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Numerical solution of shielding problem for 3D model of electrostatics in the presence of anisotropic layer

An economical numerical algorithm for solving the problem of designing a shielding device for a 3D model of electrostatics is proposed and implemented. The algorithm is based on the use of a multilayered shell. Its first layer is anisotropic, and the remaining layers are filled with one of two predefined isotropic materials according to an alternating design scheme. It is shown that the applying of the developed algorithm enables us to design easy-to-implement shielding shells with high efficiency.

Key words: *Inverse problems, shielding problem, optimization method*

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Introduction

In recent years, much attention has been paid to the development of design technologies for devices for electrical cloaking and shielding of material bodies [1–3]. An important trend in electrical shielding is associated with the use of radially anisotropic cylindrical or spherical shells (see [4–6]). It was shown in [4] that a high cloaking effect can be achieved even for a single-layer cylindrical shell, but in the case of a small diameter of the cloaked body and/or at a very high anisotropy coefficient.

We also note a series of works [9–15] related to the development of efficient numerical algorithms for solving the design problems of cylindrical or spherical cloaking and shielding devices for models of electrostatics and magnetostatics.

In this article, we will consider a more general physical scenario, when a multilayer shell with generally anisotropic layers is used for shielding. Using the results of [13–15] we propose below an economical numerical algorithm for solving design problem of shielding shell (hereafter shield) for 3D model of electrostatics. The algorithm is based on using M-layered spherical shell. The first layer of this shell is anisotropic while remaining $M-1$ layer are homogeneous, isotropic and their permittivities are obeyed the alternating

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design rule with respect to the given pair permittivities $\varepsilon_{\min}$, $\varepsilon_{\max}$. We show with the help of numerical experiments that our algorithm enables us to design layered shielding shells having high efficiency and simplicity of technical realization.

1 Statement of direct and inverse problems

We start with formulation of the direct problem of electrostatics, considered in the entire space $\mathbb{R}^3$. Let us assume that the space $\mathbb{R}^3$ is filled with a homogeneous medium with a constant permittivity $\varepsilon_0 > 0$ and that a constant electric field $\mathbf{E}_a = -\mathrm{grad}U_a$ is created in $\mathbb{R}^3$ corresponding to the electric potential U_a described in spherical coordinates r, θ, φ by the formula $U_a(r,\theta) = -\frac{E_a r\cos\theta}{b}$, where $E_a = \mathrm{const}$, $b = \mathrm{const}$. Let us further assume that an object (Ω, ε) where Ω is a spherical layer $\Omega = \{\mathbf{x} \in \mathbb{R}^3 : a < r < b\}$, ε is the permittivity of the medium filling Ω is placed into $\mathbb{R}^3$. Then the field U_a changes and takes the form $U = U_a + U_s$. Here U_s is the scattered (electrical) response of the object caused by the placing of an object (Ω, ε) into $\mathbb{R}^3$.

We assume that the medium occupying the region Ω is piecewise homogeneous in the sense that Ω can be divided into a finite number of elementary spherical layers

$$\Omega_m = \{R_{m-1} < r = |\mathbf{x}| < R_m\}, \ \ m = 1, 2, ..., M, \ \ R_0 = a, \ \ R_M = b, \tag{1}$$

of the same width $d = (b-a)/M$. Each of them is filled with a homogeneous anisotropic (generally) medium, whose constant permittivity ε_m is described by the diagonal in spherical coordinates tensor $\varepsilon_m = \mathrm{diag}(\varepsilon_{rm}, \varepsilon_{tm})$, $m = 1, 2, ..., M$. Here ε_{rm} (or ε_{tm}) is the radial (or tangential) component of the tensor ε_m. This partition of Ω into parts Ω_m corresponds to the following global radial and tangential permittivities ε_r, ε_t of Ω:

$$\varepsilon_r(\mathbf{x}) = \sum_{m=1}^{M} \varepsilon_{rm}\chi_m(\mathbf{x}), \ \ \mathbf{x} \in \Omega, \ \ \varepsilon_t(\mathbf{x}) = \sum_{m=1}^{M} \varepsilon_{tm}\chi_m(\mathbf{x}), \ \mathbf{x} \in \Omega. \tag{2}$$

Here χ_m is the characteristic function of the elementary layer Ω_m, which is equal to one in Ω_m and zero outside Ω_m. Below, to describe a piecewise homogeneous medium filling Ω, we will use the vector $\mathbf{e} = (e_{r1}, e_{t1}, ..., e_{rm}, e_{tm})$, composed of the permeabilities $\varepsilon_m = (\varepsilon_{rm}, \varepsilon_{tm})$ of individual layers Ω_m, and the pair $(\Omega, \mathbf{e})$ will be referred to as the electrical (material) shell.

In addition to the sets (1), we define the sets $\Omega_0 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < a\}$ and $\Omega_{M+1} = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| > b\}$ and set $U_m = U|_{\Omega_m}$, $m = 0, 1, ..., M+1$. Then the direct problem of finding the total field $U = (U_0, U_1, ..., U_{M+1})$ reduces to finding all $M+2$ fields U_m in the regions Ω_m, $m = 0, 1, ..., M+1$ by solving the following electrical conjugation problem:

$$\Delta U_0 = 0 \text{ in } \Omega_0, \ \Delta U_{M+1} = 0 \text{ in } \Omega_{M+1}, \ \mathrm{div}(\varepsilon_m \mathrm{grad} U_m) = 0 \text{ in } \Omega_m, \ m = \overline{1, M}, \tag{3}$$

$$\mathrm{grad}U_m \times \mathbf{n} - \mathrm{grad}U_{m+1} \times \mathbf{n} = 0 \text{ at } r = R_m, \ m = 0, 1, ..., M, \tag{4}$$

$$\varepsilon_0 \frac{\partial U_0}{\partial r} = \varepsilon_{r1}\frac{\partial U_1}{\partial r} \text{ at } r = R_0, \ \ \varepsilon_{rM}\frac{\partial U_M}{\partial r} = \varepsilon_e \frac{\partial U_{M+1}}{\partial r} \text{ at } r = R_M, \tag{5}$$

$$\varepsilon_{rm}\frac{\partial U_m}{\partial r} = \varepsilon_{r(m+1)}\frac{\partial U_{m+1}}{\partial r} \text{ at } r = R_m, \ m = 1, ..., M-1, \tag{6}$$

$$U_0(\mathbf{x}) = O(1) \text{ as } r = |\mathbf{x}| \to 0, \ \ U_{M+1}(\mathbf{x}) \to U_a(\mathbf{x}) \text{ as } r \to \infty, \tag{7}$$

considered in the space $\mathbb{R}^3$. Similarly to [6], we look for the fields U_m, $m = \overline{1, M}$ as

$$U_0(r,\theta)=\alpha_0\left(\frac{r}{b}\right)\cos\theta \text{ in } \Omega_0,\ U_m(r,\theta)=\left(\alpha_m\left(\frac{r}{b}\right)^{\nu_m}+\beta_m\left(\frac{b}{r}\right)^{\nu_m+1}\right)\cos\theta \text{ in } \Omega_m,$$

$$U_{M+1}(r,\theta)=\left(-E_a\left(\frac{r}{b}\right)+\beta_{M+1}\left(\frac{r}{b}\right)^{-2}\right)\cos\theta \text{ in } \Omega_{M+1}, \nu_m=\frac{1}{2}\left(\sqrt{1+8\left(\frac{\varepsilon_{tm}}{\varepsilon_{rm}}\right)}-1\right). \quad (8)$$

Here $\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_M, \beta_M, \beta_{M+1}$ are some coefficients. It is easy to check that functions (8) satisfy all equations in (3) and conditions (7) for any values of coefficients α_m, β_m. It remains to choose them so that the boundary conditions (4)–(6) are satisfied.

Substituting (8) into (4)–(6), we arrive at the following system of $2M + 2$ linear algebraic equations with respect to $2M + 2$ coefficients α_0, α_m, β_m, β_{M+1}, $m = \overline{1, M}$:

$$\begin{gathered}
\alpha_0-\alpha_1\left(\frac{b}{R_0}\right)^{-\nu_1+1}-\beta_1\left(\frac{b}{R_0}\right)^{\nu_1+2}=0,\\
\varepsilon_i\alpha_0-\varepsilon_{r1}\alpha_1\nu_1\left(\frac{b}{R_0}\right)^{-\nu_1+1}+\varepsilon_{r1}\beta_1(\nu_1+1)\left(\frac{b}{R_0}\right)^{\nu_1+2}=0,\\
\alpha_m\left(\frac{b}{R_m}\right)^{-\nu_m}+\beta_m\left(\frac{b}{R_m}\right)^{\nu_m+1}-\alpha_{m+1}\left(\frac{b}{R_m}\right)^{-\nu_{m+1}}-\beta_{m+1}\left(\frac{b}{R_m}\right)^{\nu_{m+1}+1}=0,\\
\varepsilon_{rm}\alpha_m\nu_m\left(\frac{b}{R_m}\right)^{-\nu_m+1}-\varepsilon_{rm}\beta_m(\nu_m+1)\left(\frac{b}{R_m}\right)^{\nu_m+2}-\varepsilon_{r(m+1)}\alpha_{m+1}\nu_{m+1}\left(\frac{b}{R_m}\right)^{-\nu_{m+1}+1}+\\
+\varepsilon_{r(m+1)}\beta_{m+1}(\nu_{m+1}+1)\left(\frac{b}{R_m}\right)^{\nu_{m+1}+2}=0,\quad m=\overline{1,M-1},\\
\alpha_M\left(\frac{b}{R_M}\right)^{-\nu_M+1}+\beta_M\left(\frac{b}{R_M}\right)^{\nu_M+2}-\beta_{M+1}\left(\frac{b}{R_M}\right)^3=-E_a,\\
\varepsilon_{rM}\alpha_M\nu_M\left(\frac{b}{R_M}\right)^{-\nu_M+1}+\varepsilon_{rM}\beta_M(\nu_M+1)\left(\frac{b}{R_M}\right)^{\nu_M+2}+2\varepsilon_e\beta_{M+1}\left(\frac{b}{R_M}\right)^3=-\varepsilon_e E_a. \quad (9)
\end{gathered}$$

Solving the system (9) and substituting the found values α_m, β_m into (8), we can find the corresponding fields U_0 in Ω_0, U_m in Ω_m, $m = \overline{1, M}$ and U_{M+1} in Ω_{M+1}, forming the desired solution of the problem (3)–(7).

Denote by $U[\mathbf{e}] = (U_0[\mathbf{e}], U_1[\mathbf{e}], \ldots, U_{M+1}[\mathbf{e}])$, where $\mathbf{e} = (e_{r1}, e_{t1}, \ldots, e_{rM}, e_{tM})$ is the solution of the problem (3)–(7) corresponding to the permittivity tensors $\varepsilon_m = \mathrm{diag}(\varepsilon_{rm}, \varepsilon_{tm})$ in Ω_m and to the constant permittivity ε_0 in Ω_0 and Ω_{M+1}. Let B_R be a ball of sufficiently large radius R containing Ω inside it. Let $\Omega_e = \Omega_{M+1} \cap B_R$.

We remind that our goal is to solve the inverse problem for the model (3)–(7) associated with designing shielding shells. This inverse problem consists of finding values e_{r1}, e_{t1},..., e_{rM}, e_{tM} from the following condition [14, 15]:

$$\nabla U_0[\mathbf{e}] = 0 \text{ in } \Omega_0, \quad \mathbf{e} \equiv (\varepsilon_{r1}, \varepsilon_{t1}, ..., \varepsilon_{rm}, \varepsilon_{tm}). \quad (10)$$

The shell $(\Omega, \mathbf{e})$, which ensures the exact fulfilment of condition (10), is called a perfect shielding shell or simply a shield.

2 Using optimization method. Numerical results

For solving our inverse problem we apply the optimization method. Similarly [14, 15] we define the bounded set $K = \{\mathbf{e} : 0 < \varepsilon_{\min} \le \varepsilon_{rm}, \varepsilon_{tm} \le \varepsilon_{\max},\ m = \overline{1, M}\}$ to which we refer to as a control set. Here given positive constants $\varepsilon_{\min}$ and $\varepsilon_{\max}$ are lower and upper boundaries of the control set K. Let us define the cost functional

$$J_i(\mathbf{e}) = \frac{\|\nabla U_0[\mathbf{e}]\|_{L^2(\Omega_0)}}{\|\nabla U^e\|_{L^2(\Omega_0)}},\ \|\nabla U_0[\mathbf{e}]\|_{L^2(\Omega_0)} = \int_{\Omega_0} |\nabla U_0[\mathbf{e}]|^2 d\mathbf{x},\ \|\nabla U^e\|^2_{L^2(\Omega_0)} = \int_{\Omega_0} |\nabla U^e|^2 d\mathbf{x}, \quad (11)$$

and formulate the following control problem:

$$J_i(\mathbf{e}) \to \min,\ \ \mathbf{e} \in K. \quad (12)$$

Problem (12) was studied in [14, 15] in the special case when all layers are isotropic. It has been shown that the optimal solution ε^{opt} has the bang-bang property and corresponds to the alternating design with respect to the pair $(\varepsilon_{\min}, \varepsilon_{\max})$. It means that

$$\varepsilon_1^{opt} = \varepsilon_3^{opt} = ... = \varepsilon_{M-1}^{opt} = \varepsilon_{\min},\ \ \varepsilon_2^{opt} = \varepsilon_4^{opt} = ... = \varepsilon_M^{opt} = \varepsilon_{\max}. \quad (13)$$

Moreover, it turned out that under the smallness condition $\varepsilon_{\min} \approx 0.01$, $J(\varepsilon^{opt})$ tends to zero with increasing contrast $\varepsilon_{\max}/\varepsilon_{\min}$ and the number of layers M. The smallness condition is restrictive, requiring the use of special materials for the technical implementation of the solutions obtained. Below we will show that the presence of anisotropic layers makes it possible to get rid of this limitation and to design a highly efficient shell.

More specifically, the designed shell will consist of the first anisotropic layer corresponding to the pair $(\varepsilon_{r1}, \varepsilon_{t1})$ and $M-1$ isotropic layers corresponding to the alternating design $(\varepsilon_2, \varepsilon_3, ..., \varepsilon_M)$. Here all parameters ε_{r1}, ε_{t1}, ε_2,..., ε_M take only one of two values $\varepsilon_{\min}$ and $\varepsilon_{\max}$. Taking into account the specified structure of the designed shell, the solution of the shielding problem consists of two stages. First, we substitute the indicated data $(\varepsilon_{r1}, \varepsilon_{t1})$, $\varepsilon_2, \varepsilon_3, ..., \varepsilon_M$ into system (9) and find the coefficient α_0 by solving it. Next, we determine the field U_0 using the first formula in (8) and calculate the value of the functional $J_i(\mathbf{e})$ using (11). A sufficiently small value of $J_i(\mathbf{e})$ will correspond to a high shielding efficiency of the shell being designed. Below we will refer to the described algorithm as Algorithm 1. The result of Algorithm 1 is an approximate optimal solution $\varepsilon^* = ((\varepsilon_{r1}, \varepsilon_{t1}), \varepsilon_2, ..., \varepsilon_M)$ to problem (12).

Let us discuss now the results of the numerical solution of the shielding problem using Algorithm 1 for the following pairs of $(\varepsilon_{\min}, \varepsilon_{\max})$: $(0.021, 2.1)$ and $(2.1, 2100)$. The externally applied field has the form: $\mathbf{E}_a = -\mathrm{grad}U_a$, $U_a(r, \theta) = -\frac{E_a r \cos\theta}{b}$. Our first test concerns to the first pair $\varepsilon_{\min} = 0.021$, $\varepsilon_{\max} = 2.1$. The results of the numerical solution of problem (12) using Algorithm 1 in the form of values of the permittivities ε_{r1}, ε_{t1}, $\varepsilon_2 = \varepsilon_{\max}$, $\varepsilon_3 = \varepsilon_{\min}$, $\varepsilon_M = \varepsilon_{\max}$ of the first, second, third and last layers, respectively, and the value of $J_i(\mathbf{e}^*)$ where $\varepsilon^* = (\varepsilon_{r1}, \varepsilon_{t1}; \varepsilon_2, ..., \varepsilon_M)$, are presented in Table 1 for even values of M varying from 2 to 16. The remaining values of the permittivities ε_m, $m = 4, 5, ..., M-1$ are determined from the relations (13). Table 1 shows that all values of $J_i(\mathbf{e}^*)$ correspond to low shielding efficiency. This can be explained by the low contrast $\varepsilon_{\max}/\varepsilon_{\min} = 100$. The second disadvantage of Table 1 is the value of $\varepsilon_{\min} = 0.021$ which corresponds to a metamaterial that is difficult to implement.

Table 1: Shielding pr.: $\varepsilon_{\min} = 0.021$, $\varepsilon_{\max} = 2.1$, $R_a = 0.03$, $R_b = 0.05$, Contrast $= 100$.

M	$(\varepsilon_{r1}, \varepsilon_{t1})$	ε_2	ε_3	ε_M	$J_i(\mathbf{e}^*)$
2	$(2.1, 0.021)$	2.1			1.222×10^0
4	$(2.1, 0.021)$	2.1	0.021	2.1	2.341×10^{-1}
8	$(2.1, 0.021)$	2.1	0.021	2.1	8.882×10^{-2}
12	$(2.1, 0.021)$	2.1	0.021	2.1	6.322×10^{-2}
16	$(2.1, 0.021)$	2.1	0.021	2.1	5.341×10^{-2}

Table 2: Shielding pr.: $\varepsilon_{\min} = 2.1$, $\varepsilon_{\max} = 2100$, $R_a = 0.03$, $R_b = 0.05$, Contrast $= 1000$.

M	$(\varepsilon_{r1}, \varepsilon_{t1})$	ε_2	ε_3	ε_M	$J_i(\mathbf{e}^*)$
2	$(2100, 2.1)$	2100			5.829×10^{-3}
4	$(2100, 2.1)$	2100	2.1	2100	3.908×10^{-4}
8	$(2100, 2.1)$	2100	2.1	2100	2.811×10^{-5}
12	$(2100, 2.1)$	2100	2.1	2100	8.679×10^{-6}
16	$(2100, 2.1)$	2100	2.1	2100	4.739×10^{-6}

In order to increase the shielding efficiency of the shell being designed, it is sufficient to increase the contrast of the pair $(\varepsilon_{\min}, \varepsilon_{\max})$. This can be seen from the analysis of Table 2 which is an analogue of Table 1 for the pair $(\varepsilon_{\min}, \varepsilon_{\max}) = (2.1, 2100)$ with contrast 1000. Note that the value $\varepsilon_{\min} = 2.1$ describes the permittivity of teflon, and $\varepsilon_{\max} = 2100$ describes the permittivity of barium titanate. It can be seen that the value of $J_i(\mathbf{e}^*)$ decreases from 5.829×10^{-3} to 4.739×10^{-6} as M increases from 2 to 16. The last value $J_i(\mathbf{e}^*)$ corresponds to the high shielding efficiency of the shell (Ω, ε^*).

3 Conclusion

The results obtained confirm the high efficiency of the shielding shell with high contrast even in the case of a small number of homogeneous layers, the first of which is filled with an anisotropic medium, while the remaining ones are isotropic.

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АННОТАЦИЯ

Предлагается и численно реализуется экономичный численный алгоритм решения задачи экранирования для трехмерной модели электростатики. Алгоритм основывается на использовании многослойной оболочки, первый слой которой является анизотропным, а остальные слои заполнены одним из двух заранее заданных изотропных материалов. Показывается на основе проведенных вычислительных экспериментов, что экранирующее устройство, спроектированное с помощью разработанного метода, обладает простотой технической реализацией и наивысшей эффективностью в рассматриваемом классе устройств.

Ключевые слова: *обратные задачи, задача экранирования, метод оптимизации.*