to see that

$$f(z) = \frac{1}{z} + \frac{z^{m+1}}{d_{m+1}} \tag{20}$$

gives sharp result, we observe that for

$$z = re^{\frac{i\pi}{m}} \quad \text{that} \quad \frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+2}}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}} \quad \text{as} \quad r \to 1^-.$$

Similarly, if we takes

$$g_2(z) = (1 + d_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1 + d_{m+1}}\right)$$

and making use of (19), we denote that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| < \frac{(1 + d_{m+1})\sum\limits_{n=m+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{m} |a_n| - (1 - d_{m+1})\sum\limits_{n=m+1}^{\infty} |a_n|}$$

which leads us immediately to the assertion (18) of Theorem 6.

The bound in (18) is sharp for each $m \in \mathbb{N}$ with extremal function $f(z)$ given by (20). □

The proof of the following theorem is analogous to that of Theorem 6, so we omit the proof.

Theorem 7. *If $f \in \sigma^*(\gamma, k, \lambda, b, s)$ be given by (4) and satisfies the condition (8) then*

$$\Re\left(\frac{f'(z)}{f'_m(z)}\right) > 1 - \frac{m+1}{d_{m+1}} \quad \text{and} \quad \Re\left(\frac{f'_m(z)}{f'(z)}\right) > \frac{d_{m+1}}{m+1+d_{m+1}},$$

where

$$d_n \geq \begin{cases} n, & \text{if} \quad n = 2, 3, \ldots, m \\ \frac{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]L(n,b,s)}{(1-\gamma)-2\lambda(k+1)}, & \text{if} \quad n = m+1, m+2, \ldots \end{cases}.$$

The bounds are sharp with the extremal function $f(z)$ of the form (10).

6 Neighbourhoods for the class $\sigma^{*\xi}(\gamma, k, \lambda, b, s)$

In this section, we determine the neighborhood for the class $\sigma^{*\xi}(\gamma, k, \lambda, b, s)$ which we define as follows:

Definition 2. A function $f \in \sigma$ is said to be in the class $\sigma^{*\xi}(\gamma, k, \lambda, b, s)$ if there exits a function $g \in \sigma^*(\gamma, k, \lambda, b, s)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \xi, \quad z \in U, 0 \leq \xi < 1.$$

Following the earlier works on neighbourhoods of analytic functions by Goodman [26] and Ruscheweyh [27], we define the $\delta-$neighbourhoods of function $f \in \sigma$ by

$$N_\delta(f) = \Big\{g \in \sigma : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \ \ and \ \ \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta\Big\}. \tag{21}$$

Theorem 8. *If $g \in \sigma^*(\gamma, k, \lambda, b, s)$ and*

$$\xi = 1 - \frac{\delta(2k + \gamma + 1)L(1, b, s)}{(2k + \gamma + 1)L(1, b, s) - (1 - \gamma) + 2\lambda(k + 1)} \tag{22}$$

then $\quad N_\delta(g) \subset \sigma^{*\xi}(\gamma, k, \lambda, b, s)$.

P r o o f. Let $f \in N_\delta(g)$. Then we find from (21) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \qquad n \in \mathbb{N}.$$

Since $g \in \sigma^*(\gamma, k, \lambda, b, s)$, we have

$$\sum_{n=1}^{\infty} b_n \leq \frac{(1 - \gamma) - 2\lambda(k + 1)}{(2k + \gamma + 1)L(1, b, s)}.$$

So that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum\limits_{n=1}^{\infty} |a_n - b_n|}{1 - \sum\limits_{n=1}^{\infty} b_n} = \frac{\delta(2k + \gamma + 1)L(1, b, s)}{(2k + \gamma + 1)L(1, b, s) - (1 - \gamma) + 2\lambda(k + 1)} = 1 - \xi$$

provided ξ is given by (22). Hence by definition, $f \in \sigma^{*\xi}(\gamma, k, \lambda, b, s)$, which completes the proof. $\square$

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References

[1] H. Silverman, "Univalent functions with negative coefficients", *Proc. Amer. Math. Soc.*, **51**, (1975), 109–116.

[2] A.W. Goodman, "On uniformly convex functions", *Ann. Polon. Math.*, **56**, (1991), 87–92.

[3] A.W. Goodman, "On uniformly starlike functions", *J. Math. Anal. Appl.*, **155**:2, (1991), 364–370.

[4] F. Ronning, "Uniformly convex functions and a corresponding class of starlike functions", *Proc. Amer. Math. Soc.*, **118**:1, (1993), 189–196.

[5] F. Ronning, "Integral representations of bounded starlike functions", *Ann. Polon. Math.*, **60**:3, (1995), 289–297.

[6] O. P. Ahuja, G. Murugusundaramoorthy and N. Magesh, "Integral means for uniformly convex and starlike functions associated with generalized hypergeometric functions", *J. Inequal. Pure Appl. Math.*, **8**:4, (2007), 1–9, Art. 118.

[7] R. Bharati, R. Parvatham and A. Swaminathan, "On subclasses of uniformly convex functions and corresponding class of starlike functions", *Tamkang J. Math.*, **28**:1, (1997), 17–32.

[8] G. Murugusundaramoorthy and N. Magesh, "Certain subclasses of starlike functions of complex order involving generalized hypergeometric functions", *Int. J. Math. Math. Sci.*, 2010, 1–12, art. ID 178605.

[9] S. O. Altinta, H. Irmak and H. M. Srivastava, "A family of meromorphically univalent functions with positive coefficients", *Panamer. Math. J.*, **5**:1, (1995), 75–81.

[10] M.K. Aouf, "On a certain class of meromorphic univalent functions with positive coefficients", *Rend. Mat.*, **11**, (1991), 209–219.

[11] M. L. Mogra, T. R. Reddy and O. P. Juneja, "Meromorphic univalent functions with positive coefficients", *Bull. Austral. Math. Soc.*, **32**:2, (1985), 161–176.

[12] H.M. Srivastava and J. Choi, *Series associated with the Zeta and related functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.

[13] J. Choi and H. M. Srivastava, "Certain families of series associated with the Hurwitz-Lerch Zeta function", *Appl. Math. Comput.*, **170**, (2005), 399–409.

[14] C. Ferreira and J. L. Lopez, "Asymptotic expansions of the Hurwitz-Lerch Zeta function Zeta function", *J. Math. Anal. Appl.*, **298**, (2004), 210–224.

[15] M. Garg, K. Jain and H.M. Srivastava, "Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions", *Integral Transforms Spec. Funct.*, **17**, (2006), 803–815.

[16] S.-D. Lin and H. M. Srivastava, "Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations", *Appl. Math. Comput.*, **154**, (2004), 725–733.

[17] Q. M. Luo and H.M. Srivastava, "Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials", *J. Math. Anal. Appl.*, **308**, (2005), 290–302.

[18] H.M. Srivastava, M.-J. Luo, and R.K. Raina, "New results involving a class of generalized Hurwitz-Lerch Zeta functions and their applications", *Turkish J. Anal. Number Theory*, **1**, (2013), 26–35.

[19] F. Ghanim, "A study of a certain subclass of Hurwitz-Lerch-Zeta function related to a linear operator", *Abstr. Appl. Anal.*, **2013**, (2013), Article ID 763756, 7 pp.

[20] H. M. Srivastava and A. A. Attiya, "An integral operator associated with the Hurwitz-Lerch Zeta function and diffierential subordination", *Integral Transforms Spec. Funct.*, **18**, (2007), 207–216.

[21] S. Sivaprasad Kumar, V. Ravichandran and G. Murugusundaramoorthy, "Classes of meromorphic p-valent parabolic starlike functions with positive coefficients", *Aust. J. Math. Anal. Appl.*, **2**:2, (2005), 1–9, art. 3.

[22] W. G. Atshan and S. R. Kulkarni, "Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative", *I. J. Rajasthan Acad. Phys. Sci.*, **6**:2, (2007), 129-140.

[23] H. Silverman, "Partial sums of starlike and convex functions", *J. Math. Anal. Appl.*, **209**:1, (1997), 221–227.

[24] E. M. Silvia, "On partial sums of convex functions of order α", *Houston J. Math.*, **11**:3, (1985), 397–404.

[25] M. K. Aouf and H. Silverman, "Partial sums of certain meromorphic $p-$valent functions", *J. Inequal. Pure Appl. Math.*, **7**:4, (2006), 1–7, art. 119.

[26] A. W. Goodman, "Univalent functions and non-analytic curves", *Proc. Amer. Math. Soc.*, **8**, (1957), 598–601.

[27] St. Ruscheweyh, "Neighbourhoods of univalent functions", *Proc. Amer. Math. Soc.*, **81**, (1981), 521–527.

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АННОТАЦИЯ

В статье вводится и изучается новый подкласс мероморфных однолистных функций, определяемых дзета-функцией Гурвица – Лерха. Получены неравенства для коэффициентов, описаны экстремальные точки, радиусы звездности и выпуклости. Наконец, изучены свойства частичных сумм и локальные свойства для функций из класса $\sigma^*(\gamma, k, \lambda, b, s)$.

Ключевые слова: *мероморфная функция, крайняя точка, частичные суммы, окрестность.*