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Theoretical analysis of cloaking problem for 3D model of heat conduction

The direct and extremal problems for the 3D heat conduction model are formulated which are associated with designing spherical thermal cloaking devices. The solvability of both problems is proved. An optimality system is constructed that describes the necessary conditions for an extremum. Some properties of optimal solutions which are consequence of the structure of the optimality system are established.

Key words: *inverse problem, heat conduction, solvability, optimality system*

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Introduction

In recent years significant research has focused on design of invisibility cloaking devices. Beginning with pioneering papers [1, 2] the large number of papers was devoted to developing different schemes of cloaking material objects.

The first works in this field were focused on the electromagnetic cloaking. Then the main results of the electromagnetic cloaking theory were expanded to an acoustic cloaking and to cloaking static (magnetic, electric and thermal) fields (see, e.g., [3, 4]).

Development of the above-mentioned approaches have opened up the opportunities for creation the invisibility cloaking design strategies. They obtained the name of direct design strategies as they were based on solving the forward electromagnetic (acoustic or static) problems. It should be noted that the invisibility devices (hereafter, cloaks) designed on the basis of direct strategies possess serious drawbacks. The main one is the difficulty of their technical realization.

That is why the another cloak design strategy began develop recently. It obtained the name of inverse design as it is related with solving inverse electromagnetic (acoustic or static) problems (see [5, 6]). The optimization method forms the core of the inverse design methodology. This enables us to solve some substantial limitations of previous

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cloaking solutions. A growing number of papers is devoted to applying the inverse design methodology in various cloaking problems. Among them we mention [7–15] where optimization method is applied for numerical solving design problems of cloaks, shields, concentrators and other special functional devices.

In this paper optimization method is applied for study inverse problems for the 3D stationary heat conduction model. These problems arise when designing thermal cloaking devices. We prove the solvability of direct and control problems for the heat conduction model under study, derive the optimality system which describes necessary conditions of extremum and establish some properties of optimal solutions.

1 Statement of direct thermal scattering problem

We begin with statement of direct problem of heat conduction in bounded domain D having the form of rectangular parallelepiped $D = \{(x, y, z) : |x| < x_0, |y| < y_0, |z| < z_0\}$ (see figure 1). We shall assume that external field T^e in D is created by two horizontal boundaries $\Gamma_1 : z = -z_0$ and $\Gamma_2 : z = z_0$ which are kept at temperatures T_1 and T_2, respectively, while the lateral boundaries are thermally insulated. By definition external field T^e satisfies equation $\kappa_0 \Delta T^e = 0$ in D and the following boundary conditions:

$$T^e\big|_{z=-z_0} = T_1, \quad T^e\big|_{z=z_0} = T_2, \quad \frac{\partial T^e}{\partial x}\bigg|_{x=\pm x_0} = 0, \quad \frac{\partial T^e}{\partial y}\bigg|_{y=\pm y_0} = 0.$$

Here κ_0 is a constant thermal conductivity of homogeneous isotropic medium (background) filling D.

We consider the scenario when a material shell (Ω, κ), where Ω is the spherical layer $a < r < b$ in spherical coordinates (r, θ, φ) which is filled by anisotropic medium characterized by heat conductivity tensor κ, is placed into D (see figure 2). Due placing the shell (Ω, κ) into D the field T^e changes and takes the form $T = T^e + T^s$, where T^s is

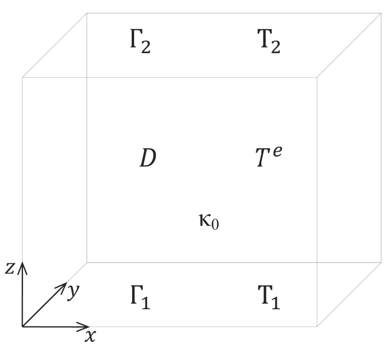

Fig. 1: The geometry of the problem without cloak.

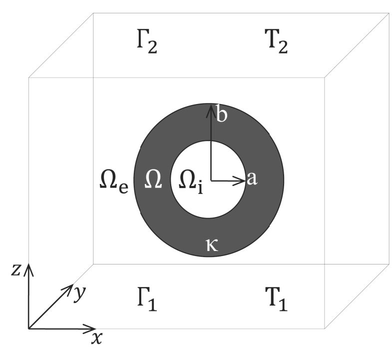

Fig. 2: The geometry of the problem for the case when a cloak is placed in D.

the thermal response of (Ω, κ) which can be determined by solving the respective direct heat conduction problem. In order to formulate the latter problem we denote by Ω_i (or Ω_e) the interior (or exterior) of Ω in D. Then the mentioned heat conduction problem consists of finding a triple of functions: T_i in the interior Ω_i of Ω, T_0 in Ω and T_e in the exterior Ω_e of Ω from the following relations [6, ch. 1]:

$$k_0 \Delta T_i = 0 \quad \text{in} \quad \Omega_i, \quad \operatorname{div}(\kappa \nabla T_0) = 0 \text{ in } \Omega, \quad k_0 \Delta T_e = 0 \text{ in } \Omega_e, \tag{1}$$

$$T_e \Big|_{z=-z_0} = T_1, \quad T_e \Big|_{z=z_0} = T_2, \quad \frac{\partial T_e}{\partial x}\Bigg|_{x=\pm x_0} = 0, \quad \frac{\partial T_e}{\partial y}\Bigg|_{y=\pm y_0} = 0, \tag{2}$$

$$T_i = T_0, \quad k_0 \frac{\partial T_i}{\partial n} = (\kappa \nabla T) \cdot \mathbf{n} \text{ at } r = a, \quad T_e = T_0, \quad k_0 \frac{\partial T_e}{\partial n} = (\kappa \nabla T) \cdot \mathbf{n} \text{ at } r = b. \tag{3}$$

Here $\mathbf{n}$ is the outward unit normal to the boundary of Ω.

We shall use a number of functional spaces when studying direct and control problems under study. In particular we shall use the space $H^1(\tilde{\Omega})$, where $\tilde{\Omega}$ is one of the domains $\Omega_i, \Omega, \Omega_e$, D, and spaces $L^\infty(\Omega)$, $H^s(\Omega)$, $L^2(Q)$, where $Q \subset D$ is an open measurable subset. The scalar products and norms in $H^1(D)$, $H^s(\Omega)$, $L^2(Q)$ will be denoted by

$$(\cdot,\cdot)_{1,D}, \ \|\cdot\|_{1,D}, \quad (\cdot,\cdot)_{s,\Omega}, \ \|\cdot\|_{s,\Omega}, \quad (\cdot,\cdot)_Q, \quad \|\cdot\|_Q.$$

Set

$$L^\infty_{\mu_0} = \{\mu \in L^\infty(\Omega) : \mu(\mathbf{x}) \ge \mu_0\}, \quad H^s_{\mu_0}(\Omega) = \{\mu \in H^s(\Omega) : \mu(\mathbf{x}) \ge \mu_0\}, \quad \mu_0 = \text{const} > 0.$$

We define a space $X = H^1(D)$ with the norm $\|\cdot\|_X^2 := \|\cdot\|_D^2 + \|\nabla \cdot\|^2_{\Omega_i \cup \Omega_e} + \|\nabla \cdot\|^2_\Omega$ and its subspace $X_0 := \{S \in X : S|_{\Gamma_1} = S|_{\Gamma_2} = 0\}$.

Now we introduce the following main assumptions to the data: T_1, T_2 and κ:

(i) $T_1 \in H^{1/2}(\Gamma_1), T_2 \in H^{1/2}(\Gamma_2)$ and there exists a function $T^0 \in X$ such that we have $T^0|_{\Gamma_1} = T_1$, $T^0|_{\Gamma_2} = T_2$, $\|T^0\|_X \le C_T := C_D \left(\|T_1\|_{1/2,\Gamma_1} + \|T_2\|_{1/2,\Gamma_2}\right)$. Here C_D is a constant which depends only on D.

(ii) tensor κ is diagonal in spherical coordinates (r, θ, φ) and its diagonal components (radial, polar and azimuthal conductivities) k_r, k_θ and k_φ satisfy $k_r \in L^\infty_{k_r^0}(\Omega)$, $k_\theta \in L^\infty_{k_\theta^0}(\Omega)$, $k_\varphi \in L^\infty_{k_\varphi^0}(\Omega)$, $k_r^0 = \text{const} > 0$, $k_\theta^0 = \text{const} > 0$, $k_\varphi^0 = \text{const} > 0$.

Now we are able to define a weak solution of problem (1)–(3). To this end we multiply every equation in (1) by $S \in X_0$, integrate by part and add the results obtained. Using Green formulae, assumption (ii) and (2), (3), we arrive at the following relations for finding a triple $T = (T_i, T_0, T_e) \in X$:

$$a(k_0, k; T, S) := a_0(k_0; T, S) + a(k; T, S) = 0 \ \ \forall S \in X_0, \ T\Big|_{\Gamma_1} = T_1, \ \ T\Big|_{\Gamma_2} = T_2. \tag{4}$$

Here, $k := (k_r, k_\theta, k_\varphi)$ while $a_0(k_0; ., .)$ and $a(k; ., .)$ are bilinear forms defined by

$$a_0(k_0; T, S) := k_0 \int\limits_{\Omega_i \cup \Omega_e} \nabla T \cdot \nabla S d\mathbf{x}, \quad a(k; T, S) := \int\limits_{\Omega} (k \nabla T) \cdot \nabla S d\mathbf{x}.$$

Using formula for grad$T$ in spherical coordinates in domain Ω we have

$$a(k;T,S) \equiv \int\limits_{\Omega} k_r \frac{\partial T}{\partial r}\frac{\partial S}{\partial r} d\mathbf{x} + \int\limits_{\Omega} \left(\frac{k_\theta}{r^2}\frac{\partial T}{\partial \theta}\frac{\partial S}{\partial \theta}\right) d\mathbf{x} + \int\limits_{\Omega} \frac{k_\varphi}{r^2 \sin^2\theta}\frac{\partial T}{\partial \varphi}\frac{\partial S}{\partial \varphi} d\mathbf{x} = \\ = a_1(k_r;T,S) + a_2(k_\theta;T,S) + a_3(k_\varphi;T,S). \tag{5}$$

Taking in account Hölder inequality, formula for the norm $\|\cdot\|_X$, Poincaré inequality $\|\nabla T\|_D^2 \geq \delta \|T\|_X^2$ for $T \in X_0$, where the constant $\delta > 0$ depends on D, and (5), one can easily derive the following estimates:

$$\big|a_0(k_0;T,S) + a(k;T,S)\big| \leq \big(k_0 + \|k_r\|_{L^\infty(\Omega)} + \|k_\theta\|_{L^\infty(\Omega)} + \|k_\varphi\|_{L^\infty(\Omega)}\big)\, \|T\|_X \|S\|_X,$$

$$a_0(k_0;T,T) + a(k;T,T) \geq k^0 \|\nabla T\|_D^2 \geq \delta k^0 \|T\|_X^2 \ \ \forall T \in X_0, \ \ k^0 = \min(k_0, k_r^0, k_\theta^0, k_\varphi^0).$$

These estimates mean that the bilinear form $a_0(k_0;\cdot,\cdot) + a(k;\cdot,\cdot)$ is continuous on X and is coercive on X_0. Based on Lax-Milgram theorem one can prove the next theorem.

Theorem 1. *Let, under assumptions (i), (ii), $K_1 \subset L^\infty_{k_r^0}(\Omega)$, $K_2 \subset L^\infty_{k_\theta^0}(\Omega)$ and $K_3 \subset L^\infty_{k_\varphi^0}$ are nonempty bounded sets. Then for any triple $(k_r, k_\theta, k_\varphi) \in K_1 \times K_2 \times K_3$ problem* (1)–(3) *has a unique weak solution $T = (T_i, T_0, T_e) \in X$ which satisfies the estimate $\|T\|_X \leq C_0 C_1 C_T$, $C_0 = \big(\delta k^0\big)^{-1}$. Here, constant C_1 depends on K_1, K_2 and K_3 but is independent of k_r, k_θ, k_φ.*

2 Statement of inverse problem. Using optimization method. Main results

As already stated our purpose is analysis of inverse problems arising when developing technologies of designing thermal cloaking devices. We recall that the general problem of thermal cloaking consists of finding the conductivities k_r, k_θ, k_φ so that two conditions are satisfied: $\nabla T_i = 0$ in Ω_i and $T_e = T^e$ in Ω_e [9]. Here $T = (T_i, T_0, T_e)$ is the corresponding solution of the direct problem (1)–(3). To solve this problem, we apply the optimization method. To this end, we introduce the following cost functionals

$$I_1(T) = \|\nabla T\|_Q^2, \ \ I_2(T) = \|T - T_d\|_Q^2 = \int\limits_Q (T - T_d)^2 d\mathbf{x}, \ \ I_3(T) = 0.5\,[I_1(T) + I_2(T)]. \tag{6}$$

We emphasize (see [6, ch. 1]) that just the functional $I_3(T)$ is used to solve the cloaking problem, while the functional $I_1(T)$ (or $I_2(T)$) is used to solve the problem of internal (or external) cloaking.

Let $K = K_1 \times K_2 \times K_3$. Define the operator

$$G := (G_0, G_1, G_2) : X \times K \to X_0^* \times H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2)$$

where $\langle G_0(T,k), S\rangle = a_0(k_0;T,S) + a(k;T,S)$ for all $S \in X_0$, $G_1 T = T|_{\Gamma_1} - T_1$, $G_2 T = T|_{\Gamma_2} - T_2$ and rewrite weak formulation (4) of problem (1)–(3) as operator equation $G(T,u) = 0$. It is assumed that controls k_r, k_θ, k_φ can change in sets K_1, K_2, K_3 and the following condition holds:

(j) $K_1 \subset H^s_{k_r^0}(\Omega)$, $K_2 \subset H^s_{k_\theta^0}(\Omega)$, $K_3 \subset H^s_{k_\varphi^0}(\Omega)$ are nonempty convex closed sets, where $s > 3/2$, $k_r^0 = \text{const} > 0$, $k_\theta^0 = \text{const} > 0$, $k_\varphi^0 = \text{const} > 0$, $\alpha_0 > 0$.

We consider the following control problem:

$$J(T,k) := \frac{\alpha_0}{2} I(T) + \frac{\alpha_1}{2}\|k_r\|^2_{s,\Omega} + \frac{\alpha_2}{2}\|k_\theta\|^2_{s,\Omega} + \frac{\alpha_3}{2}\|k_\varphi\|^2_{s,\Omega} \to \inf, \ \ G(T,k) = 0. \quad (7)$$

Here, $I(T)$ is a cost functional, α_0, α_1, α_2 and α_3 are nonnegative parameters which serve to regulate the relative importance of each of the terms in (7). Let

$$Z_{ad} = \{(T,k) \in X \times K : G(T,k) = 0, I(T) < \infty\}$$

be the set of admissible pairs for problem (7). We apply the mathematical procedure developed in [7, ch. 1] for studying control problems arising when optimization method is applied when solving inverse problems for linear heat conduction models. Based on this procedure, one can prove the next theorems.

Theorem 2. *Let, under assumptions (i), (j), $I : X \to \mathbf{R}$ be a weakly lower semicontinuous functional and Z_{ad} be a nonempty set. Let $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$ and K_1, K_2, K_3 be bounded sets or $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$ and functional $I(T)$ is bounded below. Then, control problem* (7) *has at least one solution $(T,k) \equiv (T,k_r,k_\theta,k_\varphi) \in X \times K_1 \times K_2 \times K_3$.*

Theorem 3. *Let, under assumptions (i), (j), $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$ or $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$ and K_1, K_2, K_3 be bounded sets. Then control problem* (7) *for $I = I_l(T)$, $l = 1,2,3$, has at least one solution $(T,k_r,k_\theta,k_\varphi) \in X \times K_1 \times K_2 \times K_3$.*

Theorem 4. *Let, under assumptions (i), (j), the pair $(\hat{T},\hat{k}) \equiv (\hat{T},\hat{k}_r,\hat{k}_\theta,\hat{k}_\varphi) \in X \times K_1 \times K_2 \times K_3$ be a solution of problem* (7) *and let a functional $I(T)$ be continuously differentiable at the point $\hat{T}$. Then there exists a unique Lagrange multiplier $(\hat{R},\hat{\zeta}_1,\hat{\zeta}_2) \in X_0 \times H^{1/2}(\Gamma_1)^* \times H^{1/2}(\Gamma_2)^*$ which is the solution of the Euler-Lagrange equation*

$$a_0(k_0;\Psi,\hat{R}) + a(\hat{k};\Psi,\hat{R}) + \langle\hat{\zeta}_1,\Psi\rangle_{\Gamma_1} + \langle\hat{\zeta}_2,\Psi\rangle_{\Gamma_2} = -(\alpha_0/2)\langle I'(\hat{T}),\Psi\rangle \ \forall\Psi \in X, \quad (8)$$

and the following variational inequalities hold:

$$\alpha_1(\hat{k}_r, k_r - \hat{k}_r)_{s,\Omega} + a_1((k_r - \hat{k}_r)\hat{T},\hat{R}) \geq 0 \ \ \forall k_r \in K_1, \quad (9)$$

$$\alpha_2(\hat{k}_\theta, k_\theta - \hat{k}_\theta)_{s,\Omega} + a_2((k_\theta - \hat{k}_\theta)\hat{T},\hat{R}) \geq 0 \ \ \forall k_\theta \in K_2, \quad (10)$$

$$\alpha_3(\hat{k}_\varphi, k_\varphi - \hat{k}_\varphi)_{s,\Omega} + a_3((k_\varphi - \hat{k}_\varphi)\hat{T},\hat{R}) \geq 0 \ \ \forall k_\varphi \in K_3. \quad (11)$$

Direct problem (4), the Euler-Lagrange equation (8) which has the meaning of the adjoint problem for the adjoint state $(\hat{R},\hat{\zeta}_1,\hat{\zeta}_2)$ and variational inequalities (9)–(11) with respect to controls $\hat{k}_r,\hat{k}_\theta,\hat{k}_\varphi$ constitute the optimality system for control problem (7). Based on analysis of the optimality system, one can establish sufficient conditions on the data which provide the uniqueness and stability of solutions of particular control problems. The optimality system (4), (8)–(11) can be also used to develop numerical algorithms for solving control problem (7). The simplest one for the functional $I_2(T)$ can be obtained by applying the fixed point iteration method for solving the optimality system.

3 Conclusion

In this paper, we studied control problems for the 3D heat conduction model. These problems arise when optimization method is applied for solving thermal cloaking problems. Radial, polar and asimuthal conductivities k_r, k_θ and k_φ of the inhomogeneous medium filling the cloaking shell play the role of controls. We proved the solvability of direct and control problems and derived the optimality system describing the necessary conditions of extremum. Based on analysis of the optimality system one can develop a numerical algorithm for solving particular cloaking problem. The alternative one is based on using one of the global minimization methods. We plan to devote a forthcoming paper to studying the properties of the algorithms and to the comparative analysis of results of numerical experiments performed using these algorithms.

References

[1] J. B. Pendry, D. Shurig D. R. Smith, "Controlling electromagnetic fields", *Science*, **312**, (2006), 1780–1782.

[2] U. Leonhardt, "Optical conformal mapping", *Science*, **312**, (2006), 1777–1780.

[3] F. Yang, Z. L. Mei, T. Y. Jin, T. J. Cui, "DC electric invisibility cloak", *Phys. Rev. Lett.*, **109**, (2012), 053902.

[4] T. Han, C. W. Qiu, "Transformation Laplacian metamaterials: recent advances in manipulating thermal and dc fields", *J. Opt.*, **18**, (2016), 044003.

[5] S. Xu, Y. Wang, B. Zhang, H. Chen, "Invisibility cloaks from forward design to inverse design", *Sci. China Inf. Sci.*, **56**, (2013), 120408.

[6] G. V. Alekseev, V. A. Levin, D. A. Tereshko, *Analysis and optimization in designing invisibility devices for material bodies*, Fizmatlit, Moscow, 2021.

[7] I. Peralta, V. D. Fachinotti, "Optimization-based design of heat flux manipulation devices with emphasis on fabricability", *Sci. Rep.*, **7**, (2017), 1–8.

[8] V. D. Fachinotti, A. A. Ciarbonetti, I. Peralta, I. Rintoul, "Optimization-based design of easy-to-make devices for heat flux manipulation", *Int. J. Therm. Sci.*, **128**, (2018), 38–48.

[9] G. V. Alekseev, D. A. Tereshko, "Particle swarm optimization-based algorithms for solving inverse problems of designing thermal cloaking and shielding devices", *Int. J. Heat Mass Transf.*, **135**, (2019), 1269–1277.

[10] G. V. Alekseev, D. A. Tereshko, "Optimization method for axisymmetric problems of electric cloaking of material bodies", *Comp. Math. Math. Phys.*, **59**, (2019), 207–223.

[11] G. V. Alekseev, D. A. Tereshko, "Optimization method in material bodies cloaking with respect to static physical fields", *J. Inv. Ill-posed Prob.*, **27**, (2019), 845–857.

[12] G. V. Alekseev, A. V. Lobanov, "Optimization analysis of electrostatic cloaking problems", *J. Appl. Ind. Math.*, **14**, (2020), 599–609.

[13] G. V. Alekseev, Yu. E. Spivak, "Numerical analysis of two-dimensional magnetic cloaking problems based on an optimization method", *Diff. Eq.*, **56**:9, (2020), 1219–1229.

[14] G. V. Alekseev, Yu. E. Spivak, "Optimization-based numerical analysis of three-dimensional magnetic cloaking problems", *Comp. Math. Math. Phys.*, **61**, (2021), 212–225.

[15] G. V. Alekseev, D. A. Tereshko, Y. V. Shestopalov, “Optimization approach for axisymmetric electric field cloaking and shielding”, *Inv. Prob. Sci. Eng.*, **29**:1, (2021), 40–55.

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Алексеев Г. В. Теоретический анализ задачи маскировки для трехмерной модели теплообмена. *Дальневосточный математический журнал.* 2022. Т. 22. № 2. С. 143–149.

АННОТАЦИЯ

Сформулированы прямая и экстремальная задачи для трехмерной модели теплообмена, связанные с проектированием сферического теплозащитного устройства. Доказана разрешимость обеих задач, построена система оптимальности, описывающая необходимые условия экстремума, установлены некоторые свойства оптимальных решений, являющиеся следствием структуры системы оптимальности.

Ключевые слова: *обратная задача, теплообмен, разрешимость, система оптимальности*