Two Famous Formulas (Part I)

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In this article, we will discuss *Pick's formula* for calculating the area of a lattice polygon (Part I) and *Euler's formula* for polyhedra (Part II), paying particular attention to the connection between these two formulas.

Before we proceed, let us define some terms. A *lattice* $\mathbb{Z}^2$ is the set of all points of the cartesian plane with integer coordinates. It is convenient to imagine a lattice as an infinite sheet of graph paper. A *lattice polygon* is a polygon with all its vertices on grid points. Unless otherwise specified, we consider only *simple polygons*, that is polygons that do not intersect themselves. Figure 1 shows examples of non-simple polygons.

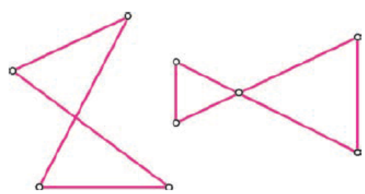

Figure 1

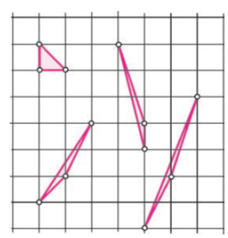

Figure 2

First of all, let us consider the smallest (and the most important) case. Suppose our lattice polygon is a triangle with no lattice points except its vertices or other lattice points on its perimeter. Such triangles are called *primitive* and example are depicted in Figure 2.

We will study their properties, beginning by showing that any triangle can be divided into primitive triangles. First, suppose that triangle ABC has no interior lattice points, but has some on at least one of its sides, say BC. Let us connect the vertex A with all lattice points on the side BC as in Figure 3. All the resulting triangles, except possibly ABP and AQC, are primitive. As for triangles ABP and AQC, they each have two sides that do not contain lattice points. Connecting points P and Q with lattice points on the sides AB and AC, we divide triangles ABP and AQC into primitive triangles.

Now suppose that the given triangle ABC has interior lattice points. Pick an arbitrary interior lattice point and connect it to the vertices A, B and C (see Figure 4). The three resulting triangles contain fewer interior lattice points than ABC. Since there are finitely many lattice points on the interior of ABC, by repeating this process, we will divide triangle ABC into triangles with no interior lattice points. To finish the decomposition into primitive triangles, we can apply the previously described process to eliminate lattice points on the sides of the resulting triangles.

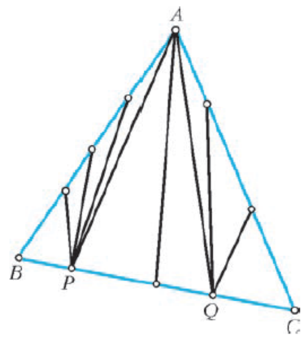

Figure 3

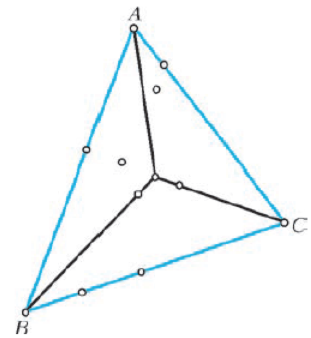

Figure 4

Theorem 1 *A triangle is primitive if and only if it has area of* $1/2$.

Proof. Let ABC be a primitive triangle. Consider the smallest lattice rectangle that contains ABC and has sides parallel to the coordinate axes. Because the rectangle is minimal, each of its four sides must pass through a vertex of the triangle, whence the pigeon-hole principle forces the rectangle to share at least one vertex with the triangle. Moreover, unless a side of the triangle is a diagonal of the rectangle, the triangle will necessarily contain a lattice point (as indicated by lattice point K in cases a) and b) of Figure 5, contrary to the assumption that it is primitive. We may therefore assume that AB is a diagonal of the rectangle $OAFB$ as shown in cases c) and d). Drop perpendiculars CD and CE to OA and OB, respectively (where C might coincide with D, E, or O).

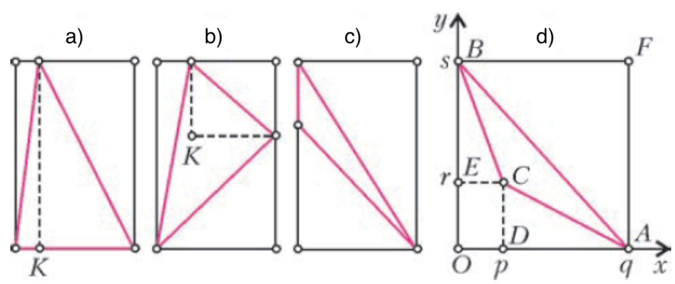

Figure 5

Suppose that the point O from Figure 5d is the origin and let $D = (p, 0), A = (q, 0), E = (0, r)$ and $B = (0, s)$. Let $I(P)$ denote the number of lattice points that lie inside a polygon P but not on its sides. Then

$$I(OAFB) = (q-1)(s-1).$$

Since AB does not contain lattice points other than A and B, we have

$$I(OAB) = I(OAFB)/2 = (q-1)(s-1)/2.$$

Similarly,

$$I(ACD) = (q-p-1)(r-1)/2 \quad \text{and} \quad I(CBE) = (s-r-1)(p-1)/2.$$

Since triangle ABC contains no interior lattice points, therefore

$$I(OAB) - I(ACD) - I(CBE) = pr,$$

the number of lattice points inside and on rectangle $ODCE$, excluding those on OD and OE. It follows that

$$(q-1)(s-1) - (q-p-1)(r-1) - (s-r-1)(p-1) = 2pr,$$

and so

$$qs - ps - qr = 1.$$

Letting square brackets denote the area of the region enclosed by the indicated polygon, it follows that

$$\begin{aligned}[ABC] &= [OAB] - [ACD] - [CBE] - [ODCE] \\ &= \frac{sq}{2} - \frac{(p-q)r}{2} - \frac{(s-r)p}{2} - pr \\ &= \frac{qs - ps - qr}{2} \\ &= \frac{1}{2},\end{aligned}$$

which proves one direction of the theorem.

Conversely, the area of a lattice triangle that is not primitive must exceed 1/2 because (as we have already seen) it can be divided into primitive triangles, each of which has an area of 1/2 by the first part of the theorem. □

Exercise 1. Prove that for any arbitrarily large number M, there exists a primitive lattice triangle such that each of its sides is larger than M.

Theorem 2 (G. Pick) *For any simple lattice polygon P, we have the following formula*

$$[P] = N_i + \frac{N_e}{2} - 1,$$

where N_i is the number of interior lattice points of P and N_e is the number of lattice points on the boundary of P.

For example, in Figure 6 we have $N_i = 9, N_e = 11$ and so $[P] = 9 + \frac{11}{2} - 1 = \frac{27}{2}$.

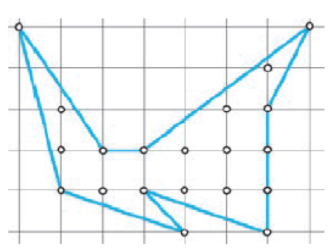

Figure 6

Proof. Assume that P is a simple polygon with k vertices. First of all, note that every simple polygon with four or more vertices has at least one diagonal that lies in its interior. This follows by definition for convex polygons; on the other hand, if the interior angle at some vertex is greater than 180°, then a ray from that vertex sweeping the interior of the polygon must strike another vertex, and these two vertices determine the desired interior diagonal. From this, by induction, it follows that any simple polygon on k vertices can be split into $k-2$ triangles whose vertices are the vertices of the original polygon and are, in particular, lattice points. Therefore, the sum of all interior angles of a simple polygon on k vertices is equal $(k-2)\pi$.

Next, divide each of the resulting $k-2$ triangles into primitive triangles. Since the area of each primitive triangle is $1/2$, the number of primitive triangles in this case is $N = 2[P]$ and therefore does not depend on the way the polygon was divided into primitive triangles.

To finish the proof, we simply need to check that

$$N = 2N_i + N_e - 2.$$

We consider three cases as shown in Figure 7:

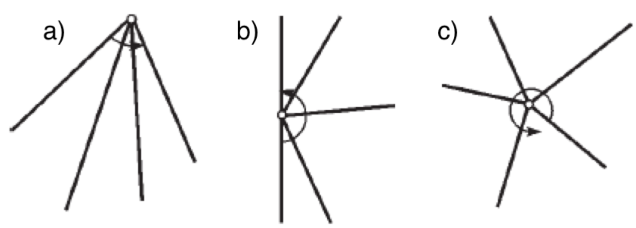

Figure 7

First of all, each vertex of P is also a vertex of one or more of our primitive triangles (see Figure 7a). The sum of the angles of all the triangles at these vertices equals to the sum of the interior angles of P and hence equals $180^\circ(k-2)$.

Secondly, a lattice point which is not a vertex of P but lies on the boundary of P also serves as a vertex of our primitive triangles (see Figure 7b), and the sum of the angles at these vertices equals $180^\circ(N_e - k)$.

Finally, we have to consider each of the N_i lattice points on the interior of P which also serve as vertices of our primitive triangles. The sum of the angles at these vertices is 360° (see Figure 7c). therefore, the sum of the angles of all primitive triangles with vertices on the interior lattice points equals $360^\circ N_i$.

On the other hand, the sum of angles of all N of our primitive triangles is $180^\circ N$, so we have

$$180^\circ N = 360^\circ N_i + 180^\circ(N_e - k) + 180^\circ(k-2).$$

Therefore, $N = 2N_i + N_e - 2$ and the proof is complete. □

Remark. Of course, we can replace the vertical lines of our lattice by any family of equally spaced parallel lines so that the square cells are replaced by congruent parallelograms. Pick's formula holds in this general case as well: For a lattice polygon P, we have

$$[P] = \left(N_i + \frac{N_e}{2} - 1\right) \cdot [a],$$

where $[a]$ is the area of each of the parallelograms. One can prove this claim with an argument similar to the one above; alternatively, one can apply a linear transformation to the cartesian lattice points by means of a 2×2 matrix whose determinant is $[a]$.

Let us summarize three proven statements:

1°. For any simple lattice polygon P, we have Pick's formula $[P] = N_i + \frac{N_e}{2} - 1$.

2°. The area of any primitive lattice triangle is $1/2$.

3°. For any decomposition of a simple polygon into N primitive triangles, we have that $N = 2N_i + N_e - 2$.

Let us consider logical connections between these statements and compare their relative strengths.

Statement 2° is an immediate consequence of 1°, whereas 3° follows from 1° and 2° (see Figure 8a). Therefore, we could get all three statements immediately if we proved Pick's formula first and without using 2° and 3° (see Exercise 3.) However, we picked a different route: we proved 2° independently, then concluded 3° and then finally got 1° as a corollary of 2° and 3° (see Figure 8b).

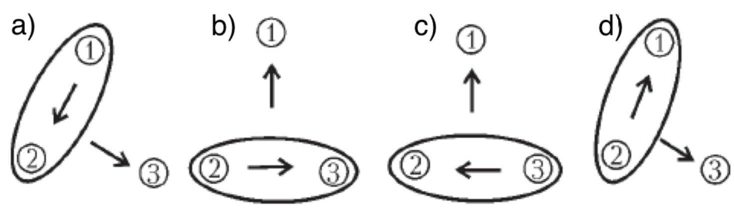

Figure 8

It is interesting to consider two other approaches, suggested by Figure 8c and 8d, to proving the three statements. Let us show how 2° follows from 3°. First note that the area of any lattice triangle can be expressed as $n/2$ (for some integer n). (To see this, use coordinates for the vertices of the triangle, or see R.W. Gaskell, M.S. Klamkin, and P. Watson, "Triangulations and Pick's Theorem", *Mathematics Magazine*, **49**:1 (Jan. 1976) 35-37.) Now let T be a primitive triangle and P be a p by q lattice rectangle (whose sides lie along the lattice lines) that encloses T. Set $T_1 = T$ and use primitive triangles $T_j,\ j = 2, \ldots, N$, to triangulate the region (or regions) inside P lying outside of T (which are bounded by $OACB$ and AFB as in Figure 5d if the smallest rectangle is used for P). The number of primitive triangles required to cover P, according to 3°, is

$$N = 2N_i + N_b - 2 = 2(p-1)(q-1) + 2p + 2q - 2 = 2pq.$$

Therefore, we have $2pq$ primitive triangles, each of area at least $1/2$, whose combined area equals

$$[P] = \sum_{j=1}^{2pq} [T_j] = pq.$$

Consequently, each of the $2pq$ triangles T_j must have area exactly $1/2$ so that their combined area does not exceed the area pq of the outer rectangle. Thus we see that $3°$ implies $2°$.

We will now prove that $2°$ implies $1°$. Consider the function

$$F(P) = N_i + \frac{N_e}{2} - 1,$$

defined on all simple lattice polygons. Split P into two lattice polygons P_1 and P_2 using a broken line passing through lattice points (see Figure 9); we write $P = P_1 + P_2$.

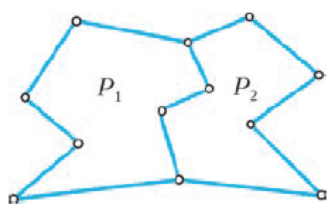

Figure 9

It is then easy to see that functions F and area are additive, that is

$$F(P_1 + P_2) = F(P_1) + F(P_2) \quad \text{and} \quad [P_1 + P_2] = [P_1] + [P_2].$$

Therefore, if Pick's formula holds for P_1 and P_2, then it also holds for $P = P_1 + P_2$. But since any simple polygon can be split into primitive triangles and by assumption Pick's formula holds for them, then it also holds for any given polygon.

To summarize, we have established that the three statements are equivalent even though a priori, $1°$ might appear to be the strongest of the three.

Exercise 2. Using the additive property of the function $F(P)$ and the proof of Theorem 1, find a proof of Pick's formula that does not require $2°$ and $3°$.

To be continued.

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