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Model predictive control of dynamic systems with mixed uncertainty and its application to supply chain management

The paper is devoted to a discrete-time linear system with constraints on states and control inputs under conditions of interval and stochastic uncertainty. We use the model predictive control approach and get the optimal control strategy that brings the system to a setpoint. The developed results are applied to the inventory control problem in a supply chain. A numerical example is studied.

Key words: *linear dynamic system, model predictive control, interval-stochastic uncertainty, interval analysis, multi-objective optimization, quadratic programming, inventory control, supply chain.*

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Introduction

The paper studies the model predictive control (MPC) [1] of a linear dynamic system with discrete time subject to constraints and mixed model uncertainty. We assume that the system is affected by additive disturbances of various nature. Some of the disturbances are random with known parameters of the probability distribution, others are given by intervals and nothing more is known about them. Both states and control actions are restricted. We minimize the expected MPC performance index subject to state and control constraints and interval-assigned uncertain inputs. We reduce the problem to a deterministic quadratic programming problem using the interval analysis tools [2] and the multiple-objective optimization techniques [3].

The results are applied to the problem of inventory control in a supply chain with an uncertain demand. The most common is a stochastic approach to modelling uncertainty in inventory control systems. The uncertain demand is assumed to be random. But what if there is not enough historical data for its probabilistic description? In these cases, we can assume that the demand uncertainty is unknown-but-bounded [4, 5], and estimate

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bounds of possible demand values based on available data or practical experience. However, in practice, we often face the situation when we have partial information about demands. For some products we do not have historical demand data, while for others we do. In addition, we can have quite stable orders, mostly within given limits, from some consumers, and random orders from others. In such cases, an uncertain aggregate demand can be decomposed in two sub-vectors, one of which is unknown-but-bounded (or interval), and the other is stochastic. These assumptions are consistent with the mixed interval-stochastic model uncertainty discussed in the paper. Finally, we consider a numerical example and show the effectiveness of the developed MPC strategy which provides the supply chain with a minimum expected level of storage, but a high level of service.

1 Model description and problem statement

We consider the linear system whose dynamics is described by the state space model:

$$x(k+1) = x(k) + Bu(k) + Cd(k) + Cw(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, the initial state $x(0)$ is assumed to be fixed and given, $u(k) \in \mathbb{R}^m$ is the control input, $d(k)$, $w(k) \in \mathbb{R}^l$ are the uncertain disturbance inputs of various nature, and k is the discrete time index. The matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times l}$ describe the system structure, they are fixed and known.

The disturbance $d(k)$ is assumed to lie within in a known interval but the rest is unknown:

$$d(k) \in \boldsymbol{D}, \tag{2}$$

where $\boldsymbol{D} \in \mathbb{IR}^l$, $\boldsymbol{D} = \left[\underline{D}, \overline{D}\right] \geq 0$. This provides an interval uncertainty in the system.

We follow the notation of the informal international standard [6]. Intervals and interval objects (vectors, matrices) are denoted in bold, $\underline{x}, \overline{x}$ are the lower and upper bounds of the interval $\boldsymbol{x}$, $\mathbb{IR}^n = \{\boldsymbol{x} = [\underline{x}, \overline{x}], \underline{x} \leq \overline{x}, \underline{x}, \overline{x} \in \mathbb{R}^n\}$ is the set of all n-dimensional intervals in the classical interval arithmetic $\mathbb{IR}$, $\mathbb{KR}^n = \{\boldsymbol{x} = [\underline{x}, \overline{x}], \underline{x}, \overline{x} \in \mathbb{R}^n\}$ is the set of all n-dimensional intervals in the Kaucher complete interval arithmetic $\mathbb{KR}$ [7].

The disturbance $w(k)$ is the vector of white noises with a zero mean and the given covariance matrix W. This forms a stochastic uncertainty in the system.

Both expected states and control actions must be non-negative and bounded:

$$\mathsf{E}\{x(k+1) \mid x(k)\} \in \boldsymbol{X}, \tag{3}$$

$$u(k) \in \boldsymbol{U}, \tag{4}$$

where $\mathsf{E}\{\cdot|\cdot\}$ denotes the conditional mean; $\boldsymbol{X} \in \mathbb{IR}^n, \boldsymbol{X} = \left[0, \overline{X}\right]$; $\boldsymbol{U} \in \mathbb{IR}^q, \boldsymbol{U} = \left[0, \overline{U}\right]$.

We define the performance index as follows:

$$\begin{aligned} J(k+p|k) = \mathsf{E}\Bigg\{ \sum_{i=1}^{p} \Big(\big(x(k+i|k) - x_s\big)^\top Q \big(x(k+i|k) - x_s\big) - \\ -Q_1\big(x(k+i|k) - x_s\big) + u(k+i-1|k)^\top R u(k+i-1|k)\Big) \;\Big|\; x(k) \Bigg\}, \end{aligned} \tag{5}$$

where $x(k+i|k)$ is the state at the time $k+i$ which is predicted at the time k, $x(k)$ or $x(k|k)$ denotes the state measured at the time k; x_s is the setpoint or target that the system will seek to reach; $u(k+i|k)$ is the predictive control at the time $k+i$ which is computed at the time k; p is the prediction horizon; $Q \in \mathbb{R}^{n\times n}$, $Q_1 \in \mathbb{R}^{1\times n}$ and $R \in \mathbb{R}^{m\times m}$ are the weighting matrices such that Q, R are symmetric positive definite matrices and $Q_1 \geq 0$; the first term $\big(x(k+i|k)-x_s\big)^\top Q\big(x(k+i|k)-x_s\big)$ penalizes the state deviation from the target, the second linear term $Q_1\big(x(k+i|k)-x_s\big)$ penalizes the state negative deviation from the target, and the last term $u(k+i-1|k)^\top Ru(k+i-1|k)$ penalizes the control efforts.

The problem to be solved at the time k is to compute a sequence of the predictive controls $u(k|k), u(k+1|k), \dots, u(k+p-1|k)$ which minimizes performance index (5) subject to system dynamics (1) and constraints (2)–(4). We reduce it to an interval quadratic programming problem where the uncertain inputs are represented by intervals. Since the input data are interval, the objective value is also interval. We calculate the lower and upper bounds of the objective values of the interval quadratic programming problem analytically using interval analysis and formulate a two-objective optimization problem. We then transform the problem into a conventional quadratic programming problem with a single objective by using multi-objective optimization.

According to the MPC approach, at the time k we calculate the predictive controls $u(k|k), u(k+1|k), \dots, u(k+p-1|k)$, but use only the first of them and obtain the feedback control $u(k) = u(k|k)$ as a function of the state $x(k)$. Then the state $x(k+1)$ is measured, the control horizon is moved by one, and the optimization is repeated at the next time $k+1$. The result is the feedback control strategy $\Phi = \{u(k) = u\,(x(k), k)\,,\, k \geq 0\}$.

2 Synthesis of predictive control strategy

The theorem gives a sequence of predictive controls at the time k.

Theorem. *The vector of predictive controls* $\tilde{u}(k) = \big(u(k|k)^\top, u(k+1|k)^\top, \dots, u(k+p-1|k)^\top\big)^\top$ *that minimizes performance index* (5) *subject to system dynamics* (1) *and constraints* (2)–(4) *is defined at the time k as a solution to the quadratic programming problem with the criterion* $Y(k+p|k) = \tilde{u}(k)^\top H\tilde{u}(k) + 2G(k)\tilde{u}(k)$ *under the constraints* $\big(B\ 0_{n\times m}\ 0_{n\times m}\dots 0_{n\times m}\big)\tilde{u}(k) \in \boldsymbol{X} \ominus \boldsymbol{CD} - x(k)$ *and* $\tilde{u}(k) \in \tilde{\boldsymbol{U}}$. *Here* $H, G(k)$ *are the block matrices of the type:*

$$H = \begin{pmatrix} H_{11} & H_{12} & \dots & H_{1p} \\ H_{21} & H_{22} & \dots & H_{2p} \\ \vdots & \ddots & \vdots & \\ H_{p1} & H_{p2} & \dots & H_{pp} \end{pmatrix}, \quad H_{ij} = \begin{cases} (p-j+1)B^\top QB, & i<j, \\ R+(p-j+1)B^\top QB, & i=j, \\ (p-i+1)B^\top QB, & i>j, \end{cases}$$

$$G(k) = \Big(\big(x(k)-x_s\big)^\top Q - \tfrac{1}{2}Q_1\Big)BK + \operatorname{mid}\boldsymbol{DF},$$

where

$$K = (K_1\ K_2 \dots K_p), \quad K_i = (p-i+1)I_m,$$

$$F = \begin{pmatrix} F_{11} & F_{12} & \dots & F_{1p} \\ F_{21} & F_{22} & \dots & F_{2p} \\ \vdots & \ddots & \vdots & \\ F_{p1} & F_{p2} & \dots & F_{pp} \end{pmatrix}, \quad F_{ij} = \begin{cases} (p-j+1)C^\top QB, & i \le j, \\ (p-i+1)C^\top QB, & i > j, \end{cases}$$

$0_{n\times m}$ is the zero matrix of the dimension $n \times m$, I_m is the unit matrix of the dimension m, $\tilde{\boldsymbol{U}} = (\boldsymbol{U}^\top, \boldsymbol{U}^\top, \dots, \boldsymbol{U}^\top)^\top$, $\tilde{\boldsymbol{D}} = (\boldsymbol{D}^\top, \boldsymbol{D}^\top, \dots, \boldsymbol{D}^\top)^\top$, $\boldsymbol{CD}$ is the result of multiplying the real matrix C by the interval vector $\boldsymbol{D}$, $\boldsymbol{DF}$ is the result of multiplying the interval vector $\tilde{\boldsymbol{D}}^\top$ by the real matrix F, $\operatorname{mid}\boldsymbol{x}$ *is the midpoint of the interval $\boldsymbol{x}$, $\boldsymbol{x} \ominus \boldsymbol{y} = [\underline{x} - \underline{y}, \overline{x} - \overline{y}]$ is the internal subtraction in $\mathbb{KR}$.*

It is worth noting that, due to the interval uncertainty in the system, we can only steer the state to a tube sufficiently close to the target x_s, and keep the state trajectory, on average, within the target tube. The target tube is a sequence of the sets that at each time contain all the states whose future trajectories can be kept inside the constraints, for all admissible disturbances [4]. It is clear that the width of this tube depends on the width of the initial uncertainty intervals. Indeed, the problem of keeping the state $x(k)$, on average, in some tube $\boldsymbol{X}(a,b) = [a,b]$ has a solution if and only if, for all $x(k) \in \boldsymbol{X}(a,b)$, there is a control $u(k) \in \boldsymbol{U}$ so that $\mathsf{E}\{x(k+1) \mid x(k)\} = x(k) + Bu(k) + Cd(k) \in \boldsymbol{X}(a,b)$ for all $d(k) \in \boldsymbol{D}$. That takes place if and only if $x(k) + Bu(k) + \boldsymbol{CD} \in \boldsymbol{X}(a,b)$, and then $x(k) + Bu(k) \in \boldsymbol{X}(a,b) \ominus \boldsymbol{CD}$. It makes sense if and only if $\boldsymbol{X}(a,b) \ominus \boldsymbol{CD} \in \mathbb{IR}^n$, that is $a - \underline{CD} \le b - \overline{CD}$. We can argue that $\overline{CD} - \underline{CD} \le b - a$ and $\operatorname{wid}\boldsymbol{CD} \le \operatorname{wid}\boldsymbol{X}(a,b)$. Therefore, the minimum width of the tube, within which, on average, the state $x(k)$ can be kept for all possible values of the demand, is given by $\operatorname{wid}\boldsymbol{CD} = \overline{CD} - \underline{CD}$.

3 Inventory control problem in a supply chain: example

We can apply the above results to multi-echelon inventory optimization in a supply chain with an uncertain demand. We describe the supply chain by the dynamic network model in which the nodes represent warehouses and the arcs are controllable and uncontrollable flows in the network. The network dynamics can be described by equation (1), where $x(k)$ represents the storage levels in the network nodes; $u(k)$ is the controllable flows which redistribute resources between the network nodes, possibly process them, and plan deliveries from outside; $d(k)$, $w(k)$ are the uncontrollable flows which represent the demand in the network nodes that can arise from outside and other nodes. The matrices B and C describe the network structure. As the unit of time k we can take, for example, a day, a week, a month, or a longer period. Constraints (3), (4) define the capacities and requirements, such as storage and order quantity limits. In (3), the lower bound equal to zero means that an out-of-stock is undesirable, but possible. The target x_s in performance index (5) defines a desired target storage level. Given the fact that we deal with storage levels, the goal of keeping the state is close to but, preferably, not below the target is consistent with (5). We suggest setting x_s at zero during the first simulation and waiting for the target tube $\boldsymbol{X}(0, \operatorname{wid}\boldsymbol{CD})$ to be received. We calculate a service level in the network nodes as the proportion of satisfied demand and if it is not high enough, we will gradually increase the target and form a safety stock until a required service level is received.

Let us consider now the supply chain represented by Figure 1. It has three interrelated production-distribution centres represented by three nodes. Nodes 1 and 2 make products A and B, these products are used later for making product AB in node 3. The controllable flows $u_1,\ u_2$ describe the production levels of A in node 1 and B in node 2, respectively, u_3 describes a production line in node 3 which takes some amount of products A and B to produce the same amount AB in node 3. The arc u_4 models an additional flexible capacity which can be split in any proportion between two production lines A and B. If u_4 works at full force, the flexible capacity is fully used to produce B, while if it works at zero force, the flexible capacity is fully used to produce A. The uncontrollable flows $d_1, w_1,\ d_2, w_2,\ d_3, w_3$ represent the demand for products A, B and AB from outside. And $d_4, w_4,\ d_5, w_5$ represent the uncontrollable redistribution flows between the nodes. The structural matrices B and C for the system have the form:

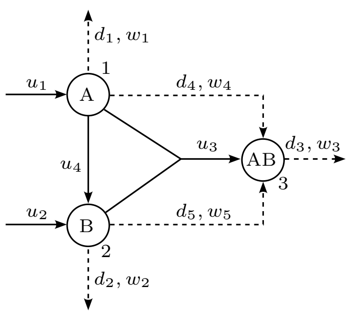

Fig. 1: Supply chain with three nodes and controllable (solid) and uncontrollable (dashed) flows between them

$$B = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

We assume that $\boldsymbol{X} = \big([0,130]\ [0,120]\ [0,150]\big)^\top$, $\boldsymbol{U} = \big([0,170]\ [0,50]\ [0,100]\ [0,70]\big)^\top$, $\boldsymbol{D} = \big([5,25]\ [20,30]\ [60,80]\ [0,20]\ [0,10]\big)^\top$.

The example is an adapted version of the example from [4]. We added the white noise w with a zero mean and the covariance $W = \operatorname{diag}\big(\sigma_1^2, \sigma_2^2, \ldots, \sigma_l^2\big)$, $\sigma_i^2 = 0.25 \operatorname{wid} \boldsymbol{D}_i$. We assume that the demand cannot be backlogged and demands during stockouts are completely lost. The initial state is $x(0) = (130\ 120\ 150)^\top$ and the target is $x_s = (0\ 0\ 0)^\top$. The weighting matrices are chosen as $Q = I_n$, $Q_1 = (1\ 1\ 1)^\top$, $R = I_m$, the prediction horizon is $p = 6$, the problem is solved for 100 time steps. We carried out modelling and simulation in MATLAB.

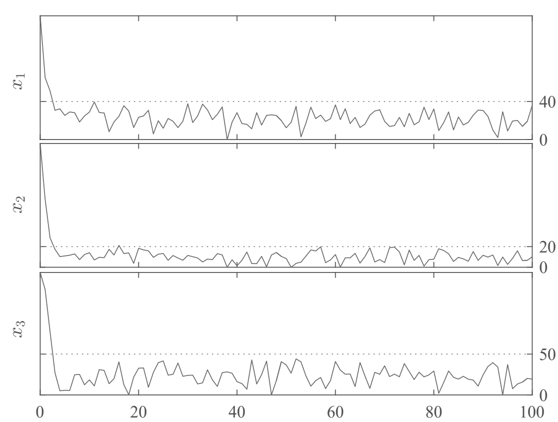

Fig. 2: Trajectories of x_i (solid) and levels $\operatorname{wid} \boldsymbol{CD}_i$ (dashed)

Figure 2 shows the inventory dynamics in the network nodes. In all the nodes, a decreasing trend of the storage levels can be observed. In our case, we get $\boldsymbol{CD} = \big([-45,-5]\ \ [-40,-20]\ \ [-80,-30]\big)^\top$ and $\operatorname{wid}\boldsymbol{CD} = (40\ \ 20\ \ 50)^\top$. Starting from some timestep, the state trajectory, on average, lies within the minimal tube $\boldsymbol{X}(0, \operatorname{wid}\boldsymbol{CD})$.

As the simulation showed, we received high levels of service in the network: 98.72% in node 1, 99.98% in node 2, and 99.67% in node 3. In this case, there is no need to increase the target x_s to form a safety stock.

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Чаусова Е. В. Управление с прогнозированием динамическими системами в условиях интервальной и стохастической неопределенности и применение к управлению цепями поставок. *Дальневосточный математический журнал.* 2022. Т. 22. № 2. С. 152–157.

АННОТАЦИЯ

В работе рассматривается линейная система в дискретном времени с ограничениями на состояния и управляющие воздействия в условиях интервальной и стохастической неопределенности. Для синтеза оптимальной стратегии управления, приводящей систему к заданному состоянию, используется подход на основе управления с прогнозирующей моделью. Полученные результаты применены к задаче управления запасами в цепи поставок. Рассмотрен численный пример.

Ключевые слова: *линейная динамическая система, управление с прогнозирующей моделью, интервально-стохастическая неопределенность, интервальный анализ, многокритериальная оптимизация, квадратичное программирование, управление запасами, цепь поставок.*