Why do we need Gauss — Kuz'min statistics?

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be standard continued fraction expansion with $a_0 \in \mathbb{Z}$, $a_1, \ldots, a_s \in \mathbb{N}$. Standard assumption $a_s > 1$ (for $s > 0$) we replace by another one: $a_s = 1$.

Notations
Continued fractions

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Standard assumption $a_s > 1$ (for $s > 0$) we replace by another one: $a_s = 1$.
Length of continued fraction will be denoted by $s(a/b)$.

For $x \in [0,1]$ and rational number $a/b = [0; a_1, \dots, a_s]$ we define **Gauss — Kuz’min statistics** $s_x(a/b)$ as

$$s_x(a/b) = \left|\left\{j : 1 \le j \le s, [0; a_j, \dots, a_s] \le x\right\}\right|.$$

In particular $s_1(a/b) = s(a/b)$ is the length of continued fraction for a/b.

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$$N_k(a/b) = \left|\{j : 1 \le j \le s, a_j = k\}\right|$$

(also known as Gauss — Kuz'min statistics) can be expressed in terms of $s_x(a/b)$:

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$$N_k(a/b) = s_{1/k}(a/b) - s_{1/(k+1)}(a/b).$$

We prefer s_x instead of N_k because function $\log_2(1+x)$ is more comfortable than a set of probabilities

$$p_k = \log_2\left(1 + \frac{1}{k(k+2)}\right).$$

Continued fractions and Kloosterman sums

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Nontrivial bounds for classical Kloosterman sums

$$K_q(1, m, n) = \sum_{\substack{x,y=1 \\ xy\equiv 1 \pmod{q}}}^{q} e^{2\pi i \frac{mx+ny}{q}}$$

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More general Kloosterman sums

$$K_q(l,m,n)=\sum_{\substack{x,y=1\\ xy\equiv l \pmod{q}}}^{q} e^{2\pi i\frac{mx+ny}{q}}$$

explain isotropic properties of sublattices in $\mathbb{Z}^2$.

- (Trivial) Euclidean algorithm, calculation of $a^{-1} \pmod n$, lattice reduction, number recognition, parametrization of solution of the equation $ad - bc = N$, calculation of convex hull of non-zero lattice points from first quadrant etc.
- Decomposition of prime $p = 4n + 1$ to the sum of two squares.
- Calculation of goodness (dicrepancy or something similar) of 2-dimesional lattice rules for numerical integration.
- Analysis of Lehmer pseudo-random number generator ($x_{n+1} = ax_n + b \pmod m$).

Applications of finite continued fractions
Usual applications

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- Calculation of Dedekind sums and Jacobi symbols.
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- A criterion for a rectangle to be tilable by rectangles of a similar shape. Construction of alternating-current circuits with given properties (Skopenkov).
- Asymptotic behavior of a curve in $\mathbb{R}^n$ with constant curvature k_1, constant second curvature $k_2, \ldots$ (till constant curvature k_{n-1}). (Arnold).
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Applications of finite continued fractions

The rest part

- The way to attack RSA public key crypto system with small private exponents (Wiener).
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First result about average length of Euclidean algorithm belongs to Heilbronn (1968), who proved that

$$\frac{1}{\varphi(b)} \sum_{\substack{1 \le a \le b \\ (a,b)=1}} s(a/b) = \frac{2\log 2}{\zeta(2)} \log b + O(\log^4 \log b).$$

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Porter (1975) has shown that

$$\frac{1}{\varphi(b)} \sum_{\substack{1 \le a \le b \\ (a,b)=1}} s(a/b) = \frac{2\log 2}{\zeta(2)} \log b + C_P + O(b^{-1/6+\varepsilon}),$$

$$C_P = \frac{2\log 2}{\zeta(2)} \left(\frac{3\log 2}{2} + 2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - 1 \right) - \frac{1}{2}.$$

We can get a better estimate of the error term for the average value of $s(a/b)$ over a, b and by using elementary arguments.

Theorem (A.U., 2008)

Let $R \geq 2$. Then

$$E(R) = \frac{2}{R(R+1)} \sum_{b \leq R} \sum_{a \leq b} s(a/b) = \frac{2 \log 2}{\zeta(2)} \log R + \widetilde{C}_P + O(R^{-1+\varepsilon}),$$

where

$$\widetilde{C}_P = C_P + \frac{2 \log 2}{\zeta(2)} \left(\frac{\zeta'(2)}{\zeta(2)} - \frac{1}{2} \right)$$

Conjecture (Arnold, 1993)

Let $\Omega(R) = R \cdot \Omega$ $(R \to \infty)$ be extending region. Then elements of finite continued fractions for rational numbers a/b, $(a, b) \in \Omega(R)$ asymptotically satisfy the Gauss — Kuz'min statistic.

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Theorem (Avdeeva — Bykovskii, 2002–2004)

If $\Omega(R)$ *is a sector:*

$$\Omega(R) = \{(a, b) : a, b > 0, a^2 + b^2 \leq R^2\}$$

then

$$\frac{1}{\mathrm{Vol}(\Omega(R))} \sum_{(a,b)\in\Omega(R)} s_x(a/b) = \frac{2\log(x+1)}{\zeta(2)} \log R + O(1).$$

Theorem (A.U., 2005)

For any region Ω with "good" boundary

$$\frac{1}{\mathrm{Vol}(\Omega(R))}\sum_{(a,b)\in\Omega(R)} s_x(a/b) = \frac{2\log(x+1)}{\zeta(2)}\log R + C_\Omega(x) + O(R^{-1/5+\varepsilon}).$$

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If a notion bears a personal name, then this name is not the name of the discoverer.

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The Berry Principle

The Arnold Principle is applicable to itself.

Particular case of Arnold's conjecture was proved by Lochs.

Theorem (Lochs, 1961 (32 years before Arnold's conjecture).)

For triangle region

$$\Omega(R)=\{(a,b):0<b<a\leq R\}$$

$$\frac{1}{\mathrm{Vol}(\Omega(R))}\sum_{(a,b)\in\Omega(R)} s_x(a/b)=\frac{2\log(x+1)}{\zeta(2)}\log R+C_\Omega(x)+O(R^{-1/2+\varepsilon}).$$

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Applications: fast Euclidean algorithms.

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centered division:

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Let $s_{centered}(a/b)$ and $s_{odd}(a/b)$ be the lengths of centered and odd Euclidean algorithms. Elementary arguments allow to reduce both these algorithms to the classical one.

Theorem (A.U., 2009–2010)

Let $b \geq 1$, $1 \leq a < b$, $(a, b) = 1$, $\varphi = \frac{1+\sqrt{5}}{2}$. *Then*

$$s_{centered}(a/b) = s_{\varphi-1}(a/b).$$

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Moreover, if $b/2 \leq a$, $aa^\star \equiv 1 \pmod{b}$, $1 \leq a^\star < b$ *then*

$$s_{odd}\left(\frac{a^\star}{b}\right) + s_{odd}\left(\frac{b-a^\star}{b}\right) = s_\varphi\left(\frac{a}{b}\right) + s_{\varphi-1}\left(\frac{a}{b}\right).$$

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Here we used "reasonable" extension of Gauss — Kuz'min statistics for arbitrary $x > 0$:

$$s_x(a/b) = \left|\{(j,t) : 0 \leq j \leq s, 0 \leq t < a_j, [t; a_{j+1}, \ldots, a_s, 1] \leq x\}\right|$$

$(a_0 = +\infty)$.

Last theorem allows to improve some results of Baladi and Vallée (2005) on the average value of $s_{centered}(a/b)$ and $s_{odd}(a/b)$.

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Corollary

We have

$$\frac{1}{\varphi(b)}\sum_{\substack{a=1\\(a,b)=1}}^{b} s_{centered}(a/b) = \frac{2\log\varphi}{\zeta(2)}\log b + C_1 + O(b^{-1/6+\varepsilon}),$$

$$\frac{2}{R(R+1)}\sum_{b\le R}\sum_{a=1}^{b} s_{centered}(a/b) = \frac{2\log\varphi}{\zeta(2)}\log R + \widetilde{C}_1 + O(R^{-1+\varepsilon}),$$

where constants C_1 and $\widetilde{C}_1$ can be written in terms of singular series.

Corollary

We have

$$\frac{1}{\varphi(b)}\sum_{\substack{a=1\\(a,b)=1}}^{b} s_{odd}(a/b) = \frac{3\log\varphi}{\zeta(2)}\log b + C_2 + O(b^{-1/6+\varepsilon}),$$

$$\frac{2}{R(R+1)}\sum_{b\le R}\sum_{a=1}^{b} s_{odd}(a/b) = \frac{3\log\varphi}{\zeta(2)}\log R + \widetilde{C}_2 + O(R^{-1+\varepsilon}),$$

where constants C_2 and $\widetilde{C}_2$ can be written in terms of singular series.

Let $a_1, \ldots, a_n$ be positive integers with $a_i \geq 2$ and $(a_1, \ldots, a_n) = 1$. The following naive questions is known as **"Diophantine Frobenius problem"** (or **"Coin exchange problem"**):

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Determine the largest number which is not of the form

$$a_1 x_1 + \cdots + a_n x_n$$

where the coefficients x_i are non-negative integers. This number is denoted by $g(a_1, \ldots, a_n)$ and is called the **Frobenius number.**

Example

Let $a = 3$, $b = 5$. Then $g(a, b) =?$

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Answer: $g(a, b) = 7$:

$$7 \neq 3x + 5y \qquad (x, y \geq 0),$$

but for every $m > 7$ there are some $x, y \geq 0$ such that

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Frobenius numbers

The Diophantine Frobenius problem

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It is known that

$$g(a, b) = ab - a - a.$$

The challenge is to find g when $n \geq 3$.

The difficulty of the Diophantine Frobenius problem (**FP**) grows very fast with the number of arguments.

- For $n = 2$ problem is easy: $g(a, b) = ab - a - b$.
- For $n = 3$ problem is rather complicated (see Rödseth's formula below).
- For $n > 3$ general formula is unknown). We have only different algorithms for claculation Frobenius numbers.
- Kannan (1992) gave a polynomial time algorithm for **FP** for any fixed n.
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We shall consider

$$f(a,b,c) = g(a,b,c) + a + b + c,$$

the **positive Frobenius number** of a, b, c, defined to be the largest integer not representable as a **positive** linear combination of a, b, c

$$ax + by + cz, \qquad x, y, z \geq 1.$$

Positive Frobenius numbers are better because of Johnson's formula: for $d \mid a$, $d \mid b$

$$f(a,b,c) = d \cdot f\left(\frac{a}{d}, \frac{b}{d}, c\right).$$

Example

Let $a = 3$, $b = 5$, $c = 7$. Then $g(a, b, c) =?$

Example

Let $a = 3$, $b = 5$, $c = 7$. Then $g(a, b, c) = ?$
Answer: $g(3, 5, 7) = 4$:

$$4 \neq 3x + 5y + 7z \qquad (x, y, z \geq 0),$$

but for any $m > 4$ we can find $x, y, z \geq 0$ such that

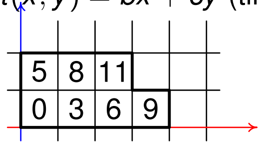
$$m \neq 3x + 5y + 7z.$$

Double loop network

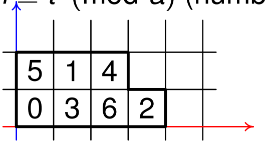
$b = 3$ (red step), $c = 5$ (blue step), $a = 7$ (number of vertices)

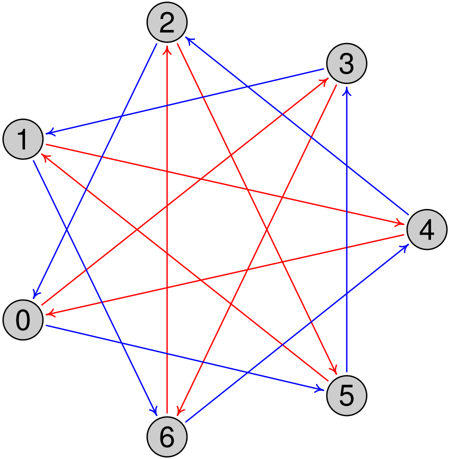
length($\color{red}{\uparrow}$)$= 3$, length($\color{blue}{\uparrow}$)$= 5$

$t(x, y) = bx + cy$ (time)



$n \equiv t \pmod{a}$ (number)





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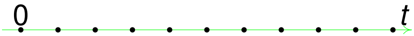

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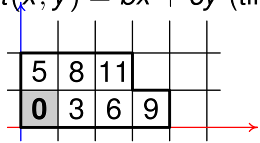

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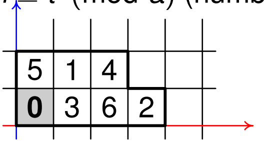

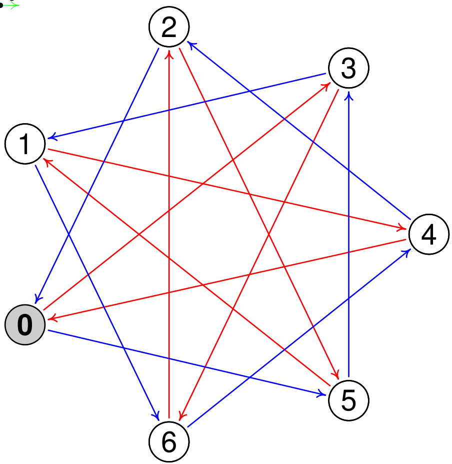

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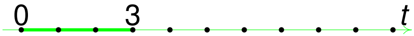

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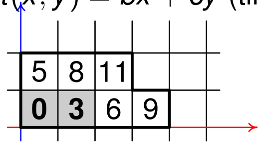

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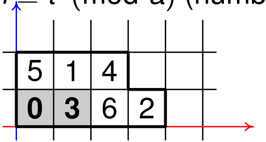
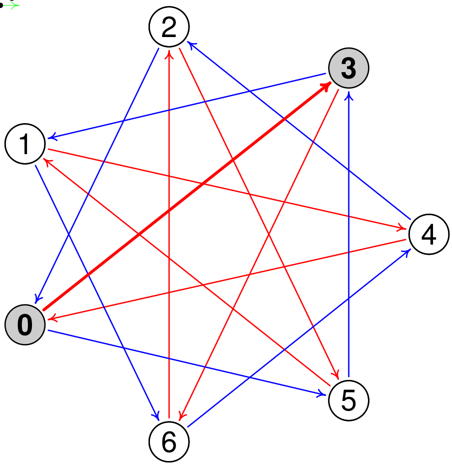




Double loop network

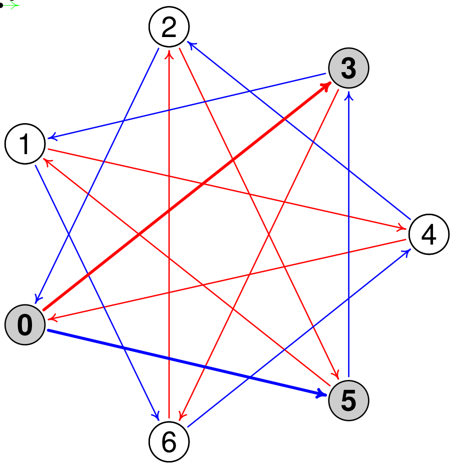
$b = 3$ (red step), $c = 5$ (blue step), $a = 7$ (number of vertices)

$t(x, y) = bx + cy$ (time)

5	8	11	
0	**3**	6	9

$n \equiv t \pmod{a}$ (number)

5	1	4	
0	**3**	6	2



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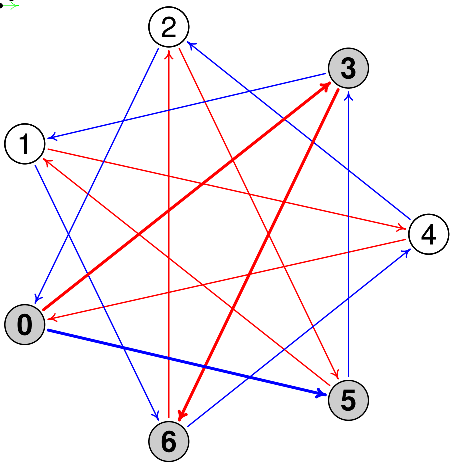

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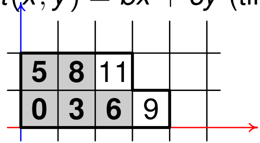

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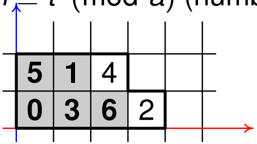

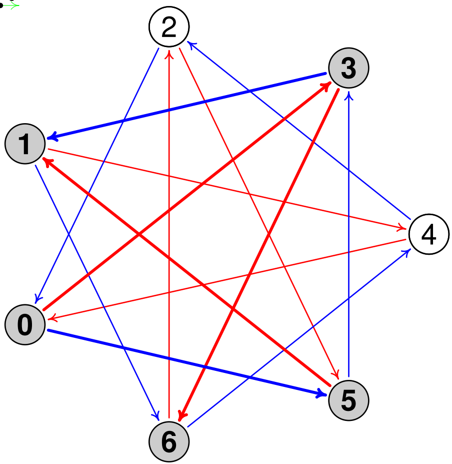

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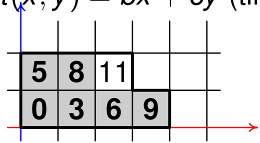

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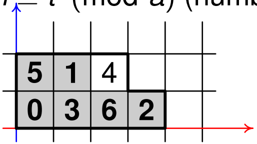

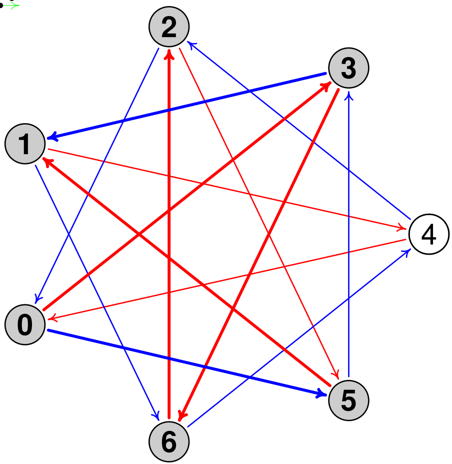

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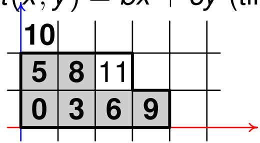

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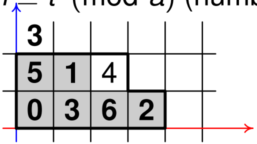

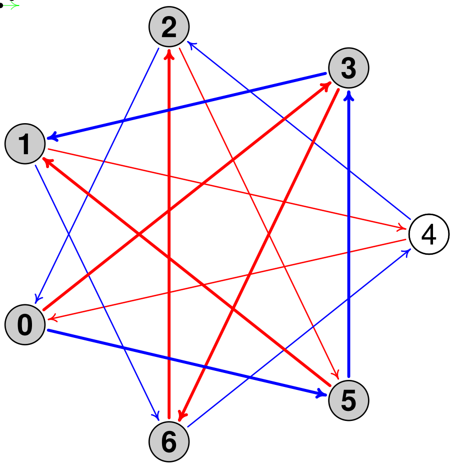

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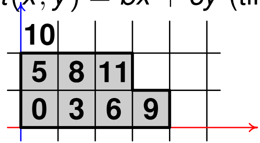

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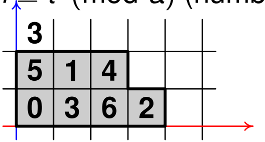
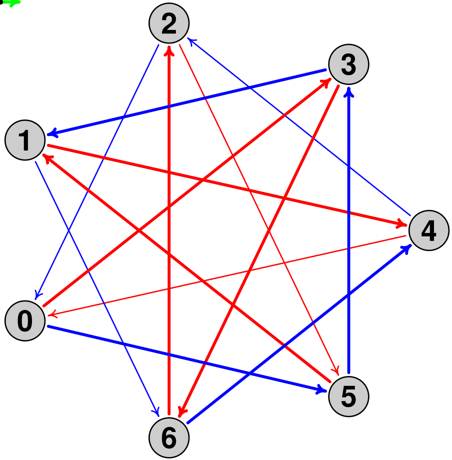
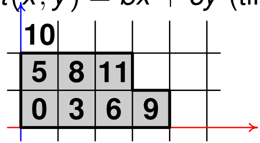




$$\mathrm{diam} = g(a, b, c) + a \quad (= 11)$$

$t(x, y) = bx + cy$ (time)



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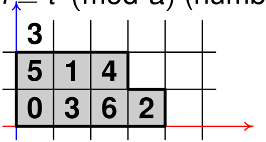

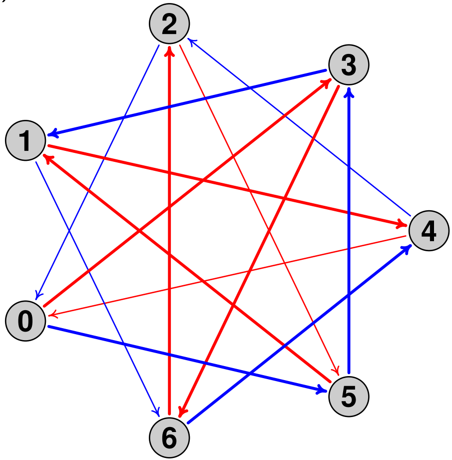

Double loop network
Properties

- Minimum distance diagram is always L-shaped (Wong, Coppersmith, 1974).
- L-shape always tessellates the plane.
- Form of L-shape depends on the properties of the lattice $\Lambda = \{(x, y) : bx + cy \equiv 0 \pmod{a}\}$.

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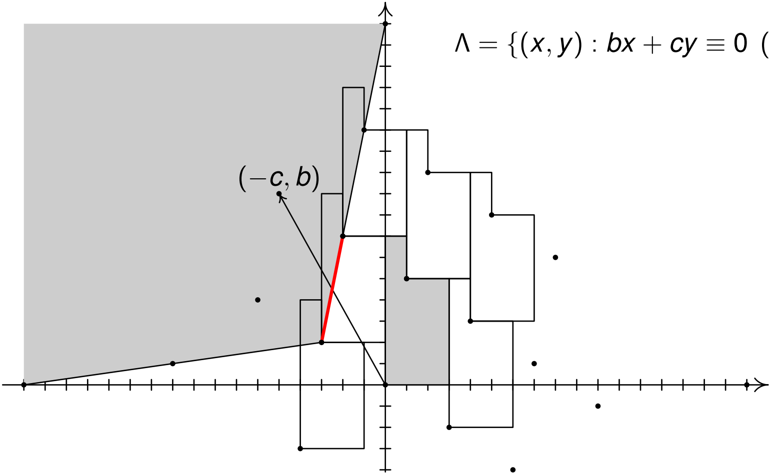

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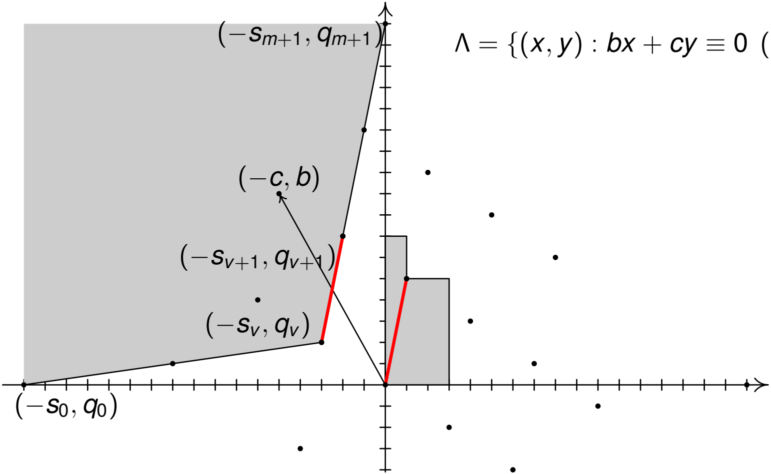

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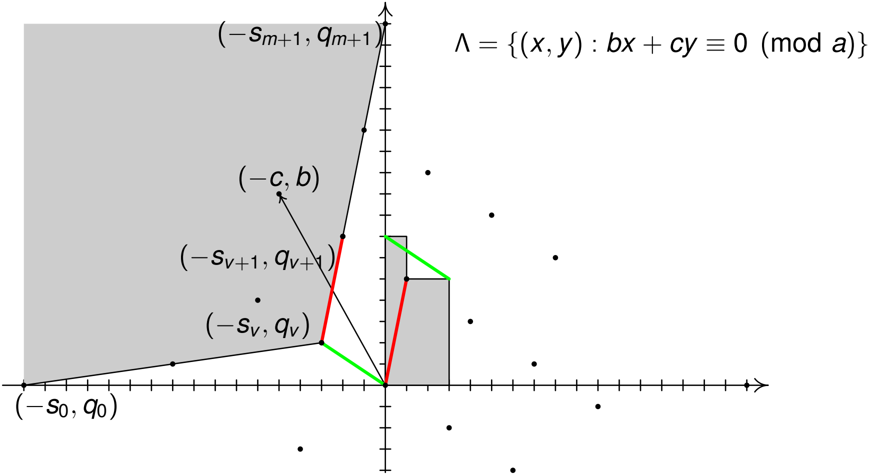

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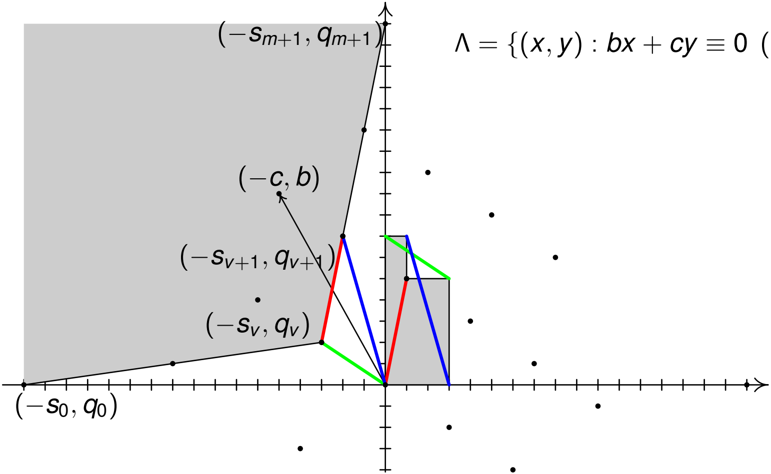

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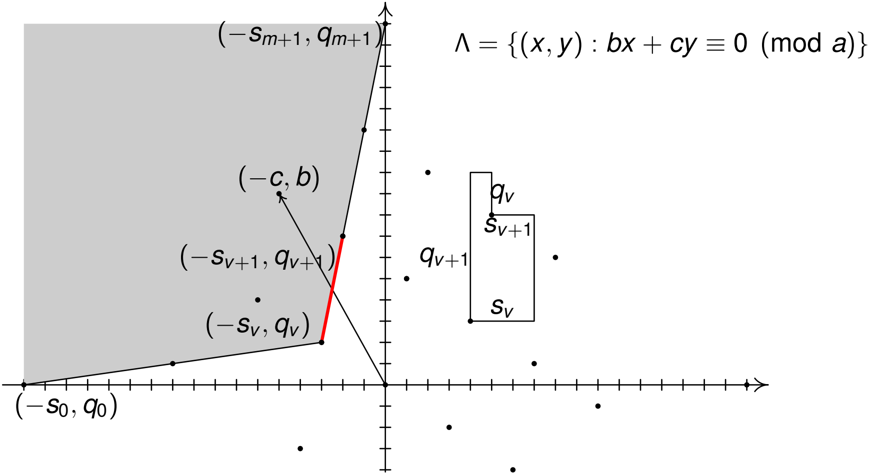

From obvious property

$$0 = \frac{s_{m+1}}{q_{m+1}} < \frac{s_{m-1}}{q_{m-1}} < \ldots < \frac{s_1}{q_1} < \frac{s_0}{q_0} = \infty$$

follows that for some n

$$\frac{s_n}{q_n} \leq \frac{c}{b} < \frac{s_{n-1}}{q_{n-1}}.$$

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Theorem (Ö. Rödseth, 1978)

$$f(a, b, c) = bs_{n-1} + cq_n - \min\{bs_n, cq_{n-1}\}.$$

Rödseth's formula can be written in terms of reduced regular continued fraction. We want to find $f(a,b,c)$ for $(a,b)=(a,c)=(b,c)=1$.
Let l is such that

$$bl \equiv c \pmod{a}, \qquad 1 \le l \le a.$$

Reduced regular continued fraction

$$\frac{a}{l} = \langle a_1, \dots, a_m \rangle = a_1 - \cfrac{1}{a_2 - \ddots - \cfrac{1}{a_m}},$$

where $a_1, \dots, a_m \geq 2$, defines sequences $\{s_j\}$, $\{q_j\}$ by

$$\frac{q_{j+1}}{q_j} = \langle a_j, \dots, a_1 \rangle, \qquad \frac{s_j}{s_{j+1}} = \langle a_{j+1}, \dots, a_m \rangle \qquad (0 \le j \le m).$$

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we have one-to-one correspondence between the set of quadruples $(q_n, s_n, q_{n-1}, s_{n-1})$ (taken for all lattices Λ_l) and the solutions of the equation

$$x_1 y_1 - x_2 y_2 = a$$

with $0 \le x_2 < x_1$, $0 \le y_2 < y_1$, $(x_1, x_2) = (y_1, y_2) = 1$:

$$(q_n, s_n, q_{n-1}, s_{n-1}) \longleftrightarrow (x_1, x_2, y_2, y_1).$$

From the equation

$$x_1 y_1 - x_2 y_2 = a$$

it follows that

$$x_1 y_1 \equiv a \pmod{x_2},$$

and **Kloosterman sums**

$$K_q(l, m, n) = \sum_{\substack{x,y=1 \\ xy \equiv l \pmod{q}}}^{q} e^{2\pi i \frac{mx+ny}{q}}$$

come into play. Solutions of the congruence $xy \equiv l \pmod{q}$ are uniformly distributed due to the bounds for Kloosterman sums.

This fact allows to calculate sums of the form

$$\sum_{xy\equiv l \pmod{q}} F(x,y)$$

and

$$\sum_{x_1y_1-x_2y_2=a} F(x_1,y_1,x_2,y_2).$$

In particular it allows to study distribution of Frobenius numbers $f(a,b,c)$.

Rödseth (1990) proved a lower bound for Frobenius numbers:

$$f(a_1, \ldots, a_n) \geq \sqrt[n-1]{(n-1)!a_1 \ldots a_n}.$$

Conjecture (Davison, 1994)

Average value of normalized Frobenius numbers $\frac{f(a,b,c)}{\sqrt{abc}}$ over cube $[1, N]^3$ tends to some constant as $N \to \infty$.

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Conjecture (Arnold, 1999, 2005)

There is weak asymptotic for Frobenius numbers: for arbitrary n average value of $f(x_1,\dots,x_n)$ over small cube with a center in $(a_1,\dots,a_n)$ approximately equal to $c_n \sqrt[n-1]{a_1\dots a_n}$ for some constant $c_n > 0$.

Frobenius numbers

Bourgain and Sinaĭ in 2007 proved (with a little gap: they used one natural assumption which was proved later) that normalized Frobenius numbers $\frac{f(a,b,c)}{\sqrt{abc}}$ have limiting density function.

Frobenius numbers
Weak asymptotic

Let $x_1, x_2 > 0$ and
$M_a(x_1, x_2) = \{(b, c) : 1 \le b \le x_1 a, 1 \le c \le x_2 a, (a, b, c) = 1\}$.

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Theorem (A.U., 2009)

Frobenius numbers $f(a, b, c)$ have weak asymptotic $\frac{8}{\pi}\sqrt{abc}$:

$$\frac{1}{a^{3/2}|M_a(x_1, x_2)|} \sum_{(b,c)\in M_a(x_1,x_2)} \left(f(a, b, c) - \frac{8}{\pi}\sqrt{abc} \right) = O_{\varepsilon,x_1,x_2}(a^{-1/6+\varepsilon}).$$

Davison's conjecture holds in a stronger form:

$$\frac{1}{|M_a(x_1, x_2)|} \sum_{(b,c)\in M_a(x_1,x_2)} \frac{f(a, b, c)}{\sqrt{abc}} = \frac{8}{\pi} + O_{\varepsilon,x_1,x_2}(a^{-1/12+\varepsilon}).$$

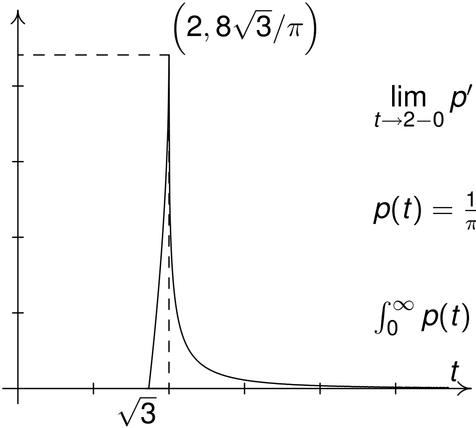
Theorem (A.U., 2010)

Normalized Frobenius numbers of three arguments have limiting density function:

$$\frac{1}{|M_a(x_1,x_2)|}\sum_{\substack{(b,c)\in M_a(x_1,x_2)\\ f(a,b,c)\le\tau\sqrt{abc}}} 1=\int_0^{\tau} p(t)\,dt+O_{\varepsilon,x_1,x_2,\tau}(a^{-1/6+\varepsilon}),$$

where

$$p(t)=\begin{cases}0, & \text{if } t\in[0,\sqrt{3}];\\ \frac{12}{\pi}\left(\frac{t}{\sqrt{3}}-\sqrt{4-t^2}\right), & \text{if } t\in[\sqrt{3},2];\\ \frac{12}{\pi^2}\left(t\sqrt{3}\arccos\frac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}+\frac{3}{2}\sqrt{t^2-4}\log\frac{t^2-4}{t^2-3}\right), & \text{if } t\in[2,+\infty).\end{cases}$$



$$\lim_{t \to 2-0} p'(t) = +\infty, \quad \lim_{t \to 2+0} p'(t) = -\infty$$

$$p(t) = \frac{18}{\pi^3} \cdot \frac{1}{t^3} + O\left(\frac{1}{t^5}\right) \qquad (t \to \infty)$$

$$\int_0^\infty p(t)\,dt = 1, \qquad \int_0^\infty tp(t)\,dt = \frac{8}{\pi}$$

Triples (α, β, r), where

$$\alpha = \frac{q_n}{\sqrt{a/\xi}}, \quad \beta = \frac{s_{n-1}}{\sqrt{a\xi}}, \quad r = \frac{s_n}{\sqrt{a\xi}} \qquad (\xi = c/b)$$

(normalized edges of L-shaped diagram) have joint limiting density function

$$p(\alpha, \beta, r) = \begin{cases} \dfrac{2}{\zeta(2)r}, & r \leq \min\{\alpha, \beta\}, 1 \leq \alpha\beta \leq 1 + r^2, \\ 0 & else. \end{cases}$$

It allows to study shortest cycles, average distances and another characteristics of L-shaped diagrams (double loop networks).

For usual Kloosterman sums

$$K_q(1,m,n) = \sum_{\substack{x,y=1 \\ xy\equiv 1 \pmod q}}^{q} e^{2\pi i \frac{mx+ny}{q}}$$

Estermann bound is known

$$|K_q(1,m,n)| \le \sigma_0(q) \cdot (m,n,q)^{1/2} \cdot q^{1/2}.$$

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This bound can be generalized for the case of sums $K_q(l,m,n)$.

Theorem (A.U., 2008)

$$|K_q(l,m,n)| \le \sigma_0(q)\cdot\sigma_0((l,m,n,q))\cdot(lm,ln,mn,q)^{1/2}\cdot q^{1/2}.$$

This estimate allows to count solutions of the congruence $xy \equiv l \pmod{a}$ in different regions.

Corollary

Let $q \geq 1$, $0 \leq P_1, P_2 \leq q$. Then for any real Q_1, Q_2

$$\sum_{\substack{Q_1 < x \leq Q_1 + P_1 \\ Q_2 < y \leq Q_2 + P_2}} \delta_q(xy - 1) = \frac{\varphi(q)}{q^2} \cdot P_1 P_2 + O\left(\sigma_0(q) \log^2(q+1) q^{1/2}\right)$$

and

$$\sum_{\substack{Q_1 < x \leq Q_1 + P_1 \\ Q_2 < y \leq Q_2 + P_2}} \delta_q(xy - l) = \frac{K_q(0,0,l)}{q^2} \cdot P_1 P_2 + O\left(q^{1/2+\varepsilon} + (q,l) q^{\varepsilon}\right).$$

General idea
Kloosterman sums

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Let $q \geq 1$, f be positive function and $T[f]$ be the number of solutions of the congruence $xy \equiv l \pmod q$ in the region $P_1 < x \leq P_2$, $0 < y \leq f(x)$:

$$T[f] = \sum_{P_1 < x \leq P_2} \sum_{0 < y \leq f(x)} \delta_q(xy - l).$$

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Let

$$S[f] = \sum_{P_1 < x \leq P_2} \frac{\mu_{q,l}(x)}{q} f(x),$$

where $\mu_{q,l}(x)$ is the number of solutions of the congruence $xy \equiv l \pmod{q}$ over y such that $1 \leq y \leq q$.

Theorem (A.U., 2008)

Let P_1, P_2 be reals, $P = P_2 - P_1 \geq 2$ and for some $A > 0$, $w \geq 1$ function $f(x)$ satisfies conditions

$$\frac{1}{A} \leq |f''(x)| \leq \frac{w}{A}.$$

Then

$$T[f] = S[f] - \frac{P}{2} \cdot \delta_q(l) + R[f],$$

where

$$R[f] \ll_w (PA^{-1/3} + A^{1/2}(l,q)^{1/2} + q^{1/2})P^{\varepsilon}.$$

Recent results

- The existence of limiting distribution for normalized Frobenius numbers of arbitrary number of arguments was proved by J. Marklof (2010).
- Distribution of diameters and distribution of shortest cycles in *circulant graphs* (often also called multi-loop networks) were studied by J. Marklof and A. Strömbergsson (2011). They proved existence of these distributions for arbitrary n and made some interesting numerical computations.
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Sinai problem

Let $0 < h < \frac{1}{8}$, $T > 0$ and $\Omega_h(T)$ is the set of angles $\varphi \in [0, 2\pi)$ such that the ray

$$\{(t\cos\varphi, t\sin\varphi) : t \geq 0\}$$

intersects h-neighborhood of some integer point $(m, n) \neq (0, 0)$ from the circle

$$\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq T^2\right\}.$$

Denote by $G_h(T)$ normalized measure of $\Omega_h(T)$:

$$G_h(T) = \frac{1}{2\pi} \operatorname{mes} \Omega_h(T) \in [0, 1].$$

In 1918 Polya proved that

$$G_h(T) = 1$$

for all $T \geq h^{-1}$.

Sinai problem

Boca, Gologan and Zaharescu (2003) proved that for all $\varepsilon > 0$ uniformly over $T \in [0, h^{-1}]$

$$G_h(T) = \int_0^{h \cdot T} \sigma(t)\, dt + O_\varepsilon(h^{1/8-\varepsilon}),$$

where

$$\sigma(t) = \begin{cases} \frac{12}{\pi^2}, & \text{if } 0 \le t \le \frac{1}{2}; \\ \frac{12}{\pi^2}\left(\frac{1}{t} - 1\right)\left(1 - \log\left(\frac{1}{t} - 1\right)\right), & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

From physical point of view $G_h(T)$ is the density function for free path lengths in 2-dimensional Lorentz gas.

Sinai problem

We considered more general situation when trajectories start from h-neighborhood of the origin. Let $v \in (-1, 1)$ be the fixed number and the particle moves along the ray

$$\left\{(-hv\sin\varphi + t\cos\varphi, hv\cos\varphi + t\sin\varphi) \in \mathbb{R}^2 : t \geq 0\right\}. \tag{1}$$

Let $(m(\varphi), n(\varphi))$ be the center of the first h-neighborhood intersected by the ray.

Sinai problem

In other words $(m(\varphi), n(\varphi))$ is the nearest to the origin point such that

$$R(m,n) > 0 \quad \text{and} \quad |U(m,n)| < h$$

where

$$R(x,y) = x\cos\varphi + y\sin\varphi,$$
$$U(x,y) = x\sin\varphi - y\cos\varphi + hv.$$

We denote by

$$r(\varphi) = h \cdot R\,(m(\varphi), n(\varphi))\,, \quad u(\varphi) = h^{-1} \cdot U\,(m(\varphi), n(\varphi))\,.$$

normalized free path length and normalized sighting (aiming?) parameter.

Sinai problem

Suppose

$$0 < r_0 < \frac{1}{1-|v|} \quad \text{and} \quad -1 < u_- < u_+ < 1.$$

Theorem (Bykovskii, A.U., 2007–2008)

Let $|v| < c < 1$. *Then for all* $\varepsilon > 0$ *for the distribution function*

$$\Phi_v(h) = \Phi_v(h; \varphi_0, r_0, u_-, u_+) = \\ = \int_0^{\varphi_0} \chi_{[0,r_0]}\left(r(\varphi)\right) \chi_{[u_-,u_+]}\left(u(\varphi)\right)\, d\varphi$$

following asymptotic formula holds $(h \to 0)$

$$\Phi_v(h) = \int_0^{\varphi_0} \int_0^{r_0} \int_{u_-}^{u_+} \rho(\varphi, r, v, u)\, d\varphi\, dr\, du + O_{\varepsilon,c}\left(h^{\frac{1}{2}-\varepsilon}\right).$$

Sinai problem

Density function has following symmetries

$$\rho(\varphi, r, v, u) = \rho(r, v, u) = \rho(r, u, v) = \rho(r, -u, -v),$$

for $u \geq |v|$ is equal to

$$\rho(r, u, v) = \begin{cases} \frac{6}{\pi^2}, & \text{if } \ 0 \leq r \leq \frac{1}{u+1}; \\ \frac{6}{\pi^2} \cdot \frac{1}{u-v} \left(\frac{1}{r} - 1 - v \right), & \text{if } \ \frac{1}{u+1} \leq r \leq \frac{1}{1+v}; \\ 0, & \text{if } \ \frac{1}{1+v} \leq r. \end{cases}$$

From physical point of view $\frac{1}{2\pi}\rho(\varphi, r, v, u)$ is the density of the particles moving along the ray (1), with unit speed after first reflection in h-neighborhood of the origin and passing distance $R = h^{-1} \cdot r$ before next reflection with sighting parameter $h \cdot u$.

Reduced bases in two-dimensional lattices

Reduced (2-dimensional) bases are important in different number theory algorithms:

- fast point multiplication on elliptic curves;
- prediction of pseudo random generators, numerical integration;
- combinatorial optimization. . .

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Work of these algorithms depends on properties of reduced basis (shorter vectors are better).

Reduced bases in two-dimensional lattices

Let $1 \le l \le a$, $(l, a) = 1$ and e_1 be the shortest vector of the lattice $\Lambda_l = \{(x, y) : lx \equiv y \pmod{a}\}$.

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$$\|e_2\| \geq \|e_1\| \quad \text{and} \quad \|e_2 \pm e_1\| \geq \|e_2\|.$$

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Moreover vector e_2 must lie on the line $l(e_1)$ defined by equation $\det(e_1, e_2) = a$. By averaging over l we can get that vectors e_2 distributed uniformly on $\Omega(e_1) \cap l(e_1)$ with weight $\|e_2\|_2^{-1}$. Suppose $e_1 = \sqrt{a}(\alpha, \beta)$, $e_2 = \sqrt{a}(\gamma, \delta)$.

Reduced bases in two-dimensional lattices

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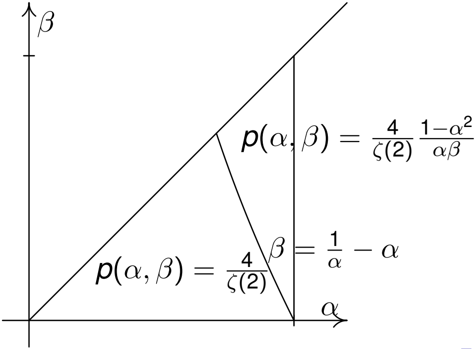

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$$p(t) = \begin{cases} 0, & \text{if } t \in \left[0, 1/\sqrt{2}\right]; \\ \frac{4}{\zeta(2)}\left(2t - \frac{1}{t} + \left(\frac{1}{t} - t\right)\log\left(\frac{1}{t^2} - 1\right)\right), & \text{if } t \in \left[1/\sqrt{2}, 1\right]; \\ \frac{4}{\zeta(2)}\left(\frac{1}{t} + \left(t - \frac{1}{t}\right)\log\left(1 - \frac{1}{t^2}\right)\right), & \text{if } t \in [1, \infty]. \end{cases}$$

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