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An extrapolation method for improving the linearity of CT-values in X-ray pulsed tomography

This paper proposes an approach for improving the quality of the attenuation coefficient reconstruction using medium irradiation with X-ray pulses of various durations. We propose a new scheme of tomographic scanning that makes it possible to reduce the contribution of the scattered component to the projection data by constructing an extrapolation approximation for a ballistic term of a radiative transfer equation solution. Numerical experiments were carried out on a specially designed digital phantom.

Key words: *extrapolation method, X-ray tomography, CT-values.*

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Conventionally, for numerical evaluation of attenuation coefficient in tomographic images Hounsfield units are used. The Hounsfield units (HU) are obtained from a linear transformation of the measured attenuation coefficients that is based on the arbitrarily-assigned densities of air and pure water [1]. For reliable and reproducible treatment planning, an accurate representation of different tissues by the Hounsfield scale is recommended. Identical values for scans obtained from different tomographic scanners are desirable. However, the presence of scattering in the medium leads to the fact that the aforementioned linear relationship is not fulfilled. As a result, inclusions of the same material can give different values in the Hounsfield units, depending on the its location within the medium. This problem is especially acute in the dental cone beam computed tomography [2, 3]. Thereby, the HU linearity is an essential parameter in a quantitative imaging and the treatment planning systems of radiotherapy.

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This study proposes a method that allows one to improve the linearity of the CT-values by the means of more accurate evaluation of the numerical values of the attenuation coefficient.

We propose a new scheme of tomographic scanning, which makes it possible to reduce the contribution of a scattered component into the projection data with decreasing pulse width of the incident radiation. As a result, in the theoretical terms, we can determine the ballistic component of the signal.

For the practical use of this approach, we build an extrapolation approximation of the ballistic component of the signal by irradiating the medium with a series of pulses with different durations. Finding the attenuation coefficient from a known ballistic component is a well-studied problem of Radon transform inversion [4]. We approve the approach proposed in numerical experiments with a specially designed digital phantom.

1 Statement of the inverse problem

Let us consider the following integro-differential radiation transfer equation [5, 6]

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \omega\cdot\nabla_r + \mu(r)\right) I(r,\omega,t) = \sigma(r)\int\limits_{\Omega} p(r,\omega\cdot\omega')I(r,\omega',t)d\omega', \tag{1}$$

where $I(r,\omega,t)$ is the radiation flux density at the point $r \in G \subset R^3$, in the direction

$$\omega \in \Omega = \{\omega \in R^3 : |\omega| = 1\}$$

at the time instant $t \in [0, T]$, μ is the attenuation factor, σ is the scattering coefficient, c is the velocity of photons, and p is the scattering phase function.

Let an irradiated object be entirely contained within a cylinder of diameter d. Let Π_ω be a plain tangent to the boundary of the domain G and perpendicular to the direction ω,

$$\Pi_\omega = \{r \in R^3 : r\cdot\omega = d/2\}.$$

We assume that the medium is irradiated with a series of pulses depending in the direction

$$\omega^* \in \Omega^* = \{\omega = (\omega_1,\omega_2,\omega_3) \in \Omega : \omega_3 = 0\}.$$

The object scanning is carried out by synchronous rotation of planes with the radiation sources $\Pi_{-\omega^*}$ and the detectors $\Pi_{+\omega^*}$. We will use the following notation:

$$X = G\times\Omega\times[0,T], \quad X_0 = G\times\Omega\times\{t=0\}, \quad \Omega_{-\omega^*} = \{\omega\in\Omega : -\omega^*\cdot\omega > 0\},$$
$$Y^- = \Pi_{-\omega^*}\times\Omega_{-\omega^*}\times[0,T], \quad X^- = Y^-\cup X_0.$$

Let us introduce the function

$$h(z,\omega,t) = \begin{cases} 0, & (z,\omega,t)\in X_0, \\ h_{ext}(z,\omega,t), & (z,\omega,t)\in Y^-, \end{cases}$$

and supplement equation (1) with the unified initial-boundary condition:

$$I|_{X^-} = h(r, \omega, t). \tag{2}$$

For brevity we will omit the parametric dependence of a direct problem solution in the ω^* direction.

The serial irradiation is proposed to be dependent on the direction ω^* and described by square impulses with the pulse duration δ

$$h(\xi, \omega, t) = \begin{cases} 1/\delta, & (\xi, \omega, t) \in \Pi_{-\omega^*} \times \Omega_{-\omega^*} \times (0, \delta), \\ 0, & (\xi, \omega, t) \notin \Pi_{-\omega^*} \times \Omega_{-\omega^*} \times (0, \delta). \end{cases}.$$

From the physical point of view, such a definition of radiation source restricts only the pulse duration without assuming collimation or spatial localization of radiation sources, which are often used in tomography to suppress the scattering effects in a medium.

We investigate the following inverse problem.

The problem is to find the function μ from (1), (2) with the additional condition

$$\int\limits_{d/c}^{d/c+\delta} I(\eta, \omega^*, t)dt = H(\eta, \omega^*), \quad (\eta, \omega^*) \in \Pi_{\omega^*} \times \Omega^*, \tag{3}$$

where c, d, δ, h, H are given.

Thus, to find the attenuation coefficient, only the averaged values of the flux density are needed over the interval equal to the pulse width shifted by the travel time of the ballistic leaving the probed signal from the source to the receiver, which somewhat reduces the requirements for the temporal resolution of the detectors.

2 Estimation of the scattered radiation contribution. The projection data extrapolation procedure

To construct a method for solving the problem, we estimate the contribution of the scattered radiation to the total radiation flux depending on a probe pulse width. In the paper we take into account only a single scattering approximation. In this framework the direct problem solution can be represented as a sum of ballistic and scattered components

$$I(\eta, \omega^*, t) = I_0(\eta, \omega^*, t) + I_1(\eta, \omega^*, t), \quad \eta \in \Pi_{\omega^*}.$$

Here I_0 means the ballistic term, and I_1 denotes the single scattered one.

The following estimations are valid

$$\int\limits_{d/c}^{d/c+\delta} I_0(\eta, \omega^*, t)dt = \exp\left(-\int\limits_0^d \mu(\eta - \tau\omega)d\tau\right),$$

$$\int\limits_{d/c}^{d/c+\delta} I_1(\eta, \omega^*, t)dt \le \mathrm{const}\left(1 + \varepsilon \ln\left|1 + \frac{1}{\varepsilon}\right| - \frac{1}{\varepsilon}\ln|1 + \varepsilon|\right),$$

where $\varepsilon(\delta) = c\delta/d$.

The latter inequality shows that the integral of the scattered term tends to zero with a lower pulse duration. In this case the integral of the ballistic component gives an exponential law of attenuation like the case of a medium without scattering.

Let a medium be irradiated with two sources of different durations δ_1 and δ_2 and $H(\delta_1), H(\delta_2)$ are corresponding outgoing radiation fluxes. With a single scattering approximation we can write down

$$H(\varepsilon) = \int\limits_{d/c}^{d(1+\varepsilon)/c} I(\eta, \omega^*, t, \varepsilon)dt = \exp\left(-\int\limits_0^d \mu(\eta - \tau\omega^*)d\tau\right) + C\Phi(\varepsilon),$$

where

$$\Phi(\varepsilon) = \left(1 + \varepsilon \ln\left|1 + \frac{1}{\varepsilon}\right| - \frac{1}{\varepsilon}\ln|1+\varepsilon|\right).$$

This equation allows us to express the Radon ray transform of the function μ based on the corresponding output signals,

$$\exp\left(-\int\limits_0^d \mu(\eta - \tau\omega^*)d\tau\right) = H(\varepsilon_1) - \frac{H(\varepsilon_1) - H(\varepsilon_2)}{\Phi(\varepsilon_1) - \Phi(\varepsilon_2)}H(\varepsilon_1). \quad (4)$$

The inverse problem solution reduces to the inversion of the Radon transform

$$\int\limits_0^d \mu(\eta - \tau\omega^*)d\tau = -\ln\left|H(\varepsilon_1) - \frac{H(\varepsilon_1) - H(\varepsilon_2)}{\Phi(\varepsilon_1) - \Phi(\varepsilon_2)}\Phi(\varepsilon_1)\right|.$$

To find the function μ, the wide-known convolution and back projection algorithm, can be used.

3 Numerical experiments and discussion

To test the method, we use the specially developed digital phantom. It is a cylinder with a radius of 10 cm and height of 10 cm, filled with water-equivalent base material (HU=0). The phantom contains cylindrical inclusions with diameter of 0.6 cm and a height of 10 cm, with given values of the attenuation coefficient in the Hounsfield units.

The inclusions contain materials with the following attenuation coefficient values: –1000HU, –600HU, –100HU, 300HU, 500HU and 2100HU which correspond to the most basic materials encountered in the dental cone beam tomography. The centers of the inclusions are located at concentric circles at different distances from the center of the phantom in order to evaluate the effects of the inclusion location on the accuracy of CT-values reconstruction. A lateral crossection of phantom is shown schematically in Fig. 1a.

To describe the serial irradiation of the medium, we used a pulsed source depending on the direction ω^* with pulse durations of 200 and 300 picoseconds ($\delta_1 = 300$, $\delta_2 = 200$).

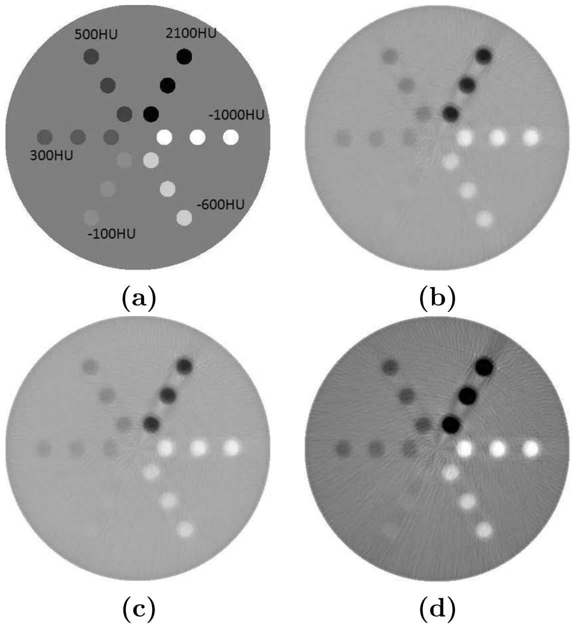

Fig. 1: Digital phantom: **(a)** – lateral crossection; **(b), (c)** – reconstruction from the "raw" data corresponding to pulse durations of 200ps and 300ps; **(d)** – reconstruction from extrapolated data.

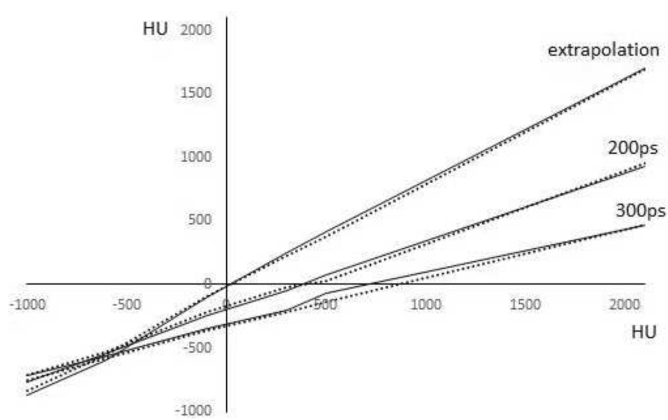

Fig. 2: Comparison of HU values reconstructed from "raw" and extrapolated data was conducted for different inclusion locations. Dotted line corresponding to the outer inclusions and solid line to the inner one.

We calculate output radiation profiles for given parameters of the medium and pulse durations using Monte Carlo method [5]. In our numerical experiments, we simulate a tomographic scanner with a data acquisition system consisting of 200 angular projections, each of which includes data from 101 detectors.

At the next step we correct the projection data using formula (4) and find the attenuation coefficient using the convolution and back projection method [4]. To control the quality improvement, a similar procedure was carried out for the projection data without correction. The phantom images reconstructed using the “raw” and extrapolated data are presented in Figure 1 b-c. It appears visually that the contrast of the tomogramm reconstructed from extrapolated data is significantly higher than on the “raw” data one.

Linearity of the CT-values was obtained from making a graph between reconstructed HU values vs referenced one. The comparison of CT-values linearity between images reconstructed using “raw” and corrected data for different inclusion locations are shown in Figure 2. The solid lines correspond to inclusions located near the phantom boundary, and a dotted lines to the inclusions located closer to the center one. It appears that the CT-values found from the corrected projection data gives a better linearity regardless of the inclusion location. It should be noted that the approach proposed uses a potentially unstable extrapolation procedure, so it is highly demanding of the accuracy of the measured data. In the results of the numerical experiments given above, a relative error in simulated outgoing radiation measurements did not exceed 1 %. At this level of error, the approach proposed gives good results. The results of additional numerical experiments have shown that with an increase in an error, the instability in the method proposed may cause artifacts on tomograms.

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Яровенко И. П., Казанцев И. Г. Экстраполяционный метод улучшения линейности восстановления коэффициента ослабления в импульсной рентгеновской томографии. *Дальневосточный математический журнал.* 2022. Т. 22. № 2. С. 269–275.

АННОТАЦИЯ

В работе предложен подход, повышающий качество реконструкции коэффициента ослабления путем облучения среды импульсами различной длительности. Новая схема томографического сканирования позволяет уменьшить вклад рассеянной составляющей в проекционные данные с помощью построения экстраполяционного приближения решения уравнения переноса излучения. Проведено численное тестирование на специальном цифровом фантоме.

Ключевые слова: *томография, экстраполяция, подавление рассеяния.*