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On the problem of determining the scattering coefficient in frequency modulated sounding of a medium

Within the framework of the kinetic model of the transfer of linear frequency modulated radiation in a scattering medium, an inverse problem is formulated, which consists in determining the volume scattering coefficient of sound. Additional information in the problem is the frequency-averaged angular distribution of the radiation flux density at a given point in space. An analytical solution of the inverse problem is obtained in the single scattering approximation.

Key words: *radiative transfer equation, linear frequency modulated sounding, scattering coefficient, inverse problem.*

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In the papers [1, 2] the problem of finding the scattering coefficient for the non-stationary radiation transfer equation with a pulse radiation source in a heterogeneous medium is studied. The applicability of the single scattering approximation for solving the inverse problem in two-dimensional and three-dimensional cases is examined.

For the processes of acoustic sounding of the ocean it is necessary for the signal strength/noise ratio to be greater than one [3]. A simple way to increase this ratio is for the source to emit a stronger pulse. However, this approach isn't always feasible to implement [3]. Because of this the only way to increase the amount of energy released is to make the pulse longer. An acceptable time-resolution can be achieved by making the frequency band wider, so it is possible to change the frequency during the pulse to study a long and wideband signal. In this paper we examine the case of a chirp signal source. An inverse problem for the non-stationary radiation transfer equation with a chirp signal source will be formulated and an explicit formula for the scattering coefficient using single-scattering approximation will be found.

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1 Formulation of an inverse problems for the equation of transfer of frequency modulated radiation

The non-stationary process of high-frequency wave fields spreading in an isotropically scattering medium can be described using the integro-differential radiation transfer equation [4–7] which has the following form in the two-dimensional case:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{k}\cdot\nabla_r I(\mathbf{r},\mathbf{k},t,\nu) + \mu I(\mathbf{r},\mathbf{k},t,\nu) = \frac{\sigma(\mathbf{r})}{2\pi}\int\limits_{\Omega} I(\mathbf{r},\mathbf{k}',t,\nu)d\mathbf{k}' + J(\mathbf{r},\mathbf{k},t,\nu), \quad (1)$$

where $\mathbf{r}\in\mathbb{R}^2, t\in[0,T]$ and the wave vector $\mathbf{k}$ belongs to a unit circle $\Omega = \{\mathbf{k}\in\mathbb{R}^2 : |\mathbf{k}| = 1\}$. The function $I(\mathbf{r},\mathbf{k},t,\nu)$ is interpreted as a wave energy flux density at the point of time t at the point $\mathbf{r}$, propagating in the direction $\mathbf{k}$ at the frequency ν at the speed c. Values μ and σ are the attenuation and scattering coefficients respectively, and the function J the sources of the sound field.

We add the following initial condition to the equation (1)

$$I^-(\mathbf{r},\mathbf{k},0,\nu) = 0, \qquad (\mathbf{r},\mathbf{k})\in G\times\Omega \quad (2)$$

and assume that the function J describes a point radiation source of unit strength located at the origin of coordinates and emitting a chirp signal with the duration Δt in the frequency range $[\nu_{\min},\nu_{\max}]$:

$$J(\mathbf{r},\mathbf{k},t,\nu) = \delta(\mathbf{r})\delta\left(t - t_0 - (\nu-\nu_0)\frac{\Delta t}{\Delta\nu}\right), \quad (3)$$

where $\Delta\nu = \nu_{\max} - \nu_{\min}$, δ is the Dirac's delta function, $\nu_0 = (\nu_{\min}+\nu_{\max})/2$ the carrier frequency and $I^{\pm}(\mathbf{r},\mathbf{k},t,\nu) = \lim\limits_{\epsilon\to-0} I(\mathbf{r}\pm\epsilon\mathbf{k},\mathbf{k},t\pm\epsilon,\nu)$.

The direct problem of the transfer equation (1) is the problem of determination of the function I using equation (1) and initial condition (2) with all coefficients given (c,μ,σ,J). It is assumed that c and μ are scalar values, and $\sigma(\mathbf{r})$ is a piecewise continuous function in $\mathbb{R}^2$, and $\sigma(\mathbf{r})\le\mu$. Inverse problems for the kinetic equations of the radiation transfer differ in formulations and research methods [6, 8–13]. We study an inverse problem that consists of determination of the function σ based on relations (1)–(3) and an additional condition

$$\frac{1}{\Delta\nu}\int\limits_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2} I^+\left(0,\mathbf{k},t+(\nu-\nu_0)\frac{\Delta t}{\Delta\nu},\nu\right)d\nu = P(\mathbf{k},t). \quad (4)$$

Values c,μ and function P are assumed to be given. In [1,2] along with the pulse radiation source an additional condition with the following form was considered

$$I^+(0,\mathbf{k},t) = P(\mathbf{k},t). \quad (5)$$

The condition (5) turns into (4) when spectral range width tends to zero.

2 Neumann series for direct problem solution

The solution to the Cauchy problem (1), (2) can be found in the form of the Neumann series [1, 2]

$$I(\mathbf{r},\mathbf{k},t,\nu)=\sum_{n=0}^{\infty} I_n(\mathbf{r},\mathbf{k},t,\nu), \tag{6}$$

where

$$I_n(\mathbf{r},\mathbf{k},t,\nu)=\int_0^{ct}\exp(-\mu\tau)\frac{\sigma(\mathbf{r}-\tau\mathbf{k})}{2\pi}\int_\Omega I_{n-1}\left(\mathbf{r}-\tau\mathbf{k},\mathbf{k}',t-\frac{\tau}{c},\nu\right)d\mathbf{k}'d\tau, \tag{7}$$

$$I_0(\mathbf{r},\mathbf{k},t,\nu)=\int_0^{ct}\exp(-\mu\tau)J\left(\mathbf{r}-\tau\mathbf{k},\mathbf{k},t-\frac{\tau}{c},\nu\right)d\tau. \tag{8}$$

Considering the form (3) of the radiation source J, singular term I_0 of the Neumann series (6) contains the product of the delta functions, other terms do not contain delta functions and thus the sum $I_1+I_2+\dots$ comprises its regular part.

From the relations (7), (8) for the first component of the Neumann series I_1 we get

$$I_1(\mathbf{r},\mathbf{k},t,\nu)=\frac{1}{2\pi}\int_0^{ct}\exp(-\mu\tau)\sigma(\mathbf{r}-\tau\mathbf{k})\int_\Omega\int_0^{ct-\tau}\exp(-\mu\tau')\times$$
$$\times J\left(\mathbf{r}-\tau\mathbf{k}-\tau'\mathbf{k}',\mathbf{k}',t-\frac{\tau+\tau'}{c},\nu\right)d\tau'd\mathbf{k}'d\tau=\frac{1}{2\pi}\int_0^{ct}\exp(-\mu\tau)\sigma(\mathbf{r}-\tau\mathbf{k})\times$$
$$\times\int_\Omega\int_0^{\infty}\chi_{ct-\tau}(\tau')\exp(-\mu\tau')J\left(\mathbf{r}-\tau\mathbf{k}-\tau'\mathbf{k}',\mathbf{k}',t-\frac{\tau+\tau'}{c},\nu\right)d\tau'd\mathbf{k}'d\tau, \tag{9}$$

where $\chi_{ct-\tau}(\tau')$ is the characteristic function of the interval $[0,ct-\tau]$.

By performing a variable substitution $\mathbf{x}=r-\tau'\mathbf{k}'$ in (9), based on a system of implicit equations

$$F_i(\mathbf{x},\mathbf{k}',t)=x_i-r_i+\tau'k_i'=0,\ i=1,2,\quad F_3(\mathbf{k}')=|\mathbf{k}'|-1=0$$

whith the Jacobian $|\mathbf{r}-\mathbf{x}|$, we find

$$I_1(\mathbf{r},\mathbf{k},t,\nu)=\frac{1}{2\pi}\int_0^{ct}\exp(-\mu\tau)\sigma(\mathbf{r}-\tau\mathbf{k})\int_{\mathbb{R}^2}\chi_{ct-\tau}(|\mathbf{r}-\mathbf{x}|)\frac{\exp(-\mu|\mathbf{r}-\mathbf{x}|)}{|\mathbf{r}-\mathbf{x}|}\times$$
$$\times\delta(\mathbf{x}-\tau\mathbf{k})\delta\left(t-t_0-\frac{\tau+|\mathbf{r}-\mathbf{x}|}{c}-(\nu-\nu_0)\frac{\Delta t}{\Delta\nu}\right)d\mathbf{x}d\tau=\frac{1}{2\pi}\int_0^{ct}\chi_{ct-\tau}(|\mathbf{r}-\tau\mathbf{k}|)\times$$
$$\times\exp(-\mu\tau)\sigma(\mathbf{r}-\tau\mathbf{k})\frac{\exp(-\mu|\mathbf{r}-\tau\mathbf{k}|)}{|\mathbf{r}-\tau\mathbf{k}|}\delta\left(t-t_0-\frac{\tau+|\mathbf{r}-\tau\mathbf{k}|}{c}-(\nu-\nu_0)\frac{\Delta t}{\Delta\nu}\right)d\tau.$$

Therefore,

$$I_1(\mathbf{r},\mathbf{k},t,\nu)=\frac{1}{2\pi}\int\limits_0^\infty \chi_{ct}(\tau)\chi_{ct-\tau}(|\mathbf{r}-\tau\mathbf{k}|)\frac{\exp(-\mu(\tau+|\mathbf{r}-\tau\mathbf{k}|))}{|\mathbf{r}-\tau\mathbf{k}|}\sigma(\mathbf{r}-\tau\mathbf{k})\times$$
$$\times\delta\left(t-t_0-\frac{\tau+|\mathbf{r}-\tau\mathbf{k}|}{c}-(\nu-\nu_0)\frac{\Delta t}{\Delta\nu}\right)d\tau. \tag{10}$$

After variable substitution

$$s=(\tau+|\mathbf{r}-\tau\mathbf{k}|)/c=(\tau+\sqrt{(\mathbf{r}-\tau\mathbf{k},\mathbf{r}-\tau\mathbf{k})})/c=(\tau+\sqrt{|\mathbf{r}|^2-2\tau(\mathbf{r},\mathbf{k})+\tau^2})/c,$$
$$\tau(\mathbf{r},\mathbf{k},s)=\frac{1}{2}\frac{(cs)^2-|\mathbf{r}|^2}{cs-(\mathbf{r},\mathbf{k})},$$
$$\frac{\partial\tau(\mathbf{r},\mathbf{k},s)}{ds}=\frac{c}{2}\frac{((cs)^2-2cs(\mathbf{r},\mathbf{k})+|\mathbf{r}|^2)}{(cs-(\mathbf{r},\mathbf{k}))^2}=\frac{c}{2}\frac{|\mathbf{r}-cs\mathbf{k}|^2}{(cs-(\mathbf{r},\mathbf{k}))^2},$$

from (10) we get

$$I_1(\mathbf{r},\mathbf{k},t,\nu)=\frac{1}{2\pi}\int\limits_0^\infty \chi_{ct}(\tau(\mathbf{r},\mathbf{k},s))\chi_{ct-\tau(\mathbf{r},\mathbf{k},s)}(cs-\tau(\mathbf{r},\mathbf{k},s))\sigma\left(\mathbf{r}-\tau(\mathbf{r},\mathbf{k},s)\mathbf{k}\right)\times$$
$$\times\frac{\exp(-\mu cs)}{|\mathbf{r}-\tau(\mathbf{r},\mathbf{k},s)\mathbf{k}|}\delta\left(t-t_0-s-(\nu-\nu_0)\frac{\Delta t}{\Delta\nu}\right)\frac{\partial\tau}{ds}ds=\chi_{ct^*}(\tau(\mathbf{r},\mathbf{k},t^*))\times$$
$$\times\chi_{ct^*-\tau(\mathbf{r},\mathbf{k},t^*)}(ct-\tau(\mathbf{r},\mathbf{k},t^*))\frac{\exp(-\mu ct^*)}{2\pi(ct^*-\tau(\mathbf{r},\mathbf{k},t^*))}\frac{\partial\tau(\mathbf{r},\mathbf{k},t)}{ds}\sigma(\mathbf{r}-\tau(\mathbf{r},\mathbf{k},t)\mathbf{k})=$$
$$=\chi_{ct^*}(\tau(\mathbf{r},\mathbf{k},t^*))\frac{\exp(-\mu ct^*)}{2\pi}\frac{2(ct^*-(\mathbf{r},\mathbf{k}))}{(ct^*)^2-2ct^*(\mathbf{r},\mathbf{k})+|\mathbf{r}|^2}\frac{c}{2}\frac{((ct^*)^2-2ct^*(\mathbf{r},\mathbf{k})+|\mathbf{r}|^2)}{(ct^*-(\mathbf{r},\mathbf{k}))^2}\times$$
$$\times\sigma(\mathbf{r}-\tau(\mathbf{r},\mathbf{k},t^*)\mathbf{k})=\chi_{ct^*}(\tau(\mathbf{r},\mathbf{k},t^*))\frac{c\exp(-\mu ct^*)}{2\pi(ct^*-(\mathbf{r},\mathbf{k}))}\sigma(\mathbf{r}-\tau(\mathbf{r},\mathbf{k},t^*)\mathbf{k}), \tag{11}$$

where $t^*=t^*(\nu,t)=t-t_0-(\nu-\nu_0)\dfrac{\Delta t}{\Delta\nu}$.

Having the analytic form of the function I_1, we can find the rest terms I_n in the same way. Unfortunately those terms do not have such a simple analytic form as I_1. This in particular is what makes the single scattering approximation compelling.

3 Single scattering approximation solutions of the inverse problems

In constructive solutions of the tomography problems methods based on single scattering approximation became widespread. The use of approximation drastically simplifies studying of inverse problems, often allowing to find an analytic solution [1, 2, 7, 14, 15]. Now we will derive an analytic formula for the scattering coefficient.

If $\mathbf{r} = 0$, then $\tau(\mathbf{r}, \mathbf{k}, t^*) = ct^*/2 = c\left(t - t_0 - (\nu - \nu_0)\dfrac{\Delta t}{\Delta \nu}\right)/2$, and (11) for single scattered radiation flux density gets the following form:

$$I_1^+(0, \mathbf{k}, t, \nu) = \frac{\exp(-\mu c t^*)}{2\pi t^*} \sigma\left(-\frac{ct^*}{2}\mathbf{k}\right) =$$
$$= \frac{\exp\left(-\mu c\left(t - t_0 - (\nu - \nu_0)\dfrac{\Delta t}{\Delta \nu}\right)\right)}{2\pi\left(t - t_0 - (\nu - \nu_0)\dfrac{\Delta t}{\Delta \nu}\right)} \sigma\left(-\frac{c\mathbf{k}}{2}\left(t - t_0 - (\nu - \nu_0)\frac{\Delta t}{\Delta \nu}\right)\right). \tag{12}$$

Taking $\mathbf{k} = -\dfrac{\mathbf{r}}{|\mathbf{r}|}$ and $t = \dfrac{2|\mathbf{r}|}{c} + t_0 + (\nu - \nu_0)\dfrac{\Delta t}{\Delta \nu}$ and using (12), we get

$$I_1^+\left(0, -\frac{\mathbf{r}}{|\mathbf{r}|}, \frac{2|\mathbf{r}|}{c} + t_0 + (\nu - \nu_0)\frac{\Delta t}{\Delta \nu}, \nu\right) = \sigma(\mathbf{r})\left(\frac{c\exp(-2\mu|\mathbf{r}|)}{4\pi|\mathbf{r}|}\right). \tag{13}$$

By integrating (13) over the interval $\nu \in [\nu_0 - \Delta/2, \nu_0 + \Delta/2]$ and expressing the function σ, we obtain

$$\sigma(\mathbf{r}) = \frac{4\pi|\mathbf{r}|\exp(2\mu|\mathbf{r}|)}{\Delta \nu} \int\limits_{\nu_0 - \Delta\nu/2}^{\nu_0 + \Delta\nu/2} I_1^+\left(0, -\frac{\mathbf{r}}{|\mathbf{r}|}, \frac{2|\mathbf{r}|}{c} + t_0 + (\nu - \nu_0)\frac{\Delta t}{\Delta \nu}, \nu\right) d\nu. \tag{14}$$

Formula (14) is an explicit solution of the inverse problem in single scattering approximation. If we replace the function I_1 in (14) with a full flux $I = I_1 + I_2 + \ldots$, then we get a formula for the scattering coefficient expressed through the function P:

$$\sigma(\mathbf{r}) = 4\pi|\mathbf{r}|\exp(2\mu|\mathbf{r}|)P\left(-\frac{\mathbf{r}}{|\mathbf{r}|}, t_0 + \frac{2|\mathbf{r}|}{c}\right). \tag{15}$$

From a practical point of view, the relation (15) serves as a basis for a way to process the data of chirp radiation flux probing of heterogeneous mediums.

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Ворновских П. А., Ермолаев Е. В., Прохоров И. В. О задаче определения коэффициента рассеяния при частотно-модулированном зондировании среды. *Дальневосточный математический журнал.* 2022. Т. 22. № 2. С. 263–268.

АННОТАЦИЯ

В рамках кинетической модели переноса линейно-частотно-модулированного излучения в рассеивающей среде сформулирована обратная задача, заключающаяся в определении объемного коэффициента рассеяния звука. Дополнительной информацией в задаче является усредненное по частоте угловое распределение плотности потока излучения в заданной точке пространства. Получено аналитическое решение обратной задачи в приближении однократного рассеяния.

Ключевые слова: *уравнение переноса излучения, линейно-частотно-модулированное зондирование, коэффициент рассеяния, обратная задача*