Voronoi – Minkowski 3-D continued fractions

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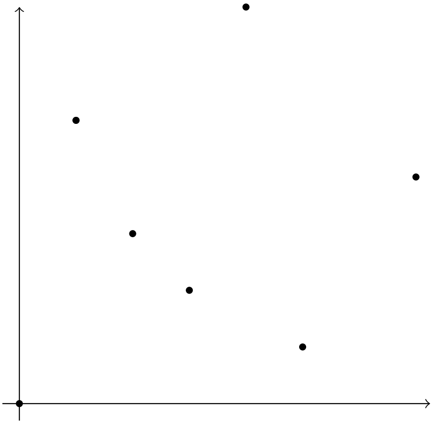
Let S be a subset of $\mathbb{R}^2_{\geqslant 0}$. Consider the boundary of the set

$$S \oplus \mathbb{R}^2_{\geqslant 0} = \{s + r \mid s \in S, r \in \mathbb{R}^2_{\geqslant 0}\}.$$

In other words, this broken line is the boundary of the union of copies of the positive quadrant shifted by vertices of the set S.



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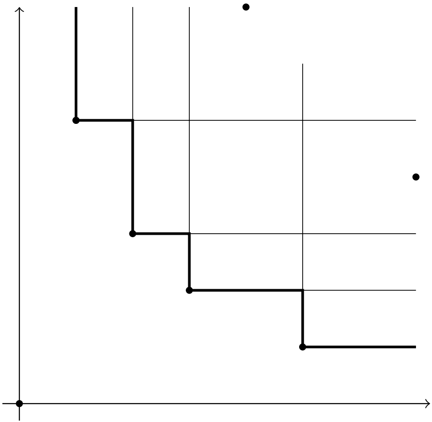

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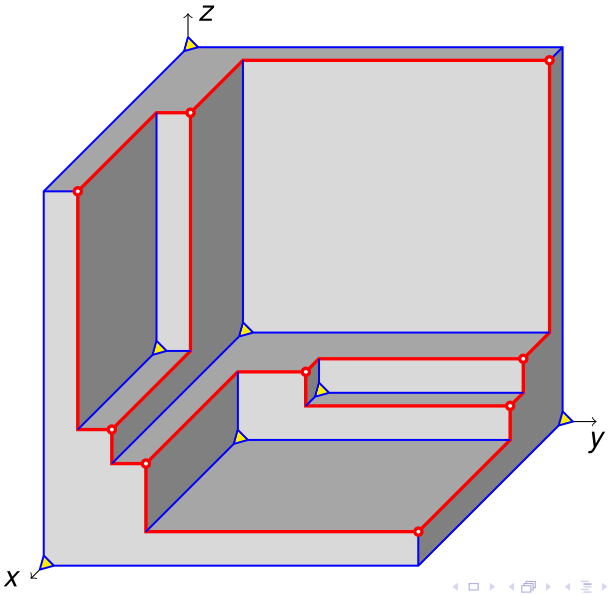
We assume that

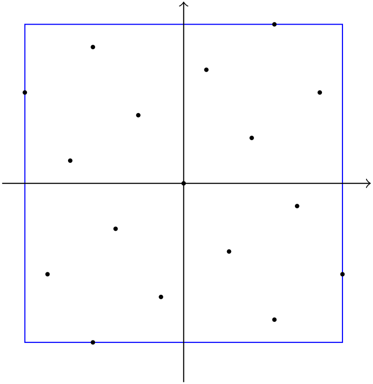
(1) S has no accumulation points;

(2) S is in general position: each plane parallel to a coordinate plane contains at most one point of S.

Voronoi-Minkowski complex



For a nonempty point set $T \subset \mathbb{R}^s$ $\mathrm{Box}(T)$ is the least possible parallelepiped circumscribed about T.



More formally: if

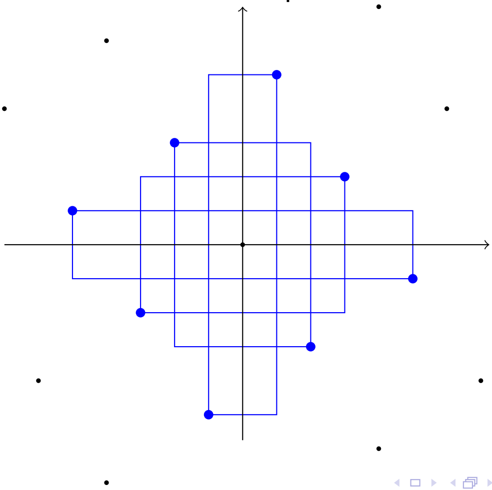
$$|T|_i = \max\{|x_i| : \ x = (x_1, \dots, x_s) \in T\} \quad (i = 1, \dots, s),$$

then

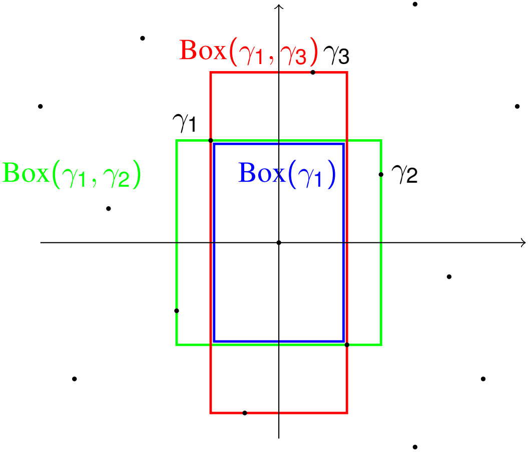
$$\mathrm{Box}(T) = [-|T|_1, |T|_1] \times \dots \times [-|T|_s, |T|_s].$$

A point γ in a lattice Γ is called a *relative (local) minimum* of the lattice Γ in the sense of Voronoi (or simply a *minimum*) if the $\mathrm{Box}(\gamma)$ is *free* (it contains no points of the lattice Γ different from its vertices and the origin).

2-D example:



The $\mathrm{Box}(\gamma_1, \gamma_2)$ is called *extreme* if it is *free* and if, at the same time, it has on each of its faces at least one lattice point.
In other words it is impossible to extend this parallelepiped in any coordinate direction so that the resulting parallelepiped still contains no nonzero lattice points.



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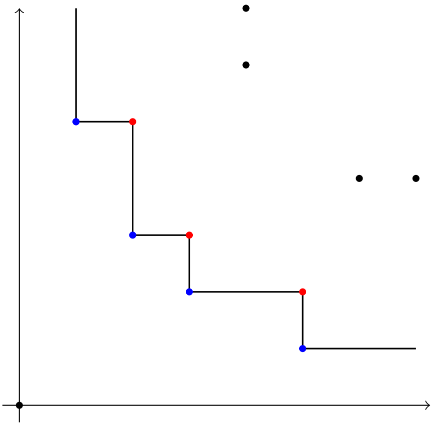
Instead of lattice Γ we can consider a set $|\Gamma| \subset \mathbb{R}^s$ where

$$|\Gamma| = \{(|x|, |y|, |z|) : (x, y, z) \in \Gamma\}.$$

As it was proved by Voronoi, we can consider a classical continued fraction as a sequence of local minima (halls) or extreme parallelepipeds (hills)



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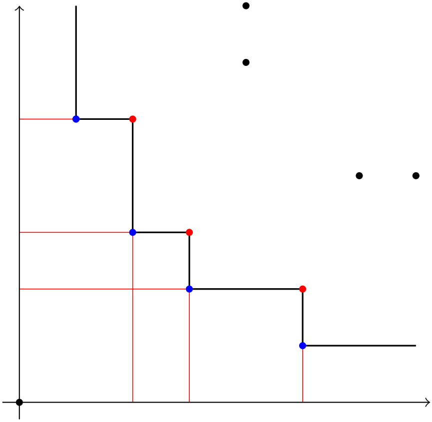

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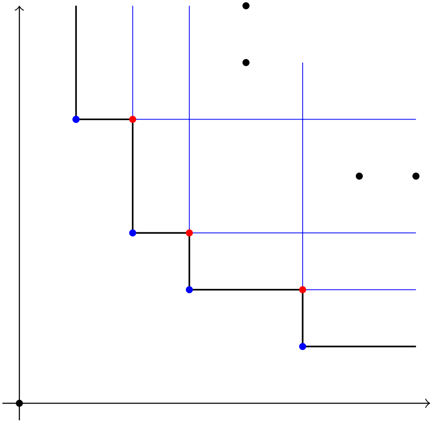

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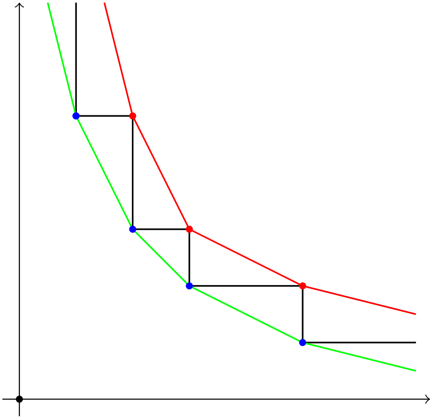
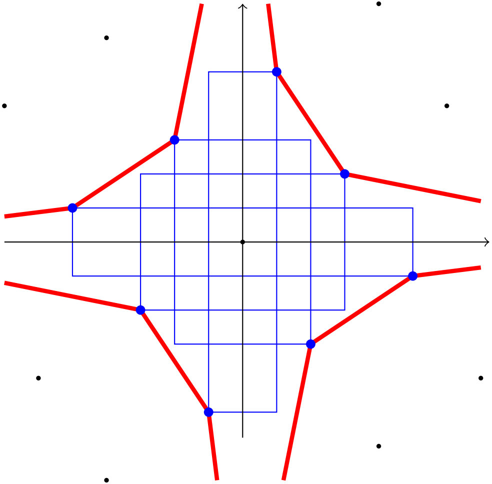


Local minima and Klein polyhedron: (in 2-D case)

local minima=vertices of Klein polyhedron



In 3-D case vertices of Klein polyhedron are always local minima, but converse is not true (Bykovski, 2006).
In other words local minima have more rich structure (they can lie on the faces of Klein polyhedron.

3-D definitions

The $\mathrm{Box}(\gamma_1, \gamma_2, \gamma_3)$ is called *extreme* if it is *free* (it contains no lattice points other than the origin) and if, at the same time, it has on each of its faces at least one lattice point.

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A set of vectors (s.t. $v_i \neq v_j$) S in the lattice Γ is said to be *minimal* if the $\mathrm{Box}(S)$ contains no points of Γ except the origin. In particular, a minimal system of order 1 is a local minimum, minimal systems of order 3 gives extreme parallelepiped.

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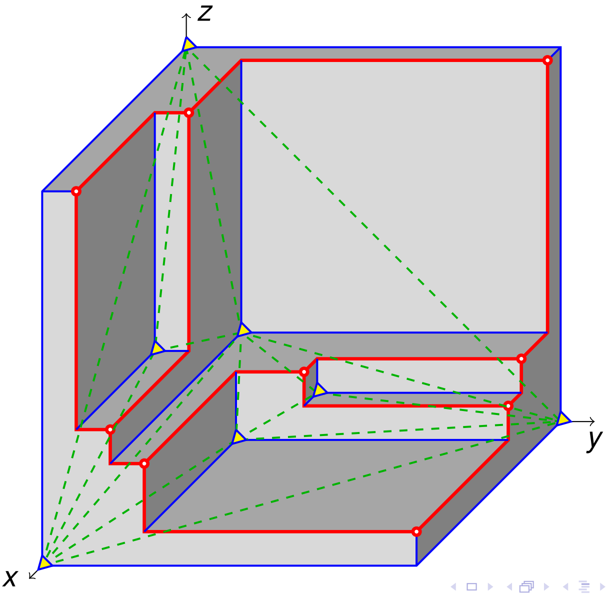
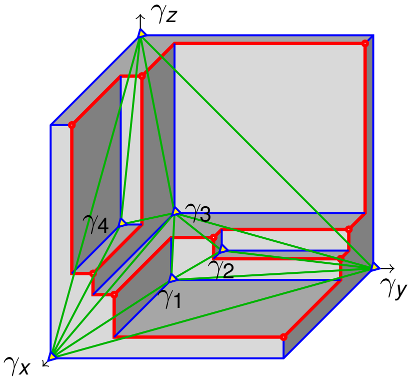
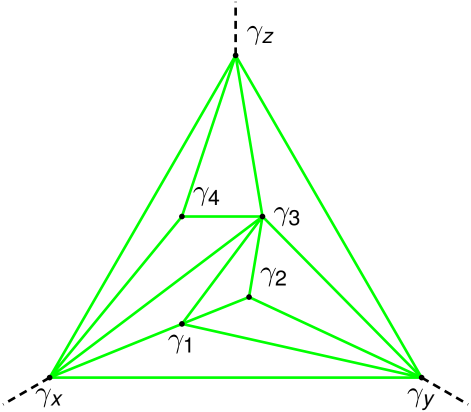
If $\{\gamma_1, \gamma_2\}$ is a minimal system of order 2 then γ_1 and γ_2 are *neighbours*.

Minkowski graph




γ_z
γ_4
γ_3
γ_2
γ_1
γ_y
γ_x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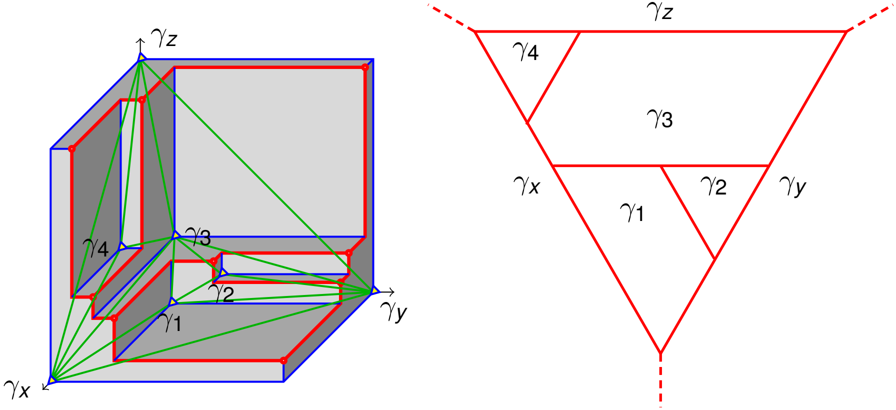
Voronoi (=Minkowski*) graph



Here coordinates of vertices in space and on the plane $x + y + z = 0$ are concordant

Some reasons

Why do this objects are interesting and important?

- Good algorithms.
- Periodicity for algebraic numbers.
- “Vahlen’s theorem”.
- “Gauss measure”.
- Possibility to apply “hard” (analytical) methods based on Kloosterman sums.

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They were able to do all calculations by hand ☺

Theorem (Lagrange's Continued Fraction Theorem.)

The real roots of quadratic expressions with integral coefficients have periodic continued fractions.

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Two main examples (the beginning of *Markov spectrum*) are

$$\frac{1+\sqrt{5}}{2} = 2\cos\frac{2\pi}{5} = [1;1,\dots,1,\dots] = 1 + \cfrac{1}{1 + \ddots + \cfrac{1}{1+\dots}},$$

$$\sqrt{2} = 2\cos\frac{2\pi}{8} = [1;2,\dots,2,\dots] = 1 + \cfrac{1}{2 + \ddots + \cfrac{1}{2+\dots}}.$$

Some reasons

Periodicity

With quadratic irrational α we can associate a lattice $\Gamma(\alpha)$ with basis $(1, 1)$ and (α, β) where β is conjugate of α (second root of the same quadratic equation.)

Periodical continued fraction of α describes periodical structure of local minima of $\Gamma(\alpha)$.

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With cubic irrationality α (from totally real cubic field) we can associate 3-D lattice with basis $(1,1,1)$, (α,β,γ), $(\alpha^2,\beta^2,\gamma^2)$, where β and γ are conjugates of α.

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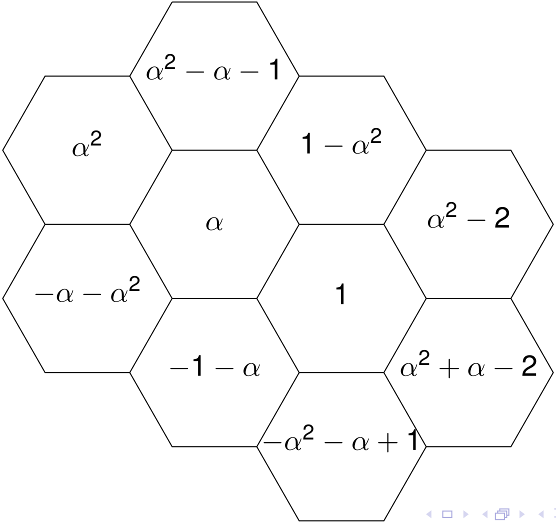
Two mains examples arise from cubic numbers $\alpha = 2\cos\frac{2\pi}{7}$ and $\alpha = 2\cos\frac{2\pi}{9}$ (associated with first two *extremal Davenport cubic forms*).

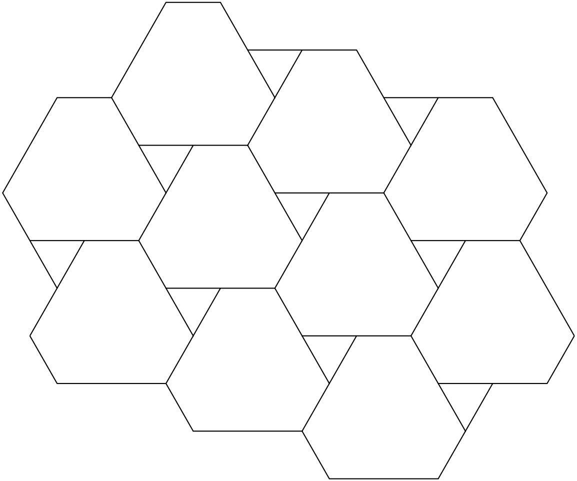
Some reasons

Periodicity

The Voronoi graph for $\alpha = 2\cos\frac{2\pi}{7}$



The Voronoi graph for $\alpha = 2\cos\frac{2\pi}{9}$



Some reasons

Vahlen's theorem

Denote by $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ convergents to a given number $\alpha = [a_0; a_1, \ldots, a_n, \ldots]$.
Vahlen's theorem: for $p/q = p_{n-1}/q_{n-1}$ or $p/q = p_n/q_n$

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{2q^2}$$

can be translated to the lattice language. The equivalent statement: $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ is a minimal system on lattice Γ, then

$$\min\{|a_1 a_2|, |b_1 b_2|\} \leqslant \frac{1}{2} \det \Gamma.$$

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Vahlen's theorem has a stronger form:

$$|a_1 a_2| + |b_1 b_2| \leqslant \det \Gamma,$$

Theorem (Avdeeva and Bykovskii, 2006)

If

$$\gamma_a = (a_1, a_2, a_3), \quad \gamma_b = (b_1, b_2, b_3), \quad \gamma_c = (c_1, c_2, c_3),$$

is a minimal system on lattice Γ, *then*

$$|a_1 a_2 a_3| + |b_1 b_2 b_3| + |c_1 c_2 c_3| \leqslant \det \Gamma.$$

Theorem (Avdeeva and Bykovskii, 2006)

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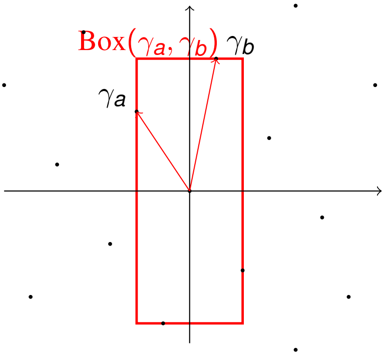
$$|a_1 a_2 a_3| + |b_1 b_2 b_3| + |c_1 c_2 c_3| \leqslant \det \Gamma.$$

This theorem can be regarded as a sharpening of the estimate

$$|a_1 a_2 a_3| + |b_1 b_2 b_3| + |c_1 c_2 c_3| \leqslant 3 \det \Gamma,$$

which follows from Minkowskis convex body theorem.

In 2-D case minimal couple $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ is always a basis of a given lattice (Voronoi):



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Gauss measure

We can associate with minimal system $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ the matrix $\left(\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right)$ with diagonal dominance: $|a_1| > |b_1|$, $|b_2| > |a_2|$.

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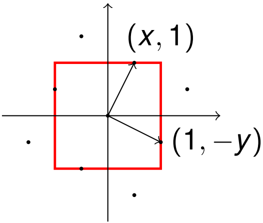
$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \to \begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix}$ where $0 < x < 1$, $0 < y < 1$. $\mathrm{Box}(\gamma_a, \gamma_b) \to [-1, 1]^2$.

Gaussian measure

$$d\mu = \frac{dx\,dy}{(1+xy)^2} = \frac{dxdy}{\begin{vmatrix} 1 & x \\ -y & 1 \end{vmatrix}^2}$$

defined for $(x, y) \in [0, 1]^2$ describes typical behavior of classical continued fractions. From geometrical point of view this density function describes distribution of vectors from bases $\begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix}$ on the sides of unit square.



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3-D surprise (Minkowski): either minimal triple $\gamma_a = (a_1, a_2, a_3)$, $\gamma_b = (b_1, b_2, b_3)$, $\gamma_c = (c_1, c_2, c_3)$ is a basis and corresponding matrix equivalent to

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$$\gamma_a \pm \gamma_b \pm \gamma_c = 0.$$

For bases $(x_2, x_3, y_1, y_3, z_1, z_2) \in$ subset of $[0, 1]^6$ defined by some simple liner inequalities depending on the sign before x_3.

Some reasons

Gauss measure

The 3-D analogue of Gaussian measure

$$d\mu = \frac{dx_2\, dx_3 \ldots dz_2}{\begin{vmatrix} 1 & x_2 & \pm x_3 \\ -y_2 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{vmatrix}^3}$$

describes distribution of basis vector on some subset of $[0, 1]^6$.

The 3-D analogue of Gaussian measure

$$d\mu = \frac{dx_2\, dx_3 \dots dz_2}{\begin{vmatrix} 1 & x_2 & \pm x_3 \\ -y_2 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{vmatrix}^3}$$

describes distribution of basis vector on some subset of $[0,1]^6$.

The same measure describes behavior of Klein polyhedrons. The difference is in measure space. Measure space varies for different types of 3-D continued fractions.

In 2-D problems we study 2×2 matrices with fixed determinant:

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Propagating a solution we have 3 degrees of freedom (u, t, $z \in \mathbb{Z}$):

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In such a situation

$$\begin{vmatrix} a & zb + ua \\ -z^{-1}c + ta & * \end{vmatrix} = N$$

we can average over z and apply estimations of Kloosterman sums

$$K_a(m,n) = \sum_{\substack{z=1 \\ (a,z)=1}}^{a} e^{2\pi i \frac{mz+nz^{-1}}{a}}.$$

Some reasons

Analytical tool: Kloosterman sums and 3-D $\to$ 2-D reduction

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $q = \det A \neq 0$ and

$$\begin{vmatrix} \multicolumn{2}{c}{\multirow{2}{*}{\Huge A}} & x_1 \\ & & x_2 \\ x_3 & x_4 & x_5 \end{vmatrix} = N.$$

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Linnik and Skubenko (1964) used this argument studying distribution of points on a variety defined by equation $\det(x_{ij}) = N$ $(i, j = 1, 2, 3)$.

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- Are there any connections with singularity resolutions in toric geometry?

Thank you for your attention!