Double Somos-4

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Somos–(4) sequence $\{s_n\}$ is defined by initial data

$$s_1 = s_2 = s_3 = s_4 = 1$$

and recurrence relation

$$s_{n+2}s_{n-2} = s_{n+1}s_{n-1} + s_n^2.$$

It begings with

$$\ldots, 2, 1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \ldots$$

(Obviously $s_n = s_{5-n}$.)

Somos–(6)

Somos first introduced the sequence Somos-(6) such that

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 1$$

and

$$s_{n+3}s_{n-3} = s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

He raised the question whether all the terms are integer:

<u>1470</u>*. *Proposed by Michael Somos, Cleveland, Ohio.*

Consider the sequence (a_n) where $a_0 = a_1 = \cdots = a_5 = 1$ and

$$a_n = \frac{a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2}{a_{n-6}}$$

for $n \geq 6$. Computer calculations show that $a_6, a_7, \ldots, a_{100}$ are all integers. Consequently it is conjectured that all the a_n are integers. Prove or disprove.

Somos M. Problem 1470. Crux Mathematicorum, 15: 7 (1989), p. 208.

Integrality

The integrality of Somos–(4) and Somos–(5) was proved by Janice Malouf, Enrico Bombieri and Dean Hickerson (1990).

The integrality of Somos–(6) was proved by Dean Hickerson (April 1990).

The integrality of Somos–(7) was proved by Ben Lotto (May 1990).

Somos–(8)

Somos–(8) is a sequence with initial data

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = s_7 = s_8 = 1$$

satisfying recurrence relation

$$s_{n+4}s_{n-4} = s_{n+3}s_{n-3} + s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

Somos–(8) is NOT an integer sequence:

$$\ldots, 1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 61, 187, 775, 5827, 14815, \frac{420514}{7}, \ldots$$

Definition

For integer $k \geq 4$ Somos–k sequence is a sequence generated by quadratic recurrence relation of the form

$$\tau_{n+k}\tau_n = \sum_{j=1}^{[k/2]} \alpha_j \tau_{n+k-j}\tau_{n+j},$$

where α_j are constants and $\tau_1, \ldots, \tau_k$ are initial data.

The problem of integrality: how to check the integrality of general Somos–k sequences?

Integrality condition

Theorem (Fomin and Zelevinsky, 2002)

For a Somos–k sequences ($k = 4, 5, 6$ and 7) all of the terms in the sequences are Laurent polynomials in these initial data whose coefficients are in $\mathbb{Z}[\alpha_1, \dots, \alpha_{[k/2]}]$, so that

$$\tau_n \in \mathbb{Z}[\alpha_1, \dots, \alpha_{[k/2]}, \tau_1^{\pm 1}, \dots, \tau_k^{\pm 1}] \text{ for all } n \in \mathbb{Z}.$$

Fomin S. and Zelevinsky A. "The Laurent Phenomenon", Adv. Appl. Math. 28 (2002) 119–144.

Integrality of original Somos–(k) sequences follows from the theorem with

$$\alpha_1 = \dots = \alpha_{[k/2]} = \tau_1 = \dots = \tau_k = 1.$$

But this Theorem is not a final step.

A stronger version of the Laurent property for Somos–4

Theorem (Hone and Swart, 2008)

Consider a Somos–4 sequence defined by a fourth-order recurrence

$$\tau_{n+4}\tau_n = \alpha\tau_{n+3}\tau_{n+1} + \beta\tau_n^2$$

with initial data $\tau_1, \tau_2, \tau_3, \tau_4$*, and translation invariant*

$$T = \frac{\tau_1^2\tau_4^2 + \alpha(\tau_2^3\tau_4 + \tau_1\tau_3^3) + \beta\tau_2^2\tau_3^2}{\tau_1\tau_2\tau_3\tau_4},$$

and $I = \alpha^2 + \beta T$*. Then for all* $n \geq 1$

$$\tau_n \in \mathbb{Z}[\alpha, \beta, I, \tau_1^{\pm 1}, \tau_2, \tau_3, \tau_4].$$

Hone A. N., Swart C. Integrality and the Laurent phenomenon for Somos–4 and Somos–5 sequences. Math. Proc. Camb. Philos. Soc., Cambridge University Press, Cambridge, 2008, 145, 65–85.

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126742987
1687054711
47301104551
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32606721084786
1662315215971057
61958046554226593
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334806306946199122193
23385756731869683322514
3416372868727801226636179
397068399296019032727466599
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2993270782370572835241882188306602
1107464108502549897934954766675333353
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By definition

$$\sigma_\Gamma(z) = z \prod_{w \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{w}\right) e^{\frac{z}{w} + \frac{1}{2}(\frac{z}{w})^2}$$

– Weierstrass sigma-function associated to lattice Γ.
In degenerate cases:
1) for $\Gamma = \{0\}$ $\qquad \sigma_\Gamma(z) = z$;
2) for $\Gamma = \{mw | m \in \mathbb{Z}\}$ with $w \in \mathbb{C} \setminus \{0\}$

$$\sigma_\Gamma(z) = z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{nw}\right) e^{\frac{z}{nw} + \frac{1}{2}(\frac{z}{nw})^2} = \frac{w}{\pi} \sin \frac{\pi z}{w} e^{\frac{\pi^2}{6}(\frac{z}{w})^2}.$$

Sigma-function is odd entire function with simple zeros at the points of lattice Γ.

A sigma-function solution for Somos–4

Theorem (Hone and Swart, 2003–2005)

The general solution of recurrence relation

$$\tau_{n+2}\tau_{n-2} = \alpha\tau_{n+1}\tau_{n-1} + \beta\tau_n^2$$

for $\alpha \neq 0$ *takes the form*

$$\tau_n = CD^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}},$$

where $z,\ z_0 \in \mathbb{C}^*$, $C = \frac{\tau_0}{\sigma(z_0)}$, $D = \frac{\sigma(z)\sigma(z_0)\tau_1}{\sigma(z+z_0)\tau_0}$.

A. Hone, *Elliptic Curves and Quadratic Recurrence Sequences* Bull. Lon, Math. Soc. 37 2 (2005) 161–171.

C. Swart *Elliptic curves and related sequences*, PhD thesis, Royal Holloway, University of London (2003).

The problem

Consider a **Double Somos–4** sequence which is a sequence of couples $(A_n, B_n) \subset \mathbb{C}^2$ defined by initial data

$$A_{\pm 2}, A_{\pm 1}, A_0, \quad B_{\pm 2}, B_{\pm 1}, B_0$$

and a fourth-order recurrences

$$A_{n+2}B_{n-2} = \alpha A_{n+1}B_{n-1} + \beta A_n B_n,$$
$$A_{n-2}B_{n+2} = \gamma A_{n-1}B_{n+1} + \delta A_n B_n.$$

Here coefficients α, β, γ, δ are NOT arbitrary. They are some functions of initial data. For example fourth-order recurrences then for $n = 0$ give linear equation on α, β, γ, δ.

The answer

Theorem (Bykovskii and Ustinov, 2015)

The general solution of recurrence relations

$$A_{n+2}B_{n-2} = \alpha A_{n+1}B_{n-1} + \beta A_n B_n,$$
$$A_{n-2}B_{n+2} = \gamma A_{n-1}B_{n+1} + \delta A_n B_n.$$

(subject to some natural restrictions on initial data) takes the form

$$A_n = C_1 D_1^n \frac{\sigma(z_1 + nz)}{\sigma(z)^{n^2}}, \qquad B_n = C_2 D_2^n \frac{\sigma(z_2 + nz)}{\sigma(z)^{n^2}},$$

where $z,\ z_1,\ z_2 \in \mathbb{C}^*$,

$$C_1 = \frac{A_0}{\sigma(z_1)}, \quad D_1 = \frac{\sigma(z)\sigma(z_1)A_1}{\sigma(z+z_1)A_0}, \quad C_2 = \frac{B_0}{\sigma(z_2)}, \quad D_2 = \dots$$

What are natural restrictions?

General Somos–4 (and Somos–5) sequence satisfy "magic determinant property" (Ma, 2010)

$$\begin{vmatrix} \tau_{m_1+n_1}\tau_{m_1-n_1} & \tau_{m_1+n_2}\tau_{m_1-n_2} & \tau_{m_1+n_3}\tau_{m_1-n_3} \\ \tau_{m_2+n_1}\tau_{m_2-n_1} & \tau_{m_2+n_2}\tau_{m_2-n_2} & \tau_{m_2+n_3}\tau_{m_2-n_3} \\ \tau_{m_3+n_1}\tau_{m_3-n_1} & \tau_{m_3+n_2}\tau_{m_3-n_2} & \tau_{m_3+n_3}\tau_{m_3-n_3} \end{vmatrix} = 0,$$

where m_i, n_i $(i = 1, 2, 3)$ are arbitrary integers or half-integers. Sequences $\{A_n\}$, $\{B_n\}$ from the Theorem satisfy similar conditions:

$$\begin{vmatrix} A_{m_1+n_1}B_{m_1-n_1} & A_{m_1+n_2}B_{m_1-n_2} & A_{m_1+n_3}B_{m_1-n_3} \\ A_{m_2+n_1}B_{m_2-n_1} & A_{m_2+n_2}B_{m_2-n_2} & A_{m_2+n_3}B_{m_2-n_3} \\ A_{m_3+n_1}B_{m_3-n_1} & A_{m_3+n_2}B_{m_3-n_2} & A_{m_3+n_3}B_{m_3-n_3} \end{vmatrix} = 0.$$

We'll use only 3 of them as assumptions.

The natural restrictions are:

For initial data

$$\begin{vmatrix} A_2B_0 & A_1B_1 & A_0B_2 \\ A_1B_{-1} & A_0B_0 & A_{-1}B_1 \\ A_0B_{-2} & A_{-1}B_{-1} & A_{-2}B_0 \end{vmatrix} = 0,$$

for A_3

$$\begin{vmatrix} A_3B_0 & A_2B_1 & A_1B_2 \\ A_2B_{-1} & A_1B_0 & A_0B_1 \\ A_1B_{-2} & A_0B_{-1} & A_{-1}B_0 \end{vmatrix} = 0,$$

and for B_3

$$\begin{vmatrix} A_2B_1 & A_1B_2 & A_0B_3 \\ A_1B_0 & A_0B_1 & A_{-1}B_2 \\ A_0B_{-1} & A_{-1}B_0 & A_{-2}B_1 \end{vmatrix} = 0.$$

If A_3 and B_3 are given then α, β are uniquely defined by equations

$$A_{n+2}B_{n-2} = \alpha A_{n+1}B_{n-1} + \beta A_n B_n \qquad (n = 0, 1)$$

and γ, δ are uniquely defined by equations

$$A_{n-2}B_{n+2} = \gamma A_{n-1}B_{n+1} + \delta A_n B_n \qquad (n = 0, 1).$$

The Laurent property for Double Somos–4

We have

$$\begin{vmatrix} A_{2n}B_0 & A_{n+1}B_{n-1} & A_nB_n \\ A_{1+n}B_{1-n} & A_2B_0 & A_1B_1 \\ A_nB_{-n} & A_1B_{-1} & A_0B_0 \end{vmatrix} = \begin{vmatrix} A_{2n-1}B_0 & A_nB_{n-1} & A_{n-1}B_n \\ A_nB_{1-n} & A_1B_0 & A_0B_1 \\ A_{n-1}B_{-n} & A_0B_{-1} & A_{-1}B_0 \end{vmatrix} = 0,$$

and the same for B_{2n}, B_{2n-1}.

Theorem (Bykovskii and Ustinov, 2015)

$$A_n, B_n \in \mathbb{Z}[A_0^{\pm 1}, B_0^{\pm 1}, A_{\pm 1}, A_{\pm 2}, B_{\pm 1}, B_{\pm 2}, \Delta_0^{\pm 1}, \Delta_1^{\pm 1}, \Delta_2^{\pm 1}],$$

where

$$\Delta_0 = \begin{vmatrix} A_1B_0 & A_0B_1 \\ A_0B_{-1} & A_{-1}B_0 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} A_2B_0 & A_1B_1 \\ A_1B_{-1} & A_0B_0 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} A_0B_2 & A_1B_1 \\ A_{-1}B_1 & A_0B_0 \end{vmatrix}.$$

Questions?