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Calculation of the variance in a problem in the theory of continued fractions

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Abstract. We study the random variable $N(\alpha,R)=\#\{j\geqslant 1: Q_j(\alpha)\leqslant R\}$, where $\alpha\in[0;1)$ and $P_j(\alpha)/Q_j(\alpha)$ is the jth convergent of the continued fraction expansion of the number $\alpha=[0;t_1,t_2,\dots]$. For the mean value

$$N(R)=\int_0^1 N(\alpha,R)\,d\alpha$$

and variance

$$D(R)=\int_0^1 \bigl(N(\alpha,R)-N(R)\bigr)^2\,d\alpha$$

of the random variable $N(\alpha,R)$, we prove the asymptotic formulae with two significant terms

$$N(R)=N_1\log R+N_0+O(R^{-1+\varepsilon}),\quad D(R)=D_1\log R+D_0+O(R^{-1/3+\varepsilon}).$$

Bibliography: 13 titles.

§ 1. Notation

1. We write $[x_0;x_1,\dots,x_s]$ to denote the continued fraction

$$x_0+\cfrac{1}{x_1+\ddots+\cfrac{1}{x_s}}$$

of length s with formal variables $x_0,x_1,\dots,x_s$.

2. For a rational number r, the representation $r=[t_0;t_1,\dots,t_s]$ is the canonical (unless additional stipulations are made) expansion of r into a continued fraction, where $t_0=[r]$ (the integer part of r), $t_1,\dots,t_s$ are positive integers, and $t_s\geqslant 2$ for $s\geqslant 1$. In certain cases the same number r is written in the form $r=[t_0;t_1,\dots,t_s-1,1]$.

3. The notation $K_n(x_1,\dots,x_n)$ (see [1]) is used for the continuants, which are defined by the initial conditions

$$K_0(\,)=1,\qquad K_1(x_1)=x_1$$

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and the recurrence relation

$$K_n(x_1,\dots,x_n)=x_nK_{n-1}(x_1,\dots,x_{n-1})+K_{n-2}(x_1,\dots,x_{n-2}),\qquad n\geqslant 2.$$

Here we always have the equality

$$[x_0;x_1,\dots,x_s]=\frac{K_{s+1}(x_0,x_1,\dots,x_s)}{K_s(x_1,\dots,x_s)}\,.$$

The lower index, which is equal to the number of arguments of a continuant, will be omitted in what follows.

4. The sign "$*$" in double sums of the form

$$\sum_n\sum_m{}^{*}\dots$$

means that the variables over which the summation is carried out are connected by the additional condition $(m,n)=1$.

5. If A is some assertion, then $[A]$ means 1 if A is true, and 0 otherwise.

6. For a positive integer q we denote by $\delta_q(a)$ the characteristic function of divisibility by q:

$$\delta_q(a)=[a\equiv 0\,(\operatorname{mod} p)]=\begin{cases}1 & \text{if } a\equiv 0\,(\operatorname{mod} q),\\ 0 & \text{if } a\not\equiv 0\,(\operatorname{mod} q).\end{cases}$$

7. The dash in sums of the form

$$\sum_{b,\,m=1}^{n}{}' \delta_n(bm\pm 1)\cdot\ldots$$

means that for $n=1$ 'minus' is chosen of the two signs in the symbol $\pm$, and for $n>1$ both signs are taken independently.

8. Finite differences of functions of one and two variables are denoted as follows:

$$\Delta a(u)=a(u+1)-a(u),$$
$$\Delta_{1,0}a(u,v)=a(u+1,v)-a(u,v),\qquad \Delta_{0,1}a(u,v)=a(u,v+1)-a(u,v),$$
$$\Delta_{1,1}a(u,v)=\Delta_{0,1}\bigl(\Delta_{1,0}a(u,v)\bigr)=\Delta_{1,0}\bigl(\Delta_{0,1}a(u,v)\bigr).$$

9. The sum of powers of divisors is denoted as

$$\sigma_\alpha(q)=\sum_{d\mid q}d^\alpha.$$

§ 2. Introduction

We denote by $s(a/b)$ the length of the continued fraction for a rational number $a/b=[t_0;t_1,\dots,t_s]$.

In 1968 Heilbronn [2] proved the asymptotic formula for the mean value of the quantity $s(a/b)$

$$\frac{1}{\varphi(b)}\sum_{\substack{1\leqslant a\leqslant b\\(a,b)=1}} s\Big(\frac{a}{b}\Big)=\frac{2\log 2}{\zeta(2)}\,\log b+O(\log^4\log b).$$

Later Porter (see [3]) obtained for the same sum the asymptotic formula with two significant terms

$$\frac{1}{\varphi(b)}\sum_{\substack{1\leqslant a\leqslant b\\(a,b)=1}} s\Big(\frac{a}{b}\Big)=\frac{2\log 2}{\zeta(2)}\,\log b+C_P-1+O(b^{-1/6+\varepsilon}),$$

where

$$C_P=\frac{\log 2}{\zeta(2)}\Big(3\log 2+4\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2\Big)-\frac{1}{2}$$

is a constant, which became known as Porter's constant (the final form of it was found by Wrench; see [4]).

For the variance of the quantity $s(a/b)$ (for a fixed value of b) only the following estimate is known, which is correct in order of magnitude and is due to Bykovskiĭ [5]:

$$\frac{1}{b}\sum_{a=1}^{b}\bigg(s\Big(\frac{a}{b}\Big)-\frac{2\log 2}{\zeta(2)}\,\log b\bigg)^2\ll\log b.$$

More exact results are obtained for averaging over both parameters a and b. For example, for the mean value of the quantity $s(a/b)$ the methods in [6], [7] yield the asymptotic formula

$$\frac{2}{R^2}\sum_{b\leqslant R}\sum_{a\leqslant b} s\Big(\frac{a}{b}\Big)=\frac{2\log 2}{\zeta(2)}\,\log b+B+O(b^{-1/2+\varepsilon}),$$

where

$$B=\frac{2\log 2}{\zeta(2)}\Big(-\frac{1}{2}+\frac{\zeta'(2)}{\zeta(2)}\Big)+C_P-\frac{3}{2}\,.$$

An asymptotic formula with two significant terms is also known for the variance (see [8]):

$$\frac{2}{R^2}\sum_{b\leqslant R}\sum_{a\leqslant b}\bigg(s\Big(\frac{a}{b}\Big)-\frac{2\log 2}{\zeta(2)}\,\log b-B\bigg)^2=\delta_1\log R+\delta_0+O(R^{-\gamma}),\qquad(1)$$

where δ_1, δ_0, and $\gamma>0$ are absolute constants.

In the case of an irrational number α, as an analogue of the quantity $s(\alpha)$ one can consider

$$N(\alpha,R)=\#\{j\geqslant 1: Q_j(\alpha)\leqslant R\},$$

where $Q_j(\alpha)$ is the denominator of the jth convergent of the continued fraction expansion of α. In the present paper we verify an asymptotic formula with two significant terms for the mean value of $N(\alpha,R)$

$$N(R)=\int_0^1 N(\alpha,R)\,d\alpha.$$

For the variance

$$D(R)=\int_0^1 \bigl(N(\alpha,R)-N(R)\bigr)^2\,d\alpha=\int_0^1 N^2(\alpha,R)\,d\alpha-N^2(R),$$

we prove the asymptotic formula

$$D(R)=D_1\log R+D_0+O(R^{-1/3}\log^5 R)$$

with absolute constants D_1, D_0.

The methods of the present paper also enable us to prove formula (1) with any $\gamma>-1/4$. The author plans to expound this result in a forthcoming paper.

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§ 3. On continued fractions

The following assertion is a modification of a well-known theorem (see [9], § 50, Theorem 1). This assertion is a basis for all the subsequent arguments.

Lemma 1. *Suppose that P is a non-negative integer, P', Q, Q' are positive integers, and $Q\leqslant Q'$. Suppose also that α is a real number in the interval $(0,1)$. Then the following two conditions are equivalent*:

(I) *P/Q and P'/Q' are consecutive convergents of the continued fraction expansion of α that are distinct from α, and the convergent P/Q precedes P'/Q'*;

(II) $PQ'-P'Q=\pm1$ *and* $0<\dfrac{Q'\alpha-P'}{-Q\alpha+P}<1.$

See the proof of Lemma 1 in [6].

Following [5] we denote by $\mathscr{M}$ the set of all integer-valued matrices

$$S=\begin{pmatrix}P & P'\\ Q & Q'\end{pmatrix}=\begin{pmatrix}P(S) & P'(S)\\ Q(S) & Q'(S)\end{pmatrix}$$

with determinant $\det S=\pm1$ such that

$$1\leqslant Q\leqslant Q',\qquad 0\leqslant P\leqslant Q,\qquad 1\leqslant P'\leqslant Q'.$$

For real $R>0$ we denote by $\mathscr{M}(R)$ the finite subset of $\mathscr{M}$ consisting of all the matrices S with the additional condition $Q'\leqslant R$.

As noted in [5], Lemma 1 implies the following properties of the set $\mathscr{M}$.

1°. The correspondence

$$(q_1,\dots,q_l)\to S=S(q_1,\dots,q_l)=\begin{pmatrix}P & P'\\ Q & Q'\end{pmatrix}, \tag{2}$$

where

$$\frac{P}{Q} = [0; q_1, \dots, q_{l-1}], \qquad \frac{P'}{Q'} = [0; q_1, \dots, q_l],$$

defines a bijection of the set of all finite tuples of positive integers onto the set $\mathscr{M}$. In particular, it follows that the set $\mathscr{M}$ is a semigroup with respect to multiplication.

2°. For real $\alpha \in (0,1)$ the inequality

$$0 < \frac{Q'\alpha - P'}{-Q\alpha + P} = S^{-1}(\alpha) < 1, \qquad S \in \mathscr{M},$$

holds if and only if for some $j \geqslant 1$

$$S = \begin{pmatrix} P_j(\alpha) & P_{j+1}(\alpha) \\ Q_j(\alpha) & Q_{j+1}(\alpha) \end{pmatrix}$$

and $j \leqslant s(r) - 2$ for rational $\alpha = r$.

3°. For every matrix $S \in \mathscr{M}$ the inequality $0 < S^{-1}(\alpha) < 1$ defines the interval

$$I(S) = \begin{cases} \left(\dfrac{P'}{Q'}, \dfrac{P+P'}{Q+Q'}\right) & \text{if } \det S = 1, \\ \left(\dfrac{P+P'}{Q+Q'}, \dfrac{P'}{Q'}\right) & \text{if } \det S = -1, \end{cases}$$

of length

$$|I(S)| = \frac{1}{Q'(Q+Q')}.$$

4°. Let $q_1, \dots, q_l$ be positive integers and let $S = S(q_1, \dots, q_l)$ in accordance with (2). Then a number α belongs to the interval $I(S)$ if and only if $s(\alpha) > l$ and in the canonical expansion $\alpha = [t_0; t_1, \dots, t_l, \dots]$

$$t_0 = 0, \qquad t_1 = q_1, \dots, t_l = q_l.$$

5°. The intersection $I(S) \cap I(S')$ is non-empty if and only if one of the intervals is contained in the other. Here, if $I(S) \subsetneq I(S')$ and $S' = S'(q_1, \dots, q_{l'})$, then for some $l > l'$ and positive integers $q_{l'+1}, \dots, q_l$ we have the equality

$$S = S'S'',$$

where $S'' = S''(q_{l'+1}, \dots, q_l)$ and $S = S(q_1, \dots, q_l)$.

6°. If $Q' \geqslant 2$, $1 \leqslant Q \leqslant Q'$, and $(Q, Q') = 1$, then there are exactly two pairs

$$(P, P') \quad \text{and} \quad (Q - P, Q' - P')$$

that can be the first row complementing the second row (Q, Q') with respect to a matrix in $\mathscr{M}$. In addition, if

$$\frac{Q}{Q'} = [0; q_s, \dots, q_1] = [0; q_s, \dots, q_1 - 1, 1], \qquad q_1 \geqslant 2,$$

then the corresponding matrices have the form

$$\begin{pmatrix}0&1\\1&q_1\end{pmatrix}\cdots\begin{pmatrix}0&1\\1&q_s\end{pmatrix}=\begin{pmatrix}K(q_2,\dots,q_{s-1})&K(q_2,\dots,q_s)\\K(q_1,\dots,q_{s-1})&K(q_1,\dots,q_s)\end{pmatrix}=\begin{pmatrix}P&P'\\Q&Q'\end{pmatrix},$$

$$\begin{aligned}&\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}0&1\\1&q_1-1\end{pmatrix}\begin{pmatrix}0&1\\1&q_2\end{pmatrix}\cdots\begin{pmatrix}0&1\\1&q_s\end{pmatrix}\\&\qquad=\begin{pmatrix}K(q_1-1,q_2,\dots,q_{s-1})&K(q_1-1,q_2,\dots,q_s)\\K(1,q_1-1,q_2,\dots,q_{s-1})&K(1,q_1-1,q_2,\dots,q_s)\end{pmatrix}\\&\qquad=\begin{pmatrix}Q-P&Q'-P'\\Q&Q'\end{pmatrix}.\end{aligned}$$

For $Q=Q'=1$ there exists only one matrix $S=\begin{pmatrix}0&1\\1&1\end{pmatrix}$ that belongs to the set $\mathscr{M}$.

§ 4. Auxiliary assertions

Lemma 2. *Let $R\geqslant 2$. Then*

$$\sum_{n\leqslant R}\frac{\varphi(n)}{n^2}=\frac{1}{\zeta(2)}\left(\log R+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+O\biggl(\frac{\log R}{R}\biggr),\tag{3}$$

$$\sum_{n\leqslant R}\frac{\varphi(n)}{n^2}\log n=\frac{1}{2\zeta(2)}\log^2 R+C_0+O\biggl(\frac{\log^2 R}{R}\biggr),$$

where

$$C_0=\gamma\,\frac{\zeta'(2)}{\zeta^2(2)}+\gamma_1\,\frac{1}{\zeta(2)}-\frac{2(\zeta'(2))^2-\zeta''(2)\zeta(2)}{2\zeta^3(2)}$$

and γ_1 is the Stieltjes constant (*see* [10], *part* 2.21), *which is defined by the equality*

$$\sum_{n\leqslant T}\frac{\log n}{n}=\frac{\log^2 T}{2}+\gamma_1+O\biggl(\frac{\log T}{T}\biggr),\qquad T\geqslant 2.\tag{4}$$

Proof. To prove equality (3) we express $\varphi(q)$ using the Möbius function:

$$\begin{aligned}\sum_{n\leqslant R}\frac{\varphi(n)}{n^2}&=\sum_{n\leqslant R}\frac{1}{n}\sum_{d\mid n}\frac{\mu(d)}{d}=\sum_{d\leqslant R}\frac{\mu(d)}{d^2}\sum_{n\leqslant R/d}\frac{1}{n}\\&=\sum_{d\leqslant R}\frac{\mu(d)}{d^2}\left(\log R-\log d+\gamma+O\biggl(\frac{d}{R}\biggr)\right).\end{aligned}$$

Since

$$\sum_{d\leqslant R}\frac{\mu(d)}{d^2}=\frac{1}{\zeta(2)}+O\biggl(\frac{1}{R}\biggr),\qquad\sum_{d\leqslant R}\frac{\mu(d)}{d^2}\log d=\frac{\zeta'(2)}{\zeta^2(2)}+O\biggl(\frac{\log R}{R}\biggr),\tag{5}$$

we have

$$\sum_{n\leqslant R}\frac{\varphi(n)}{n^2}=\frac{1}{\zeta(2)}\left(\log R+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+O\biggl(\frac{\log R}{R}\biggr).$$

We transform the second sum by the same method:

$$\sum_{n\leqslant R}\frac{\varphi(n)}{n^2}\log n=\sum_{n\leqslant R}\frac{\log n}{n}\sum_{d|n}\frac{\mu(d)}{d}=\sum_{d\leqslant R}\frac{\mu(d)}{d^2}\sum_{n\leqslant R/d}\frac{1}{n}(\log n+\log d).$$

Using equality (4) we find

$$\sum_{n\leqslant R}\frac{\varphi(n)}{n^2}\log n=\sum_{d\leqslant R}\frac{\mu(d)}{d^2}\left(\frac{\log^2 R}{2}-\frac{\log^2 d}{2}+\gamma_1+\gamma\log d\right)+O\left(\frac{\log^2 R}{R}\right).$$

The second formula of the lemma now follows from (5) and the equality

$$\sum_{d\leqslant R}\frac{\mu(d)}{d^2}\log^2 d=\frac{2(\zeta'(2))^2-\zeta''(2)\zeta(2)}{\zeta^3(2)}+O\left(\frac{\log^2 R}{R}\right).$$

Lemma 3. *For $R\geqslant 2$ the sum*

$$\Phi^*(R)=\sum_{Q'\leqslant R}\sum_{Q\leqslant Q'}^{*}\frac{1}{Q'(Q+Q')}\tag{6}$$

satisfies the asymptotic formula

$$\Phi^*(R)=\frac{\log 2}{\zeta(2)}\left(\log R+\log 2+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)-\frac{1}{2}+O\left(\frac{\log R}{R}\right).$$

Proof. First we find an asymptotic formula for the sum

$$\Phi(R)=\sum_{Q'\leqslant R}\sum_{Q\leqslant Q'}\frac{1}{Q'(Q+Q')}\,,$$

in which the summation variables Q and Q' are not connected by the coprimeness condition. We express $\Phi(R)$ in the form

$$\Phi(R)=\log 2\sum_{Q'\leqslant R}\frac{1}{Q'}+\sigma_0+O\left(\frac{1}{R}\right),$$

where

$$\sigma_0=\sum_{Q'=1}^{\infty}\frac{1}{Q'}\left(\sum_{Q=1}^{Q'}\frac{1}{Q+Q'}-\log 2\right).\tag{7}$$

The sum σ_0 is known (see [4]) to have the exact value

$$\sigma_0=\log^2 2-\frac{\zeta(2)}{2}\,;\tag{8}$$

therefore,

$$\Phi(R)=\log 2\,(\log R+\log 2+\gamma)-\frac{\zeta(2)}{2}+O\left(\frac{1}{R}\right).$$

Next, applying the formulae

$$\Phi^*(R) = \sum_{\delta\leqslant R} \frac{\mu(\delta)}{\delta^2}\,\Phi\biggl(\frac{R}{\delta}\biggr), \qquad \sum_{\delta=1}^{\infty} \frac{\mu(\delta)}{\delta^2}\log\delta = \frac{\zeta'(2)}{\zeta^2(2)}$$

we arrive at the assertion of the lemma.

Lemma 4. *Let q be a positive integer, and $a(n)$ a function defined for integer n satisfying $1\leqslant n\leqslant q$. Suppose also that this function satisfies the inequalities*

$$a(n)\geqslant 0, \quad 1\leqslant n\leqslant q, \qquad \Delta a(n)\leqslant 0, \quad 1\leqslant n\leqslant q-1.$$

Then

$$\sum_{\substack{n=1\\(n,q)=1}}^{q} a(n) = \frac{\varphi(q)}{q}\sum_{n=1}^{q} a(n) + O(A\sigma_0(q)),$$

where $A=a(1)$ is the greatest value of the function $a(n)$.

Proof. We apply the Abel transformation to this sum:

$$\begin{aligned}\sum_{\substack{n=1\\(n,q)=1}}^{q} a(n) &= \sum_{n=1}^{q} a(n)[(n,q)=1]\\ &= \varphi(q)a(q) - \sum_{k=1}^{q-1}\bigl(a(k+1)-a(k)\bigr)\sum_{n=1}^{k}[(n,q)=1].\end{aligned}$$

Next, using the equality

$$\sum_{n=1}^{k}[(n,q)=1] = \frac{\varphi(q)}{q}k + O(\sigma_0(q))$$

(see [11], Ch. II, Problem 19) we find

$$\begin{aligned}\sum_{\substack{n=1\\(n,q)=1}}^{q} a(n) &= \varphi(q)a(q) - \frac{\varphi(q)}{q}\sum_{k=1}^{q-1}\bigl(a(k+1)-a(k)\bigr)k + O(A\sigma_0(q))\\ &= \frac{\varphi(q)}{q}\sum_{n=1}^{q} a(n) + O(A\sigma_0(q)).\end{aligned}$$

The following assertion, which was proved in special cases in [12], is based on the estimates of Kloosterman's sums that belong to Estermann [13].

Lemma 5. *Let $q\geqslant 1$ be a positive integer, and $a(u,v)$ a function defined at integer points (u,v), where $1\leqslant u,v\leqslant q$. Suppose also that this function satisfies the inequalities*

$$a(u,v)\geqslant 0, \qquad \Delta_{1,0}a(u,v)\leqslant 0, \qquad \Delta_{0,1}a(u,v)\leqslant 0, \qquad \Delta_{1,1}a(u,v)\geqslant 0$$

at all the points, where these conditions are defined. Then the sum

$$W = \sum_{u,v=1}^{q} \delta_q(uv\pm 1)a(u,v)$$

(for any choice of sign in the symbol $\pm$*) satisfies the asymptotic formula*

$$W=\frac{\varphi(q)}{q^2}\sum_{u,v=1}^{q}a(u,v)+O\bigl(A\psi(q)\sqrt{q}\bigr),$$

where $\psi(q)=\sigma_0(q)\sigma_{-1/2}(q)\log^2(q+1)$ *and* $A=a(1,1)$ *is the greatest value of the function* $a(u,v)$*.*

See the proof of Lemma 5 in [6].

§ 5. On the quantities $N(R)$ and $D(R)$

Lemma 6. *For* $R\geqslant 1$ *the quantity*

$$N(R)=\int_0^1 N(\alpha,R)\,d\alpha$$

can be represented in the form

$$N(R)=2\Phi^*(R)-\frac{1}{2}\,, \tag{9}$$

where the function $\Phi^*(R)$ *is defined by the series* (6)*.*

In addition, $N(R)$ *satisfies the asymptotic formula*

$$N(R)=\frac{2\log 2}{\zeta(2)}\log R+\frac{2\log 2}{\zeta(2)}\biggl(\log 2+\gamma-\frac{\zeta'(2)}{\zeta(2)}\biggr)-\frac{3}{2}+O\biggl(\frac{\log(R+1)}{R}\biggr). \tag{10}$$

Proof. By Lemma 1, for irrational $\alpha\in(0,1)$ the quantity $N(\alpha,R)$ coincides with the number of solutions of the system

$$\begin{cases} PQ'-P'Q=\pm1,\\ 0<S^{-1}(\alpha)<1\end{cases}$$

with respect to the unknowns P, P', Q, and Q' that are connected by the inequalities

$$1\leqslant Q\leqslant Q'\leqslant R,\qquad 0\leqslant P\leqslant Q,\qquad 1\leqslant P'\leqslant Q'.$$

Hence,

$$N(\alpha,R)=\sum_{S\in\mathscr{M}(R)}[0<S^{-1}(\alpha)<1]=\sum_{S\in\mathscr{M}(R)}\chi_{I(S)}(\alpha), \tag{11}$$

$$N(R)=\sum_{S\in\mathscr{M}(R)}\int_0^1\chi_{I(S)}(\alpha)\,d\alpha=\sum_{S\in\mathscr{M}(R)}\frac{1}{Q'(Q+Q')}\,, \tag{12}$$

where $\chi_{I(S)}(\alpha)$ is the characteristic function of the interval $I(S)$.

Suppose that $Q'\geqslant 2$, $1\leqslant Q<Q'$, and $(Q,Q')=1$. Then by property 6° of the set $\mathscr{M}$ the fraction $1/\bigl(Q'(Q+Q')\bigr)$ appears in the sum (12) exactly two times. For the pair $(Q',Q)=(1,1)$ the corresponding fraction appears once. Consequently, equality (9) holds. Applying Lemma 3 to (9) we arrive at the asymptotic formula for $N(R)$.

Lemma 7. *For $R \geqslant 1$ the quantity*

$$D(R) = \int_0^1 \big(N(\alpha, R) - N(R)\big)^2 \, d\alpha = \int_0^1 N^2(\alpha, R) \, d\alpha - N^2(R)$$

satisfies the representation

$$D(R) = 4\sigma(R) - N^2(R) + \frac{1}{2}, \tag{13}$$

where

$$\sigma(R) = \sum_{Q' \leqslant R} \sum_{Q \leqslant Q'}^{*} \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}} \frac{[mQ + nQ' \leqslant R]}{(mQ + nQ')\big((a+m)Q + (b+n)Q'\big)}.$$

Proof. By formulae (11) and (12) we have

$$\int_0^1 N^2(\alpha, R) \, d\alpha = \int_0^1 \Big(\sum_{S \in \mathscr{M}(R)} \chi_{I(S)}(\alpha) \Big)^2 d\alpha$$

$$= \sum_{S \in \mathscr{M}(R)} |I(S)| + 2 \sum_{\substack{S, S' \in \mathscr{M}(R) \\ I(S) \subsetneq I(S')}} |I(S)| = N(R) + 2 \sum_{\substack{S, S' \in \mathscr{M}(R) \\ I(S) \subsetneq I(S')}} |I(S)|.$$

Using property 5° we express the matrices S and S' in the form

$$S' = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix}, \qquad S = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \begin{pmatrix} a & m \\ b & n \end{pmatrix},$$

where the matrix $\begin{pmatrix} a & m \\ b & n \end{pmatrix}$ also belongs to the set $\mathscr{M}$. Therefore,

$$\int_0^1 N^2(\alpha, R) \, d\alpha = N(R)$$

$$+ 2 \sum_{\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathscr{M}} \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}} \frac{[mQ + nQ' \leqslant R]}{(mQ + nQ')\big((a+m)Q + (b+n)Q'\big)}.$$

Considering separately the case $Q = Q' = 1$ and using property 6° we find

$$\int_0^1 N^2(\alpha, R) \, d\alpha = N(R) + 4\sigma(R) - 2 \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}} \frac{[m + n \leqslant R]}{(m+n)(a+b+m+n)}.$$

The equality

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & m \\ b & n \end{pmatrix} = \begin{pmatrix} b & n \\ a+b & m+n \end{pmatrix}$$

and property 6° imply that each pair of numbers (q, q') such that $1 \leqslant q < q'$ and $(q, q') = 1$ is the second row of the matrix $\begin{pmatrix} b & n \\ a+b & m+n \end{pmatrix}$ for exactly one matrix $\begin{pmatrix} a & m \\ b & n \end{pmatrix} \in \mathscr{M}$.

Therefore, in view of equality (9), we have

$$2\sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathscr{M}} \frac{[m+n\leqslant R]}{(m+n)(a+b+m+n)} = 2\left(\Phi^*(R)-\frac{1}{2}\right) = N(R)-\frac{1}{2},$$

$$\int_0^1 N^2(\alpha,R)\,d\alpha = 4\sigma(R)+\frac{1}{2}, \tag{14}$$

which proves Lemma 7.

Remark 1. The sum $\sigma(R)$ has another representation:

$$\sigma(R) = \sum_{2\leqslant Q'\leqslant R}\ \sideset{}{^*}\sum_{Q\leqslant Q'} \frac{s(Q/Q')}{Q'(Q+Q')}. \tag{15}$$

Indeed, if $S=S(q_1,\dots,q_n)$, then a matrix $S'\in\mathscr{M}$ such that $I(S)\subsetneq I(S')$ can be chosen in $n-1$ ways. Therefore,

$$\int_0^1 N^2(\alpha,R)\,d\alpha = N(R)+2\sum_{\substack{S\in\mathscr{M}\\ Q'\geqslant 2}} \frac{n-1}{Q'(Q+Q')}.$$

By property 5° of the set $\mathscr{M}$, for fixed Q and Q' such that $1\leqslant Q<Q'$ and $(Q,Q')=1$, the parameter n can take two values: $s(Q/Q')$ and $s(Q/Q')+1$. Thus,

$$\begin{aligned}\int_0^1 N^2(\alpha,R)\,d\alpha &= N(R)+2\sum_{2\leqslant Q'\leqslant R}\ \sideset{}{^*}\sum_{Q\leqslant Q'} \frac{2s(Q/Q')-1}{Q'(Q+Q')}\\ &= 4\sum_{2\leqslant Q'\leqslant R}\ \sideset{}{^*}\sum_{Q\leqslant Q'} \frac{s(Q/Q')}{Q'(Q+Q')}+\frac{1}{2},\end{aligned}$$

which, in view of equality (14), proves formula (15).

To find $D(R)$ we introduce a parameter U satisfying $2\leqslant U\leqslant R$. We represent the sum $\sigma(R)$ in the form

$$\sigma(R) = \sigma_1-\sigma_2+\sigma_3,$$

where

$$\sigma_1 = \sum_{Q'\leqslant R}\ \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathscr{M}(U)}\ \sideset{}{^*}\sum_{Q\leqslant Q'} \frac{1}{(mQ+nQ')\big((a+m)Q+(b+n)Q'\big)}, \tag{16}$$

$$\sigma_2 = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathscr{M}(U)}\ \sum_{Q'\leqslant R}\ \sideset{}{^*}\sum_{Q\leqslant Q'} \frac{[mQ+nQ'>R]}{(mQ+nQ')\big((a+m)Q+(b+n)Q'\big)}, \tag{17}$$

$$\sigma_3 = \sum_{Q'\leqslant R}\ \sideset{}{^*}\sum_{Q\leqslant Q'}\ \sum_{\substack{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathscr{M},\\ n>U}} \frac{[mQ+nQ'\leqslant R]}{(mQ+nQ')\big((a+m)Q+(b+n)Q'\big)}. \tag{18}$$

We analyse separately each of the quantities σ_1, σ_2, and σ_3.

§ 6. Calculating the sum σ_1

For a matrix $S=\begin{pmatrix} a & m \\ b & n \end{pmatrix} \in \mathscr{M}$, we denote by $f_S(\xi)$ the function

$$f_S(\xi)=\frac{1}{(m\xi+n)\bigl((a+m)\xi+(b+n)\bigr)},$$

and by $J_1(a,b,m,n)$ the integral

$$J_1(a,b,m,n)=\int_0^1 f_S(\xi)\,d\xi.$$

Lemma 8. *Let n be a positive integer. Then the sum*

$$w_1(n)=\sum_{b,m=1}^{n}{}' \delta_n(bm\pm 1)J_1(a,b,m,n)$$

(*henceforth,* $a=(bm\pm 1)/n$) *satisfies the asymptotic formula*

$$w_1(n)=2\log^2 2\,\frac{\varphi(n)}{n^2}+O\biggl(\frac{\psi(n)}{n^{3/2}}\biggr),$$

where $\psi(n)$ is the function defined in the hypothesis of Lemma 5.

Proof. The assertion of the lemma is obvious for $n=1$. Therefore we assume that $n\geqslant 2$. Since

$$\frac{1}{\xi\bigl((bm\pm 1)/n+m\bigr)+(b+n)}-\frac{1}{\xi(bm/n+m)+(b+n)}=O\biggl(\frac{1}{n^3}\biggr), \tag{19}$$

the sum $w_1(n)$ has a simpler representation:

$$w_1(n)=\sum_{b,m=1}^{n}\delta_n(bm\pm 1)\int_0^1\frac{d\xi}{\bigl(b/n+1\bigr)(m\xi+n)^2}+O\biggl(\frac{1}{n^3}\biggr).$$

By Lemma 5,

$$w_1(n)=2\frac{\varphi(n)}{n^2}\sum_{b,m=1}^{n}\frac{1}{b+n}\int_0^1\frac{n\,d\xi}{(m\xi+n)^2}+O\biggl(\frac{\psi(n)}{n^{3/2}}\biggr).$$

Substituting into the last equality the asymptotic formulae

$$\sum_{b=1}^{n}\frac{1}{b+n}=\log 2+O\biggl(\frac{1}{n}\biggr), \tag{20}$$

$$\sum_{m=1}^{n}\int_0^1\frac{n\,d\xi}{(m\xi+n)^2}=\log 2+O\biggl(\frac{1}{n}\biggr)$$

we arrive at the assertion of the lemma.

Corollary 1. *For any real $U \geqslant 2$ the sum*

$$W_1(U) = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}(U)} J_1(a,b,m,n) \tag{21}$$

satisfies the asymptotic formula

$$W_1(U) = \frac{2\log^2 2}{\zeta(2)} \left(\log U + \gamma - \frac{\zeta'(2)}{\zeta(2)}\right) + C_1 + O\left(\frac{\log^5 U}{U^{1/2}}\right), \tag{22}$$

where

$$C_1 = \sum_{n=1}^{\infty} \left(\sideset{}{'}\sum_{b,m=1}^{n} \delta_n(bm \pm 1) J_1(a,b,m,n) - 2\log^2 2\, \frac{\varphi(n)}{n^2} \right). \tag{23}$$

Proof. We express the sum $W_1(U)$ in the form

$$W_1(U) = \sum_{n \leqslant U} \sideset{}{'}\sum_{b,m=1}^{n} \delta_n(bm \pm 1) J_1(a,b,m,n).$$

By Lemma 8,

$$\begin{aligned} W_1(U) &= \sum_{n \leqslant U} \left(\sideset{}{'}\sum_{b,m=1}^{n} \delta_n(bm \pm 1) J_1(a,b,m,n) - 2\log^2 2\, \frac{\varphi(n)}{n^2} \right) + 2\log^2 2 \sum_{n \leqslant U} \frac{\varphi(n)}{n^2} \\ &= 2\log^2 2 \sum_{n \leqslant U} \frac{\varphi(n)}{n^2} + C_1 + O\left(\frac{\log^5 U}{U^{1/2}}\right). \end{aligned}$$

Substituting formula (3) into the last equality we arrive at the assertion of the corollary.

Remark 2. One can verify in similar fashion the equalities

$$\sideset{}{'}\sum_{b,m=1}^{n} \delta_n(bm \pm 1)\, \frac{1}{(m+n)(a+b+m+n)} = \log 2\, \frac{\varphi(n)}{n^2} + O\left(\frac{\psi(n)}{n^{3/2}}\right),$$

$$\sideset{}{'}\sum_{b,m=1}^{n} \delta_n(bm \pm 1) f_S'(\xi) = -\frac{2\log 2}{(\xi+1)^2}\, \frac{\varphi(n)}{n^2} + O\left(\frac{\psi(n)}{n^{3/2}}\right).$$

For the sums

$$A(U) = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}(U)} \frac{1}{(m+n)(a+b+m+n)}, \qquad B(U,\xi) = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}(U)} f_S'(\xi),$$

this yields the asymptotic formulae

$$A(U) = \frac{\log 2}{\zeta(2)} \left(\log U + \gamma - \frac{\zeta'(2)}{\zeta(2)}\right) + C_2 + O\left(\frac{\log^5 R}{U^{1/2}}\right), \tag{24}$$

$$B(U,\xi) = -\frac{2\log 2}{\zeta(2)(\xi+1)^2} \left(\log U + \gamma - \frac{\zeta'(2)}{\zeta(2)}\right) + C_3(\xi) + O\left(\frac{\log^5 R}{U^{1/2}}\right), \tag{25}$$

where

$$C_2 = \sum_{n=1}^{\infty} \biggl(\sideset{}{'}\sum_{b,m=1}^{n} \frac{\delta_n(bm \pm 1)}{(m+n)(a+b+m+n)} - \log 2 \frac{\varphi(n)}{n^2} \biggr),$$

$$C_3(\xi) = \sum_{n=1}^{\infty} \biggl(\sideset{}{'}\sum_{b,m=1}^{n} \delta_n(bm \pm 1) f_S'(\xi) + \frac{2\log 2}{(\xi+1)^2} \frac{\varphi(n)}{n^2} \biggr). \tag{26}$$

Lemma 9. *Let* $\rho(x) = 1/2 - \{x\}$ *and*

$$h(x) = \sum_{q=1}^{\infty} \frac{\rho(qx)}{q^2}\,. \tag{27}$$

Then

$$\int_0^1 \frac{h(x)}{(x+1)^2}\, dx = \log^2 2 - \frac{\zeta(2)}{4}\,.$$

Proof. The assertion of the lemma follows from the definition of the function $\rho(x)$ and formula (8):

$$\begin{aligned} \int_0^1 \frac{h(x)}{(x+1)^2}\, dx &= \sum_{q=1}^{\infty} \frac{1}{q^2} \sum_{a=0}^{q-1} \int_{a/q}^{(a+1)/q} \biggl(\frac{1}{2} + a - qx \biggr) \frac{dx}{(x+1)^2} \\ &= \sum_{q=1}^{\infty} \frac{1}{q} \biggl(\frac{1}{q+1} + \dots + \frac{1}{2q} - \log 2 + \frac{1}{4q} \biggr) \\ &= \sigma_0 + \frac{\zeta(2)}{4} = \log^2 2 - \frac{\zeta(2)}{4}\,. \end{aligned}$$

Theorem 1. *Let* $2 \leqslant U \leqslant R$. *Then the sum*

$$\sigma_1 = \sum_{Q' \leqslant R} \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}(U)} \sideset{}{^*}\sum_{Q \leqslant Q'} \frac{1}{(mQ + nQ')\bigl((a+m)Q + (b+n)Q'\bigr)}$$

satisfies the asymptotic formula

$$\begin{aligned} \sigma_1 = \frac{2\log^2 2}{\zeta^2(2)} \log R \log U + \frac{1}{\zeta(2)} \biggl(\frac{2\log^2 2}{\zeta(2)} \biggl(\gamma - \frac{\zeta'(2)}{\zeta(2)} \biggr) + C_1 \biggr) \log R \\ + \frac{2\log^2 2}{\zeta^2(2)} \biggl(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \log 2 - \frac{\zeta(2)}{2\log 2} \biggr) \log U + C_1' + O\biggl(\frac{\log^6 R}{U^{1/2}} \biggr), \end{aligned}$$

where the constant C_1 *is defined by the series* (23) *and*

$$\begin{aligned} C_1' = \biggl(\gamma - \frac{\zeta'(2)}{\zeta(2)} \biggr) \biggl(\frac{2\log^2 2}{\zeta^2(2)} \biggl(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \log 2 - \frac{\zeta(2)}{2\log 2} \biggr) + \frac{C_1}{\zeta(2)} \biggr) \\ - \frac{1}{\zeta^2(2)} \int_0^1 h(\xi) C_3(\xi)\, d\xi + \frac{C_2}{2} + \frac{3}{4} - \frac{\log^2 2}{\zeta(2)}\,. \end{aligned} \tag{28}$$

Proof. The summation formula

$$\sum_{0<x\leqslant q} g(x) = \int_0^q g(x)\,dx + \frac{1}{2}\big(g(q)-g(0)\big) - \int_0^q \rho(x)g'(x)\,dx$$

applied to the function

$$g(x) = \frac{1}{(mx+nq)\big((a+m)x+(b+n)q\big)} = \frac{1}{q^2}\,f_S\Big(\frac{x}{q}\Big),$$

results in the equality

$$\sum_{x=1}^{q} g(x) = \frac{1}{q}\,J_1(a,b,m,n) + \frac{1}{2q^2}\left(\frac{1}{(m+n)(a+b+m+n)} - \frac{1}{n(m+n)}\right) - \frac{1}{q^2}\int_0^1 \rho(q\xi)f_S'(\xi)\,d\xi.$$

Hence,

$$\sum_{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathscr{M}(U)} \sum_{x=1}^{q} \frac{1}{(mx+nq)((a+m)x+(b+n)q)} = \frac{1}{q}\,W_1(U) + \frac{1}{2q^2}\,\big(A(U)-N(U)\big) - \frac{1}{q^2}\int_0^1 \rho(q\xi)B(U,\xi)\,d\xi. \qquad (29)$$

We apply this formula for calculating σ_1. For that we preliminarily transform the sum σ_1:

$$\sigma_1 = \sum_{Q'\leqslant R} \sum_{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathscr{M}(U)} \sum_{Q=1}^{Q'} \frac{1}{(mQ+nQ')\big((a+m)Q+(b+n)Q'\big)} \sum_{\delta\mid(Q,Q')} \mu(\delta)$$
$$= \sum_{Q'\leqslant R}\sum_{\delta\mid Q'} \frac{\mu(\delta)}{\delta^2} \sum_{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathscr{M}(U)} \sum_{x=1}^{Q'/\delta} \frac{1}{(mx+nQ'/\delta)\big((a+m)x+(b+n)Q'/\delta\big)}.$$

By formula (29) we have

$$\sigma_1 = \sum_{Q'\leqslant R}\sum_{\delta\mid Q'} \frac{\mu(\delta)}{\delta^2}\left(\frac{\delta}{Q'}W_1(U) + \frac{\delta^2}{2(Q')^2}\big(A(U)-N(U)\big) + \frac{\delta^2}{(Q')^2}\int_0^1 \rho\Big(\frac{Q'\xi}{\delta}\Big)B(U,\xi)\,d\xi\right)$$
$$= W_1(U)\sum_{Q'\leqslant R}\frac{\varphi(Q')}{(Q')^2} + \frac{1}{2}\big(A(U)-N(U)\big) - \frac{1}{\zeta(2)}\int_0^1 h(\xi)B(U,\xi)\,d\xi + O\Big(\frac{\log^2 R}{R}\Big),$$

where the function $h(x)$ is defined by equality (27).

Substituting the asymptotic formulae (22), (24), (10), (25) for the quantities involved in the last equality and applying Lemma 9 we obtain the assertion of Theorem 1.

§ 7. Calculating the sum σ_2

For a matrix $S = \begin{pmatrix} a & m \\ b & n \end{pmatrix} \in \mathscr{M}$ we denote by $J_2(a,b,m,n)$ the integral

$$J_2(a,b,m,n) = \int_0^1 \frac{\log(m\xi+n)d\xi}{(m\xi+n)\big((a+m)\xi+(b+n)\big)}\,.$$

Lemma 10. *Let n be a positive integer. Then the sum*

$$w_2(n) = \sideset{}{'}\sum_{b,m=1}^{n} \delta_n(bm\pm 1)J_2\biggl(\frac{bm\pm 1}{n}\,,b,m,n\biggr)$$

satisfies the asymptotic formula

$$w_2(n) = 2\log^2 2\frac{\varphi(n)\log n}{n^2} + \log^2 2\biggl(2+\log 2-\frac{\zeta(2)}{\log 2}\biggr)\frac{\varphi(n)}{n^2} + O\biggl(\frac{\psi(n)\log(n+1)}{n^{3/2}}\biggr),$$

where $\psi(n)$ is the function defined in the hypothesis of Lemma 5.

Proof. The assertion of the lemma is obvious for $n=1$. Therefore we assume that $n\geqslant 2$. It follows from equality (19) that

$$w_2(n) = \int_0^1 d\xi \sum_{b,m=1}^{n} \delta_n(bm\pm 1)\frac{\log(m\xi+n)}{(1+b/n)(m\xi+n)^2} + O\biggl(\frac{\log(n+1)}{n^3}\biggr).$$

Applying Lemma 5 we obtain

$$\begin{aligned} w_2(n) &= 2\frac{\varphi(n)}{n^2}\int_0^1 d\xi \sum_{b=1}^{n}\frac{1}{1+b/n}\sum_{m=1}^{n}\frac{\log(m\xi+n)}{(m\xi+n)^2} + O\biggl(\frac{\psi(n)\log(n+1)}{n^{3/2}}\biggr)\\ &= 2\log 2\frac{\varphi(n)}{n}\int_0^1 d\xi\int_0^n dm\,\frac{\log(m\xi+n)}{(m\xi+n)^2} + O\biggl(\frac{\psi(n)\log(n+1)}{n^{3/2}}\biggr)\\ &= 2\log 2\frac{\varphi(n)}{n^2}\int_0^1 d\xi\int_0^1 dz\,\frac{\log n+\log(z\xi+1)}{(z\xi+1)^2} + O\biggl(\frac{\psi(n)\log(n+1)}{n^{3/2}}\biggr). \end{aligned}$$

To complete the proof it remains to use the equalities

$$\int_0^1\int_0^1 \frac{d\xi\,dz}{(z\xi+1)^2} = \log 2, \qquad \int_0^1\int_0^1 \frac{\log(z\xi+1)\,d\xi\,dz}{(z\xi+1)^2} = \frac{\log 2}{2}\biggl(2+\log 2-\frac{\zeta(2)}{\log 2}\biggr).$$

Similarly to Corollary 1, the following assertion is a consequence of Lemmas 2 and 10.

Corollary 2. *For any real* $U \geqslant 2$ *the sum*

$$W_2(U) = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}(U)} J_2(a,b,m,n)$$

satisfies the asymptotic formula

$$W_2(U) = \frac{\log^2 2}{\zeta(2)} \log^2 U + \frac{\log^2 2}{\zeta(2)} \left(2 + \log 2 - \frac{\zeta(2)}{\log 2}\right) \log U + C_4 + O\left(\frac{\log^6 U}{U^{1/2}}\right),$$

where

$$C_4 = \frac{\log^2 2}{\zeta(2)} \left(2 + \log 2 - \frac{\zeta(2)}{\log 2}\right)\left(\gamma - \frac{\zeta'(2)}{\zeta(2)}\right) + 2\log^2 2\, C_0 + C_4', \tag{30}$$

C_0 *is the constant in Lemma 2, and* C_4' *is the sum of the series*

$$C_4' = \sum_{n=1}^{\infty} \left(w_2(n) - 2\log^2 2 \frac{\varphi(n)}{n^2} \left(\log n + 2 + \log 2 - \frac{\zeta(2)}{\log 2} \right)\right).$$

Theorem 2. *Let* $2 \leqslant U \leqslant R$. *Then the sum* σ_2 *defined by equality (17) satisfies the asymptotic formula*

$$\sigma_2 = \frac{\log^2 2}{\zeta^2(2)} \log^2 U + \frac{\log^2 2}{\zeta^2(2)} \left(2 + \log 2 - \frac{\zeta(2)}{\log 2}\right) \log U + \frac{C_4}{\zeta(2)} + O\left(\frac{\log^6 R}{U^{1/2}}\right) + O\left(\frac{U \log R}{R}\right),$$

where C_4 *is the constant in Corollary 2.*

Proof. Applying Lemma 4 to the inner sum over the variable Q we obtain

$$\begin{aligned}
\sigma_2 &= \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}(U)} \sum_{Q' \leqslant R} \sum_{Q \leqslant Q'}{}^{*} \frac{[mQ + nQ' > R]}{(mQ + nQ')\big((a+m)Q + (b+n)Q'\big)} \\
&= \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}(U)} \sum_{Q' \leqslant R} \frac{\varphi(Q')}{Q'} \sum_{Q \leqslant Q'} \frac{[mQ + nQ' > R]}{(mQ + nQ')\big((a+m)Q + (b+n)Q'\big)} \\
&\quad + O\left(\frac{U \log R}{R}\right).
\end{aligned}$$

Replacing the sum over the variable Q by the integral and performing the change of variables $Q = \xi Q'$ we obtain

$$\sigma_2 = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathscr{M}(U)} \sum_{Q' \leqslant R} \frac{\varphi(Q')}{(Q')^2} \int_0^1 \frac{[m\xi + n > R/Q']\, d\xi}{(m\xi + n)\big((a+m)\xi + b + n\big)} + O\left(\frac{U \log R}{R}\right).$$

Next, since

$$\sum_{Q'\leqslant R}\frac{\varphi(Q')}{(Q')^2}\left[m\xi+n>\frac{R'}{Q}\right]=\sum_{\delta\leqslant R}\frac{\mu(\delta)}{\delta^2}\sum_{Q'\leqslant R/\delta}\frac{[m\xi+n>R/(\delta Q')]}{Q'}$$
$$=\sum_{\delta\leqslant R}\frac{\mu(\delta)}{\delta^2}\biggl(\log(m\xi+n)+O\biggl(\frac{n\delta}{R}\biggr)\biggr)=\frac{\log(m\xi+n)}{\zeta(2)}+O\biggl(\frac{n\log R}{R}\biggr),$$

we have

$$\sigma_2=\frac{1}{\zeta(2)}\sum_{\left(\begin{smallmatrix}a&m\\b&n\end{smallmatrix}\right)\in\mathscr{M}(U)}\int_0^1\frac{\log(m\xi+n)\,d\xi}{(m\xi+n)\bigl((a+m)\xi+(b+n)\bigr)}+O\biggl(\frac{U\log R}{R}\biggr).$$

Applying Corollary 2 we arrive at the assertion of Theorem 2.

§ 8. Calculating the sum σ_3

Lemma 11. *For $N\geqslant 2$ the sum*

$$F^*(N)=\sum_{n<N}\sum_{m\leqslant n}{}^{*}\frac{1}{m}\biggl(\frac{1}{n}-\frac{1}{m+n}\biggr)-\sum_{n<N}\sum_{\substack{m\leqslant n\\ m+n>N}}{}^{*}\frac{1}{m}\biggl(\frac{1}{N}-\frac{1}{m+n}\biggr)$$

satisfies the asymptotic formula

$$F^*(N)=\frac{\log 2}{\zeta(2)}(\log N+H)+O\biggl(\frac{\log^2 N}{N}\biggr),\tag{31}$$

where

$$H=\gamma-\frac{\zeta'(2)}{\zeta(2)}+\frac{\log 2}{2}-1.\tag{32}$$

Proof. The substitution of $x=1$ into Lemma 10 in [6] results in equality (31) with the constant

$$H=\gamma-\frac{\zeta'(2)}{\zeta(2)}-\frac{\log 2}{2}-1+\frac{1}{\log 2}\biggl(\sigma_0+\frac{\zeta(2)}{2}\biggr),$$

where σ_0 is defined by the series (7). Substituting the value of σ_0 in (8) into the last formula we arrive at the assertion of Lemma 11.

Theorem 3. *Let $2\leqslant U\leqslant R$. Then the sum σ_3 given by equality (18) satisfies the asymptotic formula*

$$\sigma_3=\frac{\log^2 2}{\zeta^2(2)}\log\frac{R}{U}\biggl(\log\frac{R}{U}+2H\biggr)+O\biggl(\frac{\log^6 R}{U^{1/2}}\biggr),$$

where the constant H is given by equality (32).

Proof. Since for any matrix $\begin{pmatrix}a&m\\b&n\end{pmatrix}\in\mathscr{M}$ we have

$$\frac{1}{(a+m)Q+(b+n)Q'}-\frac{1}{(bm/n+m)Q+(b+n)Q'}\ll\frac{1}{n^3Q'}$$

and

$$\sum_{Q'\leqslant R}\ \sum^{*}_{Q\leqslant Q'}\ \sum_{\substack{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathscr{M},\\ n>U}} \frac{1}{n^4(Q')^2} \ll \frac{\log R}{U^2},$$

the sum σ_3 can be rewritten in the form

$$\sigma_3 = \sum_{Q'\leqslant R}\ \sum^{*}_{Q\leqslant Q'}\ \sum_{n>U}\ \sum_{b,m=1}^{n} \delta_n(bm\pm 1)\,\frac{[mQ+nQ'\leqslant R]}{(b/n+1)(mQ+nQ')^2} + O\biggl(\frac{\log R}{U^2}\biggr).$$

Applying Lemma 5 we obtain

$$\sigma_3 = 2\sum_{Q'\leqslant R}\ \sum^{*}_{Q\leqslant Q'}\ \sum_{n>U} \frac{\varphi(n)}{n} \sum_{b=1}^{n} \frac{1}{b+n} \sum_{m=1}^{n} \frac{[mQ+nQ'\leqslant R]}{(mQ+nQ')^2} + O\biggl(\frac{\log^6 R}{U^{1/2}}\biggr).$$

Next, by formula (20) we have

$$\sigma_3 = 2\log 2\sigma_4 + O\biggl(\frac{\log^6 R}{U^{1/2}}\biggr), \tag{33}$$

where

$$\begin{aligned}
\sigma_4 &= \sum_{Q'\leqslant R}\ \sum^{*}_{Q\leqslant Q'}\ \sum_{n>U} \frac{\varphi(n)}{n} \sum_{m=1}^{n} \frac{[mQ+nQ'\leqslant R]}{(mQ+nQ')^2}\\
&= \sum_{Q'\leqslant R}\ \sum^{*}_{Q\leqslant Q'}\ \sum_{U<n\leqslant R/(Q+Q')} \frac{\varphi(n)}{n} \sum_{m=1}^{n} \frac{1}{(mQ+nQ')^2}\\
&\quad + \sum_{Q'\leqslant R}\ \sum^{*}_{Q\leqslant Q'}\ \sum_{\max\{U,R/(Q+Q')\}<n\leqslant R/Q'} \frac{\varphi(n)}{n} \sum_{m\leqslant (R-nQ')/Q} \frac{1}{(mQ+nQ')^2}.
\end{aligned}$$

Replacing the inner sums over the variable m by the corresponding integrals we obtain

$$\begin{aligned}
\sigma_4 &= \sum_{Q'\leqslant R}\ \sum^{*}_{Q\leqslant Q'}\ \sum_{U<n\leqslant R/(Q+Q')} \frac{\varphi(n)}{n}\,\frac{1}{Q}\biggl(\frac{1}{nQ}-\frac{1}{nQ+nQ'}\biggr)\\
&\quad + \sum_{Q'\leqslant R}\ \sum^{*}_{Q\leqslant Q'}\ \sum_{\max\{U,R/(Q+Q')\}<n\leqslant R/Q'} \frac{\varphi(n)}{n}\,\frac{1}{Q}\biggl(\frac{1}{nQ}-\frac{1}{R}\biggr).
\end{aligned}$$

By making the summation over n the outer one we arrive at the equality

$$\sigma_4 = \sum_{U<n\leqslant R} \frac{\varphi(n)}{n^2}\,F^*\biggl(\frac{R}{n}\biggr) + O\biggl(\frac{\log R}{U}\biggr) + O\biggl(\frac{\log R}{U}\biggr),$$

where

$$\begin{aligned}
F^*(\xi) &= \sum_{Q'<\xi}\ \sum^{*}_{Q\leqslant Q'} \frac{1}{Q}\biggl(\frac{1}{Q'}-\frac{1}{Q+Q'}\biggr)[\xi\geqslant Q+Q']\\
&\quad + \sum_{Q'<\xi}\ \sum^{*}_{Q\leqslant Q'} \frac{1}{Q}\biggl(\frac{1}{Q'}-\frac{1}{\xi}\biggr)[\xi< Q+Q'].
\end{aligned}$$

By Lemma 11,

$$\sigma_4 = \frac{\log 2}{\zeta(2)} \sum_{U<n\leqslant R} \frac{\varphi(n)}{n^2}\left(\log\frac{R}{n} + H\right) + O\left(\frac{\log^3 R}{U}\right).$$

Next, by using the formulae of Lemma 2 we obtain the following asymptotic formula for the sum σ_4:

$$\sigma_4 = \frac{\log 2}{2\zeta^2(2)} \log\frac{R}{U}\left(\log\frac{R}{U} + 2H\right) + O\left(\frac{\log^3 R}{U}\right).$$

Substituting it into equality (33) we arrive at the assertion of Theorem 3.

§ 9. Main result

Theorem 4. *For $R \geqslant 2$ we have*

$$D(R) = D_1 \log R + D_0 + O(R^{-1/3}\log^5 R),$$

where

$$D_1 = \frac{8\log^2 2}{\zeta^2(2)}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{\log 2}{2} - 1\right) + \frac{4}{\zeta(2)}\left(C_1 + \frac{3\log 2}{2}\right),$$

$$D_0 = 4\left(C_1' - \frac{C_4}{\zeta(2)}\right) - \left(\frac{2\log 2}{\zeta(2)}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \log 2\right) - \frac{3}{2}\right)^2 + \frac{1}{2},$$

while the constants C_1, C_1', and C_4 are defined by equalities (23), (28), *and* (30), *respectively.*

Proof. Combining the results of Theorems 1–3, for the sum $\sigma(R) = \sigma_1 - \sigma_2 + \sigma_3$ we obtain the asymptotic formula

$$\sigma(R) = \frac{\log^2 2}{\zeta^2(2)}\log^2 R + \frac{\log R}{\zeta(2)}\left(\frac{2\log^2 2}{\zeta(2)}\left(2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} + \frac{\log 2}{2} - 1\right) + C_1\right) + C_1' - \frac{C_4}{\zeta(2)} + O\left(\frac{U\log R}{R}\right) + O\left(\frac{\log^6 R}{U^{1/2}}\right). \tag{34}$$

Choosing $U = R^{2/3}\log^4 R$ and applying Lemmas 6, 7 we arrive at the assertion of Theorem 4.

Remark 3. Computer calculations give the following approximate value of the constant D_1:

$$D_1 = 0.51606\dots.$$

Remark 4. Equality (34) in the proof of Theorem 4 gives an asymptotic formula with three significant terms for the sum in (15).

Remark 5. The constant C_1 defined by equality (23) also appears in the averaging of $N(\alpha, R)$ with respect to the Gaussian measure

$$d\mu(\alpha) = \frac{1}{\log 2}\,\frac{d\alpha}{1+\alpha}.$$

Indeed, straightforward calculations based on the representation (11) lead to the equality

$$\frac{1}{\log 2}\int_0^1 N(\alpha,R)\,\frac{d\alpha}{1+\alpha} = \frac{1}{\log 2}\,W_1(R),$$

where $W_1(R)$ is the sum defined by formula (21). Therefore, according to the corollary of Lemma 8 we have

$$\frac{1}{\log 2}\int_0^1 N(\alpha,R)\,\frac{d\alpha}{1+\alpha} = \frac{2\log 2}{\zeta(2)}\biggl(\log R+\gamma-\frac{\zeta'(2)}{\zeta(2)}\biggr)+\frac{C_1}{\log 2}+O\bigl(R^{-1/2+\varepsilon}\bigr).$$

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