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A Short Proof of Euler's Identity for Continuants

A. V. Ustinov

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By *continuants* we mean the polynomials $K_n(x_1,\dots,x_n)$ defined by the initial conditions

$$K_0(\,)=1,\qquad K_1(x_1)=x_1$$

and the recurrence relation

$$K_n(x_1,\dots,x_n)=x_nK_n(x_1,\dots,x_{n-1})+K_n(x_1,\dots,x_{n-2}),\qquad n\ge 2$$

(it is also convenient to assume that $K_{-1}=0$). They arise in the solution of problems related to continued fractions and Euclid's algorithm. Continuants satisfy various relations; the most general among them is Euler's identity

$$\begin{aligned}K_{m+n}(x_1,\dots,x_{m+n})K_l(x_{m+1},\dots,x_{m+l})-K_{m+l}(x_1,\dots,x_{m+l})K_n(x_{m+1},\dots,x_{m+n})\\ =(-1)^nK_{m-1}(x_1,\dots,x_{m-1})K_{n-l-1}(x_{m+l+2},\dots,x_{m+n}),\qquad m\ge 1,\quad l\ge 0,\quad n\ge l+1.\end{aligned}\tag{1}$$

It is usually verified by induction on two out of the three parameters l,m,n. The proof given in the present paper is based on the fact that the determinant of a skew-symmetric 4×4 matrix (just as of any matrix of even order) is the square of some polynomial (the Pfaffian of the matrix):

$$\det\begin{pmatrix}0&a&b&c\\-a&0&d&e\\-b&-d&0&f\\-c&-e&-f&0\end{pmatrix}=(af+cd-be)^2.$$

The arguments are given for the polynomials $k_n(x_1,\dots,x_n)$ defined by the initial conditions

$$k_0(\,)=1,\quad k_1(x_1)=x_1$$

and the recurrence relation

$$k_n(x_1,\dots,x_n)=x_nk_n(x_1,\dots,x_{n-1})-k_n(x_1,\dots,x_{n-2}),\qquad n\ge 2.\tag{2}$$

These polynomials occur as the numerators and denominators of a continued fraction of the form

$$x_1-\cfrac{1}{x_2-\ddots-\cfrac{1}{x_n}}$$

and are related to the polynomials $K_n(x_1,\dots,x_n)$ by

$$k_n(x_1,\dots,x_n)=i^{-n}K_n(ix_1,\dots,ix_n),\qquad\text{where}\quad i=\sqrt{-1}.\tag{3}$$

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Theorem. *Suppose that* $m \geq 1$, $l \geq 0$, *and* $n \geq l+1$. *Then*

$$\begin{aligned} &k_{m+n}(x_1, \dots, x_{m+n})k_l(x_{m+1}, \dots, x_{m+l}) - k_{m+l}(x_1, \dots, x_{m+l})k_n(x_{m+1}, \dots, x_{m+n}) \\ &\quad + k_{m-1}(x_1, \dots, x_{m-1})k_{n-l-1}(x_{m+l+2}, \dots, x_{m+n}) = 0. \end{aligned} \tag{4}$$

Proof. Consider the skew-symmetric matrix $A = (a_{u,v})_{u,v=1}^{m+n+2}$ whose elements, for $1 \leq u < v \leq m+n+2$, are the quantities

$$a_{u,v} = k_{v-u-1}(x_u, \dots, x_{v-2}).$$

For example, for $m = n = 1$, we have

$$A = \begin{pmatrix} 0 & 1 & x_1 & x_1x_2 - 1 \\ -1 & 0 & 1 & x_2 \\ -x_1 & -1 & 0 & 1 \\ 1 - x_1x_2 & -x_2 & -1 & 0 \end{pmatrix}.$$

It follows from the recurrence relations (2) that each column of the matrix A is the linear combination of the two previous ones (if such exist). Hence any minor of order 3 of the matrix A is either greater than zero or equal to zero. Considering one of the principal minors, we obtain

$$\begin{aligned} &\det \begin{pmatrix} 0 & a_{1,m+1} & a_{1,m+l+2} & a_{1,m+n+2} \\ -a_{1,m+1} & 0 & a_{m+1,m+l+2} & a_{m+1,m+n+2} \\ -a_{1,m+l+2} & -a_{m+1,m+l+2} & 0 & a_{m+l+2,m+n+2} \\ -a_{1,m+n+2} & -a_{m+1,m+n+2} & -a_{m+l+2,m+n+2} & 0 \end{pmatrix} \\ &\quad = (a_{1,m+1}a_{m+l+2,m+n+2} + a_{1,m+n+2}a_{m+1,m+l+2} - a_{1,m+l+2}a_{m+1,m+n+2})^2 = 0, \end{aligned}$$

which is equivalent to relation (4). The theorem is now proved. □

Euler's identity (1) can be obtained from (4) by applying relation (3).

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M. V. Lomonosov Moscow State University
E-mail: ustinov@mech.math.msu.su