Two Famous Formulas (Part II)

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Part I of this article appears in ***Crux*** *43(2).*

The first part of this article discussed *Pick's formula* for calculating the area of a lattice polygon. Here, we will discuss *Euler's formula* for polyhedra and the connection between these two formulas.

We define a *polygonal map* to be a partition of a simple plane polygon into non-overlapping simple polygons that connect along entire sides (more precisely, any two of the partition's polygons that touch may do so either at a vertex or along a common side). A polygonal map is a special case of a *planar graph*, a graph that can be drawn on the plane without its edges crossing (that is, its edges intersect only at their endpoints). For connected planar graphs, and in particular for polygonal maps, we have the famous *Euler's formula*

$$V - E + F = 1,$$

where V is the number of vertices of the graph, E is the number of its edges and F is the number of its faces (not counting the outer face). See Figure 10 for an example. [Ed.: In Western literature the outer face of the planar graph is normally included in the count so that the right-hand side of the formula equals 2.]

The word "map" emphasizes the fact that Euler's formula holds also for topological partitions, where the sides of the regions are allowed to be curves (and not just straight lines) as long as the neighbouring regions share only a common side or a common vertex (see Figure 11).

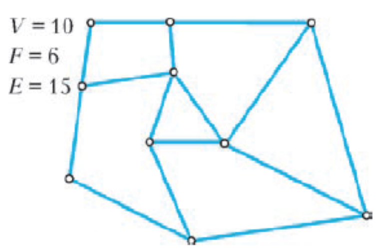

Figure 10

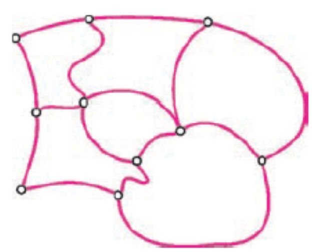

Figure 11

When all the polygons of the partition are triangles, the polygonal map is called a *triangulation*. Clearly, we can triangulate a given simple polygon in many different ways. However, for N triangles in a triangulation (not necessarily on a lattice), we always have that

$$N = 2N_i + N_e - 2,$$

where N_i is the number of vertices of the graph that lie inside the polygon and N_e is the number of its boundary vertices. (We proved this in Part I for lattice polygons.)

Exercise 3. Prove this formula for planar graph triangulations.

We also have that

$$E = 3N_i + 2N_e - 3,$$

where E is the total number of edges in the triangulation. Indeed, there are N_e edges that form the border of the polygon and every one of those edges appears in exactly one triangle. Therefore, there are $E - N_e$ other edges and each of those, being an inside edge, appears in exactly two triangles. In total, we have $3N$ triangle sides. Hence,

$$N = \frac{2(E - N_e) + N_e}{3} = \frac{2E - N_e}{3}.$$

Therefore,

$$E = \frac{3N + N_e}{2} = \frac{3(2N_i + N_e - 2) - N_e}{2} = 3N_i + 2N_e - 3,$$

as desired.

Theorem 3 (L. Euler) *For any polygonal map, we have*

$$V - E + F = 1.$$

Proof. Given a polygonal map P, introduce a new vertex inside each of the partition polygons. Connect these points (using either lines or curves) to the vertices of the corresponding polygon (see Figure 12) to form its triangulation.

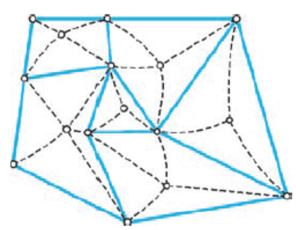

Figure 12

Now note that the formula $E' = 3N_i + 2N_e - 3$ holds for the number of edges E' of the newly created curved triangulation R of the polygon P. For this triangulation, we clearly have $N_i + N_e = V + F$. By construction, one edge of each triangle in R belongs to the original map P. Each of the newly drawn edges belongs to two triangles in R. Therefore, double the number of the new edges equals double the number of triangles. So for the number E' of all edges in R, we have $E' = E + N$, where N is the number of triangles in R. Now using the equation $N = \frac{2E' - N_e}{3}$ derived above, we have

$$V + F - E = (N_i + N_e) - (E' - N) = N_i + N_e - \frac{3N_i + 3N_e - 3}{3} = 1,$$

as desired. $\square$

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Exercise 4. Using Euler's formula, derive the equation $N = 2N_i + N_e - 2$.

Exercise 5. Prove Euler's formula without using the equation $N = 2N_i + N_e - 2$.

These two exercises together with the final section of Part I show that Pick's and Euler's formulas are equivalent – each can be derived from the other. They can also be generalized to maps with polygonal holes (see Figure 13) as follows:

Theorem 4 *For any simple lattice polygon P with n holes, we have*

$$[P] = N_i + \frac{N_e}{2} - 1 + n,$$

where $[P]$ denotes the area of P, N_i is the number of lattice points interior to P and exterior to all the holes, while N_e is the number of lattice points on the boundary of P and the boundaries of all the holes.

Theorem 5 *For any polygonal map with n holes, we have*

$$V - E + F = 1 - n.$$

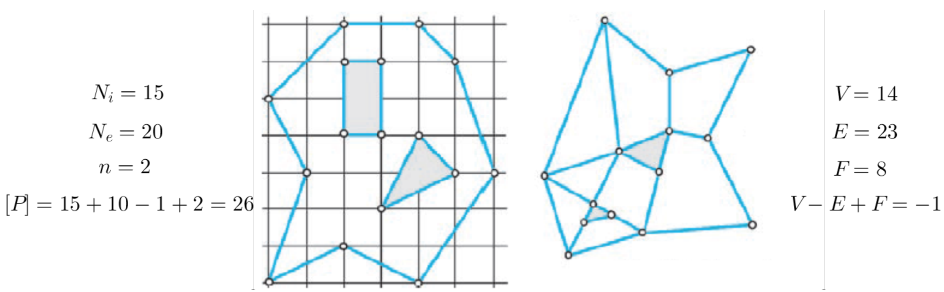

Figure 13

The proofs of these two generalized theorems are left as exercises for the reader.

Exercises.

6. Three frogs sit on the vertices of a lattice square and play leapfrog: each frog can jump over another frog and land on the point on the other side symmetrical to its original position (see Figure 14). Can any of the frogs end up on the fourth vertex of the original square? If they start on a lattice triangle, are there any points which frogs can never land on?

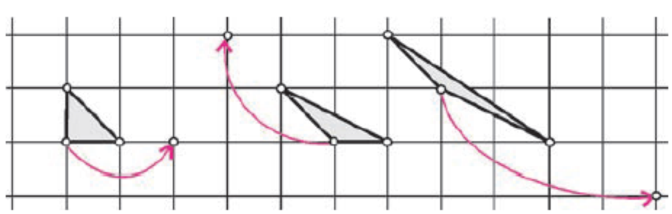

Figure 14

7. Consider a lattice triangle with no lattice points on its sides other than its vertices. Prove that if such a triangle contains exactly one lattice point in its interior, then this point is the point of intersection of its medians.

8. Consider a convex lattice n-gon with no lattice points on its sides other than its vertices and with no lattice points in its interior. Prove that $n \leq 4$.

9. Consider any two lattice points A and B such that there are no other lattice points on the line segment AB. Prove that there exists a lattice point C so that the triangle ABC is primitive (that is, contains no lattice points, except its vertices, inside or on its perimeter). If $|AB| = d$, find the distance from C to AB.

10. On a lattice, mark n ($n \geq 3$) lattice points so that any three of them form a triangle whose medians do not intersect at a lattice point. Find the largest value of n for which this is possible.

11. A king goes for a tour around an 8×8 chessboard; it visits every square exactly once before returning to the square where it started. A zigzag line that connects the centers of the squares in the order which the king visited them is non-intersecting. What is the maximum area of the polygon which this line borders?

12. Prove that an $n \times n$ square arbitrarily placed on a lattice will cover no more than $(n+1)^2$ lattice points.

13. For two situations in Figure 15, calculate the area of the shaded parallelograms if the sides of the parallelogram $ABCD$ are divided by the given lattice into n and m equal parts as indicated.

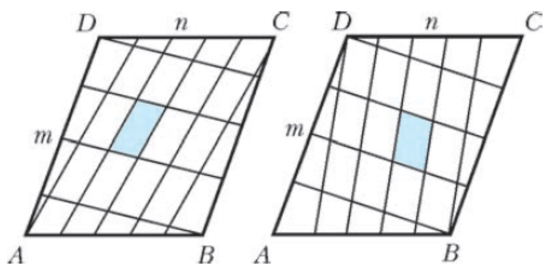

Figure 15

14. Each side of a triangle ABC is divided into three equal parts and one of the points on each side is connected to the opposite vertex as in Figure 16. Find the area of the shaded triangle in terms of the area of ABC.

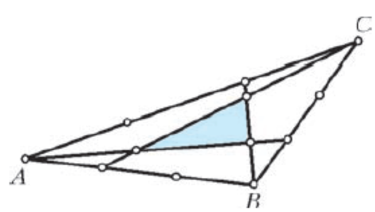

Figure 16

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15. Midpoints of the sides of a square are connected as shown in Figure 17. Find the area of the shaded octagon in terms of the area of the original square.

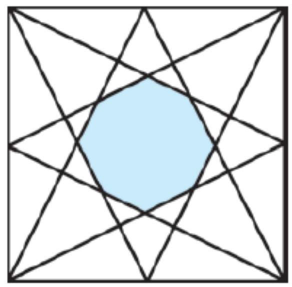

Figure 17

16. Let $f(P)$ be the function defined on all simple lattice polygons P as follows:

$$f(P) = aN_i(P) + bN_e(P) + c,$$

where a, b and c are some constants. Suppose also that $f(P) = f(P_1) + f(P_2)$ if the polygon P is divided into simple lattice polygons P_1 and P_2 by some zigzag line connecting lattice points in the interior of P. Prove that $b = a/2$ and $c = -a$.

17. Prove that for any simple lattice polygon P we have

$$2[P] = N(2P) - 2N(P) + 1,$$

where $N(P)$ denotes the total number of lattice points in the interior and on the boundary of P and $2P$ denotes a polygon that was obtained from P by a stretch with a scale factor 2 with respect to the origin.

18. Find one thousand points in the plane, no three of them collinear, so that the distance between any two of them is irrational and the area of a triangle formed by any three of them is rational.

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