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The Statistics of Particle Trajectories in the Homogeneous Sinai Problem for a Two-Dimensional Lattice*

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Abstract. In this paper, we generalize and refine some results by F. P. Boca, R. N. Gologan, and A. Zaharescu on the asymptotic behavior as $h \to 0$ of the statistics of the free path length until the first hit of the h-neighborhood (a disk of radius h) of a nonzero integer for a particle issuing from the origin. The established facts imply that the limit distribution function for the free path length and for the sighting parameter (the distance from the trajectory to the integer point in question) does not depend on the particle escape direction (the property of isotropy).

Key words: integer lattice, continued fraction, Kloosterman's sum.

Notation

1. $\|x\|$ is the distance from a real number x to the nearest integer.
2. $\varphi(d)$ denotes the number of integers from 1 to d coprime with d (the Euler function).
3. $\mu(d)$ is the Möbius function.
4. For a positive integer q and an integer a, we denote by $\delta_q(a)$ the characteristic function for divisibility by q,

$$\delta_q(a) = \begin{cases} 1 & \text{if } a \equiv 0 \pmod{q}, \\ 0 & \text{if } a \not\equiv 0 \pmod{q}. \end{cases}$$

5. Finite differences for a function $a(u,v)$ will be denoted as follows:

$$\Delta_{1,0}a(u,v) = a(u+1,v) - a(u,v), \quad \Delta_{0,1}a(u,v) = a(u,v+1) - a(u,v),$$
$$\Delta_{1,1}a(u,v) = \Delta_{0,1}(\Delta_{1,0}a(u,v)) = \Delta_{1,0}(\Delta_{0,1}a(u,v)).$$

Introduction

Let $0 < h < 1/(2\sqrt{2})$ and $T > 0$. An open disk of radius h with center at some point will be called the h-neighborhood of this point. We define $\Omega_h(T)$ as the subset of $[0, 2\pi)$ consisting of the angles φ such that the ray

$$\{(t\cos\varphi, t\sin\varphi) \mid t \geqslant 0\} \tag{1}$$

meets the h-neighborhood of some integer point $(m,n) \neq (0,0)$ in the disk

$$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant T^2\}.$$

Let $G_h(T)$ denote the normalized measure of $\Omega_h(T)$,

$$G_h(T) = \frac{1}{2\pi} \operatorname{mes} \Omega_h(T) \in [0,1].$$

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In 1918, Pólya proved (see [2], Number Theory, Problem 239) that $G_h(T) = 1$ for all $T \geqslant h^{-1}$. Answering the question posed by Ya. G. Sinai in 1981, Boca, Gologan, and Zaharescu [4] proved that, for an arbitrary $\varepsilon > 0$, the relation

$$G_h(T) = \int_0^{hT} \sigma(t)\,dt + O_\varepsilon(h^{1/8-\varepsilon}), \tag{2}$$

where

$$\sigma(t) = \begin{cases} \frac{12}{\pi^2} & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ \frac{12}{\pi^2}\left(\frac{1}{t}-1\right)\left(1-\log\left(\frac{1}{t}-1\right)\right) & \text{for } \frac{1}{2} < t \leqslant 1, \end{cases}$$

holds uniformly with respect to $T \in [0, h^{-1}]$.

From the physical viewpoint, the expression $G_h(T)$ can be interpreted as the distribution function for free path lengths of particles moving rectilinearly from the origin until the first hit of the h-neighborhood of a nonzero integer point. We speak of the homogeneous two-dimensional model known as the "periodic Lorentz gas."

In this connection, the following more general problem seems to be of interest. Let $(m, n) = (m_h(\varphi), n_h(\varphi)) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ be the center of the first h-neighborhood hit by the ray (1). We denote by $T_h(\varphi)$ the distance from the origin to the projection $M_h(\varphi)$ of the point $(m_h(\varphi), n_h(\varphi))$ onto the ray (1) and also introduce the function $U_h(\varphi) \in [-h, h]$ whose absolute value coincides with the distance from $(m_h(\varphi), n_h(\varphi))$ to $M_h(\varphi)$ and which satisfies $U_h(\varphi) > 0$ (respectively, $U_h(\varphi) < 0$) if the point $(m_h(\varphi), n_h(\varphi))$ remains on the right (respectively, left) as a particle moves along the ray (1). It is convenient to consider the normalized values of these expressions,

$$t_h(\varphi) = h \cdot T_h(\varphi) \in [0,1], \qquad u_h(\varphi) = h^{-1} \cdot U_h(\varphi) \in [-1,1].$$

Resorting to the terminology of nuclear physics, we refer to $u_h(\varphi)$ as the normalized sighting parameter, and $t_h(\varphi)$ will be called the normalized free path length.

Let $0 \leqslant t_0 \leqslant 1$ and $-1 \leqslant u_1 \leqslant u_2 \leqslant 1$. As usual, $\chi_I(\dots)$ is the characteristic function of an interval I on the real line. The following assertion is the main result of this paper.

Theorem. *For every* $\varepsilon > 0$, *the asymptotic formula*

$$\Phi(h) = \int_0^{\varphi_0}\int_0^{t_0}\int_{u_1}^{u_2} \rho(\varphi, t, u)\,d\varphi\,dt\,du + O_\varepsilon(h^{1/2-\varepsilon}), \quad h \to 0,$$

where

$$\rho(\varphi, t, u) = \rho(t, u) = \begin{cases} \frac{3}{\pi^3} & \text{for } |u| \leqslant \frac{1}{t} - 1, \\ \frac{3}{\pi^3}\,\frac{1}{|u|}\left(\frac{1}{t}-1\right) & \text{for } |u| > \frac{1}{t} - 1, \end{cases}$$

holds uniformly with respect to t_0, u_1, u_2, *and* $\varphi_0 \in [0, 2\pi]$ *for the distribution function*

$$\Phi(h) = \Phi(h; \varphi_0, t_0, u_1, u_2) = \frac{1}{2\pi}\int_0^{\varphi_0} \chi_{[0,t_0]}(t_h(\varphi))\chi_{[u_1,u_2]}(u_h(\varphi))\,d\varphi.$$

Remark 1. If we set $\varphi_0 = 2\pi$ and also $u_1 = -1$ and $u_2 = 1$, then we obtain a refined expression for the remainder term in the asymptotic formula (2) proved in [4].

Remark 2. From the physical viewpoint, the function $\rho(\varphi, t, u)$ can be interpreted as the density of particles that move rectilinearly from the origin at an angle φ to the abscissa axis, travel the distance $T = h^{-1} \cdot t$ (the free path length) until the first scattering in the h-neighborhoods of integer points, and hit the h-neighborhoods with sighting parameter $U = h \cdot u$.

Remark 3. The distribution density $\rho(\varphi, t, u)$ does not depend on the angle φ (the property of isotropy).

1. Application of Continued Fractions

The obvious symmetries in the problem under consideration lead to the relations

$$t_h(\varphi) = t_h(-\varphi) = t_h(\varphi + \pi/2), \qquad u_h(\varphi) = -u_h(-\varphi) = u_h(\varphi + \pi/2).$$

Therefore, it suffices to prove the theorem only for $\varphi_0 \in [0, \pi/4]$, which is assumed in what follows. In this case, we have $\alpha = \tan^{-1}\varphi \in [0,1]$.

Set

$$l_\varphi(x,y) = x\sin\varphi - y\cos\varphi = \frac{\alpha x - y}{\sqrt{1+\alpha^2}}, \tag{3}$$

$$l^*_\varphi(x,y) = x\cos\varphi + y\sin\varphi = \frac{x + \alpha y}{\sqrt{1+\alpha^2}}\,. \tag{4}$$

Remark 4. It readily follows from the definitions of $l_\varphi(x,y)$ and $l^*_\varphi(x,y)$ that $(m_h(\varphi), n_h(\varphi))$ is an integer point (m,n) with $|l_\varphi(m,n)| < h$, $m > 0$, and $n \geqslant 0$ minimizing $l^*_\varphi(m,n)$; the minimum value is equal to

$$h^{-1} \cdot t_0(\varphi) = l^*_\varphi(m_h(\varphi), n_h(\varphi)).$$

Furthermore,

$$h \cdot u_0(\varphi) = l_\varphi(m_h(\varphi), n_h(\varphi)).$$

Lemma 1. *The integer pair $(m_h(\varphi), n_h(\varphi))$ is uniquely determined by the conditions*

$$m_h(\varphi) = \min\bigl\{m \in \mathbb{N} \mid \|\alpha m\| < h\sqrt{1+\alpha^2}\bigr\}, \qquad |\alpha m_h(\varphi) - n_h(\varphi)| < \frac{1}{2}\,.$$

Proof. Assume that, for some positive integer m less than $m_h(\varphi)$, the inequality

$$\|\alpha m\| \leqslant \|\alpha m_h(\varphi)\|$$

holds. Then there exists an $n \geqslant 0$ such that

$$|\alpha m - n| = \|\alpha m\| \leqslant \|\alpha m_h(\varphi)\| = |\alpha m_h(\varphi) - n_h(\varphi)| < h\sqrt{1+\alpha^2} \leqslant \frac{1}{2\sqrt{2}}\sqrt{2} = \frac{1}{2}\,.$$

It follows that

$$n < \alpha m + \frac{1}{2} \leqslant \alpha m_h(\varphi) + \frac{1}{2} = \alpha m_h(\varphi) - n_h(\varphi) + n_h(\varphi) + \frac{1}{2}$$
$$< n_h(\varphi) + h\sqrt{1+\alpha^2} + \frac{1}{2} < n_h(\varphi) + 1.$$

This means that $n \leqslant n_h(\varphi)$, and therefore,

$$m\cos\varphi + n\sin\varphi < m_h(\varphi)\cos\varphi + n_h(\varphi)\sin\varphi.$$

This inequality contradicts Remark 4, which means that our assumption is untrue, and hence the proof of Lemma 1 is complete. □

Recall that an arbitrary real number x admits a canonical continued fraction expansion

$$x = [q_0; q_1, \ldots, q_i, \ldots] = q_0 + \cfrac{1}{q_1 + \ddots + \cfrac{1}{q_i + \ddots}}$$

with integer part $q_0 = [x]$ and partial quotients $q_i = q_i(x) \in \mathbb{N}$ for $i \geqslant 1$. The continued fraction is finite only for rational x, and in this case its terminal partial quotient (if it is present in the expansion) is greater than 1. By definition,

$$P_i = P_i(x) \quad \text{and} \quad Q_i = Q_i(x) \qquad (i = 1, 2, \ldots)$$

are, respectively, the numerator (an integer) and the denominator (a positive integer) of the ith convergent of x reduced to lowest terms,

$$\frac{P_i}{Q_i} = [q_0; q_1, \dots, q_{i-1}].$$

Here $P_1 = q_0$ and $Q_1 = 1$.

Lemma 2. *The integers $n_h(\varphi)$ and $m_h(\varphi)$ are, respectively, the numerator and the denominator of some convergent of $\alpha = \tan\varphi \in [0,1]$.*

Proof. Lemma 1 implies that there exist no positive integers m less than $m_h(\varphi)$ such that $\|\alpha m\| \leqslant \|\alpha m_h(\varphi)\|$, whence, using the Lagrange theorem on best approximations (see [3]), we conclude that the assertion of Lemma 2 is true. □

We denote by $\mathscr{M}$ the set of all integer matrices

$$S = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} = \begin{pmatrix} P(S) & P'(S) \\ Q(S) & Q'(S) \end{pmatrix} \tag{5}$$

such that $\det S = \pm 1$ and

$$1 \leqslant Q \leqslant Q', \quad 0 \leqslant P \leqslant Q, \quad 1 \leqslant P' \leqslant Q'.$$

It splits into two disjoint subsets $\mathscr{M}_+$ and $\mathscr{M}_-$ consisting of matrices with determinant $+1$ and -1, respectively. In what follows, it will be assumed that $*$ is either $+$ or $-$ or an empty symbol. In accordance with (5), depending on the context, we shall use the notation P, P', Q, and Q' for the entries of the matrix S instead of $P(S)$, $P'(S)$, $Q(S)$, and $Q'(S)$, respectively, provided that this does not lead to ambiguity.

Let $X \geqslant 1$, and let

$$\mathscr{M}_*(X) = \{S \in \mathscr{M}_* \mid (P')^2 + (Q')^2 \leqslant X^2\}.$$

For $\alpha \in (0,1)$, the matrix

$$S_i(\alpha) = \begin{pmatrix} P_i(\alpha) & P_{i+1}(\alpha) \\ Q_i(\alpha) & Q_{i+1}(\alpha) \end{pmatrix}$$

is an element of $\mathscr{M}$ with $\det S_i(\alpha) = (-1)^i$. Let $\widetilde{\mathbb{N}}$ denote the set of all nonempty finite n-tuples $(q_1, \dots, q_n)$ of positive integers. We construct a map

$$\mathscr{B}: \quad \widetilde{\mathbb{N}} = \bigcup_{k=1}^{\infty} \mathbb{N}^k \to \mathscr{M}$$

by setting

$$\mathscr{B}(q_1, \dots, q_n) = S = S(q_1, \dots, q_n) = \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & q_n \end{pmatrix}.$$

Here S has the form (5) with

$$\frac{P}{Q} = [0; q_1, \dots, q_{n-1}] \quad \text{and} \quad \frac{P'}{Q'} = [0; q_1, \dots, q_n].$$

The assertion below can readily be verified on the basis of properties of continued fractions.

Lemma 3. *The map $\mathscr{B}$ is a bijection.*

It is of interest to note (although this will not be needed in what follows) that $\mathscr{M}$ is a semigroup with respect to the usual multiplication of matrices and the bijection $\mathscr{B}$ is actually a semigroup isomorphism. Moreover,

$$(q_1, \dots, q_n) * (q_{n+1}, \dots, q_{n+m}) = (q_1, \dots, q_n, q_{n+1}, \dots, q_{n+m})$$

is the corresponding operation on finite n-tuples of positive integers.

With every matrix

$$S = S(q_1, \dots, q_n) = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathscr{M},$$

we associate the half-open interval

$$I(S)=\{\alpha=[0;q_1,\dots,q_{n-1},q_n+\beta]\mid 0\leqslant\beta<1\}=$$
$$=\left\{\alpha=\frac{P'+\beta P}{Q'+\beta Q}\;\middle|\;0\leqslant\beta<1\right\}=\left\{\alpha\in(0,1]\;\middle|\;-1<\frac{\alpha Q'-P'}{\alpha Q-P}\leqslant 0\right\}. \qquad (6)$$

Here if $S\in\mathscr{M}_+$, then

$$I(S)=\left[\frac{P'}{Q'},\frac{P'+P}{Q'+Q}\right)=\left[\frac{P'}{Q'},\frac{P'}{Q'}+\frac{1}{Q'(Q'+Q)}\right),$$

and if $S\in\mathscr{M}_-$, then

$$I(S)=\left(\frac{P'+P}{Q'+Q},\frac{P'}{Q'}\right]=\left(\frac{P'}{Q'}-\frac{1}{Q'(Q'+Q)},\frac{P'}{Q'}\right].$$

Furthermore, we set

$$I_h(S)=\left\{\alpha\in(0,1]\;\middle|\;0\leqslant\alpha Q'-P'<h\sqrt{1+\alpha^2}\leqslant-\alpha Q+P\right\}$$

for $S\in\mathscr{M}_+$ and

$$I_h(S)=\left\{\alpha\in(0,1]\;\middle|\;0\leqslant-\alpha Q'+P'<h\sqrt{1+\alpha^2}\leqslant\alpha Q-P\right\}$$

for $S\in\mathscr{M}_-$.

For $I(S)$ in (6), the above representation readily implies the inclusion

$$I_h(S)\subset I(S) \qquad (7)$$

and also the assertion below.

Remark 5. The expressions P/Q and P'/Q' are consecutive convergents of α and are uniquely determined by α and h.

Let

$$f_S(\beta)=Q'\beta-h\sqrt{1+\left(\frac{P'}{Q'}+\det S\cdot\beta\right)^2}.$$

Since

$$f_S'(\beta)=Q'-h\frac{\det S\cdot(P'/Q')+\beta}{\sqrt{1+(\det S\cdot(P'/Q')+\beta)^2}}>Q'-h>0$$

and

$$f_S(0)=-h\sqrt{1+\left(\frac{P'}{Q'}\right)^2}<0,$$

it follows that the equation $f_S(\beta)=0$ has a unique positive root, which we denote by $\lambda=\lambda_S(h)$. Note that

$$\left(\frac{Q'\lambda}{h}\right)^2=1+\left(\frac{P'}{Q'}+\det S\cdot\lambda\right)^2\leqslant 3+2\lambda^2.$$

Since

$$2\leqslant\frac{1}{4}\left(\frac{1}{h}\right)^2\leqslant\frac{1}{4}\left(\frac{Q'}{h}\right)^2,$$

we have

$$\lambda^2\leqslant\frac{3}{(Q'/h)^2-2}\leqslant\frac{3}{(Q'/h)^2-\frac{1}{4}(Q'/h)^2}=\left(\frac{2h}{Q'}\right)^2.$$

Therefore,

$$\lambda_S(h)\leqslant\frac{2h}{Q'}. \qquad (8)$$

Now consider the function

$$g_S(\beta) = Q\beta + h\sqrt{1 + \left(\frac{P'}{Q'} + \det S \cdot \beta\right)^2},$$

for which

$$g'_S(\beta) = Q + h\frac{\det S \cdot (P'/Q') + \beta}{\sqrt{1 + (\det S \cdot (P'/Q') + \beta)^2}} > Q - h > 0.$$

This is an increasing function,

$$g_S(0) = h\sqrt{1 + \left(\frac{P'}{Q'}\right)^2},$$

and hence the assertion below is true.

Remark 6. The equation $g_S(\beta) = 1/Q'$ has a unique (nonnegative) root $\eta = \eta_S(h)$ only if

$$h\sqrt{1 + \left(\frac{P'}{Q'}\right)^2} \leqslant \frac{1}{Q'}, \quad \text{that is,} \quad S \in \mathscr{M}(h^{-1}).$$

Summarizing what has been said and using the definition of the set $I_h(S)$, we arrive at the assertion below.

Remark 7. The set $I_h(S)$ in nonempty only if $S \in \mathscr{M}(h^{-1})$, and it coincides with

$$\left[\frac{P'}{Q'}, \frac{P'}{Q'} + \lambda_S(h)\right) \cap \left[\frac{P'}{Q'}, \frac{P'}{Q'} + \eta_S(h)\right]$$

for $S \in \mathscr{M}_+(h^{-1})$ and with

$$\left(\frac{P'}{Q'} - \lambda_S(h), \frac{P'}{Q'}\right] \cap \left[\frac{P'}{Q'} - \eta_S(h), \frac{P'}{Q'}\right]$$

for $S \in \mathscr{M}_-(h^{-1})$.

Lemma 4. *The sets $I_h(S)$ are pairwise disjoint, and*

$$\bigcup_{S \in \mathscr{M}(h^{-1})} I_h(S) = \left[\frac{h}{\sqrt{1-h^2}}, 1\right].$$

Proof. The first part of the assertion of the lemma readily follows from Remark 5. Furthermore, let $P_j = P_j(\alpha)$ (respectively, $Q_j = Q_j(\alpha)$) be the numerator (respectively, denominator) of the jth convergent of $\alpha \in (0, 1]$. Since the absolute values of the elements of the sequence

$$\alpha = \alpha_1 = \alpha Q_1 - P_1, \quad \ldots, \quad \alpha_j = \alpha Q_j - P_j, \quad \ldots \tag{9}$$

monotonically decrease to zero, there exists an

$$i = i_h(\alpha) = \min\left\{j \;\middle|\; |\alpha Q_j - P_j| < h\sqrt{1+\alpha^2}\right\}.$$

If $i = 1$, then (recall that $P_1 = 0$ and $Q_1 = 1$)

$$|\alpha Q_1 - P_1| = \alpha < h\sqrt{1+\alpha^2}, \quad \text{that is,} \quad \alpha \in \left(0, \frac{h}{\sqrt{1-h^2}}\right).$$

For all other α, we always have $i \geqslant 2$, and in this case $|\alpha Q_{i-1} - P_{i-1}| \geqslant h\sqrt{1+\alpha^2}$. By taking into account Remark 6 and the fact that (9) is an alternating sequence and by setting

$$S = \begin{pmatrix} P_{i-1} & P_i \\ Q_{i-1} & Q_i \end{pmatrix},$$

we prove the second part of the assertion of Lemma 4. □

2. Auxiliary Transformations

In accordance with the notation introduced earlier, we set

$$\xi_S(\alpha) = \chi_{[u_1,u_2]}\left(\frac{\alpha Q' - P'}{h\sqrt{1+\alpha^2}}\right), \qquad I_h^{(\alpha_0)}(S) = [0,\alpha_0] \cap I_h(S).$$

Passing to the variable $\alpha = \tan\varphi \in [0,1]$ in the integral determining $\Phi(h)$ (see the statement of the theorem in the Introduction) and using Lemma 4, we obtain

$$\Phi(h) = \frac{1}{2\pi} \sum_{S\in\mathscr{M}(h^{-1})} \int_{I_h^{(\alpha_0)}(S)} \chi_{[0,t_0]}\left(\frac{h(Q'+\alpha P')}{\sqrt{1+\alpha^2}}\right)\xi_S(\alpha)\,\frac{d\alpha}{1+\alpha^2} + O(h).$$

Since

$$\left(\frac{Q'+\alpha P'}{\sqrt{1+\alpha^2}}\right)^2 + \left(\frac{\alpha Q'-P'}{\sqrt{1+\alpha^2}}\right)^2 = (P')^2 + (Q')^2,$$

it follows that the inequalities

$$(P')^2 + (Q')^2 - h^2 < \left(\frac{Q'+\alpha P'}{\sqrt{1+\alpha^2}}\right)^2 \leqslant (P')^2 + (Q')^2$$

hold for every $\alpha \in I_h(S)$. Therefore, if

$$(t_0 h^{-1})^2 < (P')^2 + (Q')^2$$

and if there exists an $\alpha \in I_h(S)$ such that

$$\chi_{[0,t_0]}\left(\frac{h(Q'+\alpha P')}{\sqrt{1+\alpha^2}}\right) = 1,$$

then

$$(t_0 h^{-1})^2 < (P')^2 + (Q')^2 < (t_0 h^{-1})^2 + h^2 \leqslant (t_0 h^{-1})^2 + 1/8.$$

Now we use the estimate

$$\int_{I_h^{(\alpha_0)}(S)} d\alpha \leqslant \int_{I_h(S)} d\alpha \leqslant \lambda_S(h) \leqslant \frac{2h}{Q'}$$

(also see (8)) to obtain the relation

$$\Phi(h) = \frac{1}{2\pi} \sum_{S\in\mathscr{M}(t_0 h^{-1})} \int_{I_h^{(\alpha_0)}(S)} \xi_S(\alpha)\,\frac{d\alpha}{1+\alpha^2} + R_1,$$

where

$$R_1 \ll h + \sum_{\substack{(t_0/h)^2 \leqslant (P')^2+(Q')^2 < (t_0/h)^2+1/8 \\ 1\leqslant P' \leqslant Q'}} \frac{h}{Q'} \ll h + \frac{h}{t_0/h + 1/2} \sum_{Q' \leqslant t_0/h+1/2} 1 \ll h.$$

By the Lagrange theorem, in accordance with the estimate (8),

$$\left|\frac{1}{1+\alpha^2} - \frac{1}{1+(P'/Q')^2}\right| \leqslant \left|\alpha - \frac{P'}{Q'}\right| \leqslant \lambda_S(h) \leqslant \frac{2h}{Q'}$$

for $\alpha \in I_h(S)$. Consequently, the formula

$$\Phi(h) = \frac{1}{2\pi} \sum_{S\in\mathscr{M}(t_0 h^{-1})} \frac{1}{1+(P'/Q')^2} \int_{I_h^{(\alpha_0)}(S)} \xi_S(\alpha)\,d\alpha + R_2$$

holds, where

$$R_2 \ll h + \sum_{1\leqslant P' \leqslant Q' \leqslant h^{-1}} \frac{h}{Q'}\cdot\frac{h}{Q'} \ll h.$$

On the basis of the inequality

$$\left|\alpha - \frac{P'}{Q'}\right| \leqslant \frac{2h}{Q'} \qquad \forall \alpha \in I_h(S),$$

by setting

$$\mathscr{M}_*^{(\alpha_0)}(X) = \{S \in \mathscr{M}_*(X) \mid P' \leqslant \alpha_0 Q'\},$$

we conclude that

$$\Phi(h) = \frac{1}{2\pi} \sum_{S \in \mathscr{M}^{(\alpha_0)}(t_0 h^{-1})} \frac{1}{1 + (P'/Q')^2} \int_{I_h(S)} \xi_S(\alpha)\, d\alpha + R_3, \tag{10}$$

where

$$R_3 \ll h + \sum_{\substack{1 \leqslant P' \leqslant Q' \leqslant h^{-1} \\ |\alpha_0 Q' - P'| \leqslant 2h}} \frac{h}{Q'} \ll h + h \sum_{Q' \leqslant h^{-1}} \frac{1}{Q'} \ll h \cdot \log \frac{1}{h}\,.$$

Note that

1. For $0 \leqslant u_1 \leqslant u_2 \leqslant 1$, we have

$$\Phi(\ldots, u_1, u_2) = \Phi(\ldots, 0, u_2) - \Phi(\ldots, 0, u_1).$$

2. For $-1 \leqslant u_1 \leqslant 0 \leqslant u_2 \leqslant 1$, we have

$$\Phi(\ldots, u_1, u_2) = \Phi(\ldots, u_1, 0) + \Phi(\ldots, 0, u_2).$$

3. For $-1 \leqslant u_1 \leqslant u_2 \leqslant 0$, we have

$$\Phi(\ldots, u_1, u_2) = \Phi(\ldots, u_1, 0) - \Phi(\ldots, u_2, 0).$$

Therefore, it suffices to prove the theorem only in the cases $u_1 = 0$, $u_2 = u_0$ or $u_1 = -u_0$, $u_2 = 0$ with $u_0 \in [0, 1]$. Here, in accordance with relation (10), we have

$$\Phi_+(h) = \Phi(h; \varphi_0, t_0, 0, u_0) = \widetilde{\Phi}_+(h) + O\left(h \log \frac{1}{h}\right) \tag{11}$$

and

$$\Phi_-(h) = \Phi(h; \varphi_0, t_0, -u_0, 0) = \widetilde{\Phi}_-(h) + O\left(h \log \frac{1}{h}\right), \tag{12}$$

where

$$\widetilde{\Phi}_\pm(h) = \frac{1}{2\pi} \sum_{S \in \mathscr{M}_\pm^{(\alpha_0)}(t_0 h^{-1})} \frac{\min\{\lambda_S(u_0 h), \eta_S(h)\}}{1 + (P'/Q')^2}.$$

The equation determining $\lambda_S(h)$ and the estimate (8) imply that

$$\lambda_S(h) = \widetilde{\lambda}_S(h) + O\left(\left(\frac{h}{Q'}\right)^2\right), \quad \text{where} \quad \widetilde{\lambda}_S(h) = \frac{h}{Q'}\sqrt{1 + \left(\frac{P'}{Q'}\right)^2}.$$

Likewise, for $\eta_S(h) \leqslant \lambda_S(h)$, the asymptotic relation

$$\eta_S(h) = \widetilde{\eta}_S(h) + O\left(\frac{h^2}{QQ'}\right), \quad \text{where} \quad \widetilde{\eta}_S(h) = \frac{1}{QQ'} - \frac{h}{Q}\sqrt{1 + \left(\frac{P'}{Q'}\right)^2},$$

holds.

Therefore,

$$\min\{\lambda_S(u_0 h), \eta_S(h)\} = \min\{\widetilde{\lambda}_S(u_0 h), \widetilde{\eta}_S(h)\} + O\left(\frac{h^2}{QQ'}\right).$$

By setting

$$\Psi(x, y) = \Psi(Q'; x, y) = \frac{\min\{u_0 h Q'\sqrt{1 + x^2}, \frac{1}{y}(1 - hQ'\sqrt{1 + x^2})\}}{1 + x^2},$$

we finally conclude that

$$\widetilde{\Phi}_{\pm}(h) = \frac{1}{2\pi} \sum_{S\in\mathscr{M}_{\pm}^{(\alpha_0)}(t_0h^{-1})} \frac{1}{(Q')^2}\Psi\left(\frac{P'}{Q'}, \frac{Q}{Q'}\right) + R_4, \tag{13}$$

where

$$R_4 \ll \sum_{1\leqslant Q\leqslant Q'\leqslant h^{-1}} \frac{h^2}{QQ'} \ll h^2\log^2\frac{1}{h} \ll h.$$

We introduce the function

$$\Psi_0(x,y) = \Psi_0(Q';x,y) = \begin{cases} \Psi(Q';x,y) & \text{for } \ x\leqslant\alpha_0, \sqrt{1+x^2}\leqslant t_0(hQ')^{-1}, \\ 0 & \text{otherwise.}\end{cases}$$

For a fixed $Q'=q$, the numbers P' and Q must be solutions of the congruence $ab\equiv\pm1 \pmod q$. Therefore, the expressions $\widetilde{\Phi}_{\pm}(h)$ can be rewritten as

$$\widetilde{\Phi}_{\pm}(h) = \frac{1}{2\pi}\sum_{q\leqslant h^{-1}}\frac{1}{q^2}\sum_{a,b=1}^{q}\delta_q(ab\pm1)\Psi_0\left(\frac{a}{q},\frac{b}{q}\right) + O(h) = \frac{1}{2\pi}\sum_{q\leqslant h^{-1}}\frac{1}{q^2}W_{\pm}(q) + O(h), \tag{14}$$

where

$$W_{\pm}(q) = \sum_{a,b=1}^{q}\delta_q(ab\pm1)\Psi_0\left(\frac{a}{q},\frac{b}{q}\right). \tag{15}$$

3. Application of Estimates for Kloosterman's Sums

In what follows, it is assumed that ε is an arbitrarily small positive constant. By using the estimates found by Estermann [5] for Kloosterman's sums, one can prove the assertion below in a standard way.

Lemma 5. *Let q, k, and l be some positive integers. Then*

$$\sum_{a=1}^{k}\sum_{b=1}^{l}\delta_q(ab\pm1) = \frac{\varphi(q)}{q^2}kl + O_{\varepsilon}(q^{1/2+\varepsilon}).$$

For the proof of the lemma, see [1].

Lemma 6. *Let $q\ll h^{-1}$ be a positive integer. Then the following asymptotic formula holds for the sum $W_{\pm}(q)$ determined by relation* (15):

$$W_{\pm}(q) = \varphi(q)\int_0^1\int_0^1 \Psi_0(q;x,y)\,dx\,dy + O_{\varepsilon}(hq^{3/2+\varepsilon}).$$

Proof. Performing the Abel transformation

$$\sum_{n=1}^{q} f(n)g(n) = g(q+1)\sum_{n=1}^{q}f(n) - \sum_{k=1}^{q}\left(\sum_{n=1}^{k}f(n)\right)(g(k+1)-g(k))$$

of the sum $W_{\pm}(q)$ consecutively with respect to the variables a and b (we first choose $f(a) = \delta_q(ab-1)$ and $g(a)=\Psi_0(a/q,b/q)$ and then $f(b)=\sum_{a=1}^{k}\delta_q(ab-1)$ and $g(b)=\Delta_{1,0}\Psi_0(a/q,b/q)$), we conclude that

$$W_{\pm}(q) = \sum_{k,l=1}^{q}\Delta_{1,1}\Psi_0\left(\frac{k}{q},\frac{l}{q}\right)\sum_{a=1}^{k}\sum_{b=1}^{l}\delta_q(ab-1).$$

By applying Lemma 5 to the inner double sum, we obtain

$$W_{\pm}(q) = \frac{\varphi(q)}{q^2} \sum_{k,l=1}^{q} \Delta_{1,1}\Psi_0\left(\frac{k}{q}, \frac{l}{q}\right) kl + O_\varepsilon(Aq^{1/2+\varepsilon})$$
$$= \frac{\varphi(q)}{q^2} \sum_{k,l=1}^{q} \Delta_{1,1}\Psi_0\left(\frac{k}{q}, \frac{l}{q}\right) \sum_{u=1}^{k}\sum_{v=1}^{l} 1 + O_\varepsilon(Aq^{1/2+\varepsilon}),$$

where

$$A = \sum_{k,l=1}^{q} \left|\Delta_{1,1}\Psi_0\left(\frac{k}{q}, \frac{l}{q}\right)\right|.$$

Passing to outer summation with respect to a and b and then summing with respect to k and l, we find

$$W_{\pm}(q) = \frac{\varphi(q)}{q^2} \sum_{a,b=1}^{q} \Psi_0\left(\frac{a}{q}, \frac{b}{q}\right) + O_\varepsilon(Aq^{1/2+\varepsilon})$$
$$= \varphi(q) \int_0^1 \int_0^1 \Psi_0(x,y)\,dx\,dy + O_\varepsilon(Aq^{1/2+\varepsilon} + hq).$$

It readily follows from the definition of the function $\Psi_0(x,y)$ that the inequality

$$\Delta_{1,1}\Psi_0\left(\frac{k}{q}, \frac{l}{q}\right) \geqslant 0$$

holds at all points $(k/q, l/q)$ except possibly at the points such that the curve

$$u_0 hq\sqrt{1+x^2} = \frac{1}{y}(1 - hq\sqrt{1+x^2}) \qquad (0 \leqslant x,\ y \leqslant 1)$$

meets the square $[k/q,(k+1)/q] \times [l/q,(l+1)/q]$. The number of these points is $O(q)$, and we have

$$\Delta_{1,1}\Psi_0\left(\frac{k}{q}, \frac{l}{q}\right) \ll \Delta_{1,0}\Psi_0\left(\frac{k}{q}, \frac{l}{q}\right) \ll h$$

at every such point. Therefore,

$$A = \sum_{k,l=1}^{q} \Delta_{1,1}\Psi_0\left(\frac{k}{q}, \frac{l}{q}\right) + O(hq)$$
$$= \Psi_0(0,0) - \Psi_0\left(\frac{q+1}{q}, 0\right) - \Psi_0\left(0, \frac{q+1}{q}\right) + \Psi_0\left(\frac{q+1}{q}, \frac{q+1}{q}\right) + O(hq) = O(hq)$$

and

$$W_{\pm}(q) = \varphi(q) \int_0^1 \int_0^1 \Psi_0(q; x,y)\,dx\,dy + O_\varepsilon(hq^{3/2+\varepsilon}). \qquad \square$$

Now we are in a position to prove the main result. In view of formulas (11) and (12), it suffices to verify the relation

$$\widetilde{\Phi}_{\pm}(h) = \int_0^{\alpha_0} \frac{dx}{1+x^2} \int_0^{t_0} dr \int_0^{u_0} \rho(r,u)\,du + O_\varepsilon(h^{1/2-\varepsilon}),$$

where $\rho(r,u)$ is defined in the statement of the theorem.

Substituting the result established in Lemma 6 into (14), we obtain

$$\widetilde{\Phi}_\pm(h) = \frac{1}{2\pi} \sum_{q\leqslant h^{-1}} \frac{\varphi(q)}{q^2} \int_0^1 \int_0^1 \Psi_0(q;x,y)\,dx\,dy + O_\varepsilon(h^{1/2-\varepsilon})$$

$$= \frac{1}{2\pi} \sum_{d\leqslant h^{-1}} \frac{\mu(d)}{d^2} \sum_{n\leqslant (dh)^{-1}} \int_0^1 \int_0^1 \frac{\Psi_0(nd;x,y)}{n}\,dx\,dy + O_\varepsilon(h^{1/2-\varepsilon}). \tag{16}$$

Let us transform the inner sum using the definition of the function $\Psi_0(x,y)$:

$$\sum_{n\leqslant (dh)^{-1}} \int_0^1 \int_0^1 \frac{\Psi_0(nd;x,y)}{n}\,dx\,dy$$

$$= dh \int_0^{\alpha_0} \frac{dx}{1+x^2} \int_0^1 dy \sum_{n\leqslant (dh\sqrt{1+x^2})^{-1}} \min\left\{u_0\sqrt{1+x^2}, \frac{1}{y}\left(\frac{1}{dhn} - \sqrt{1+x^2}\right)\right\}.$$

By replacing summation with respect to n by integration and by introducing the new integration variable $r = ndh\sqrt{1+x^2}$, we find

$$\sum_{n\leqslant (dh)^{-1}} \int_0^1 \int_0^1 \frac{\Psi_0(nd;x,y)}{n}\,dx\,dy = \int_0^{\alpha_0} \frac{dx}{1+x^2} \int_0^{t_0} dr \int_0^1 dy \min\left\{u_0, \frac{1}{y}\left(\frac{1}{r} - 1\right)\right\} + O(dh)$$

$$= -\frac{\pi^3}{3} \int_0^{\alpha_0} \frac{dx}{1+x^2} \int_0^{t_0} dr \int_0^{u_0} \rho(r,u)\,du + O(dh).$$

The substitution of the above formula into (16) gives

$$\widetilde{\Phi}_\pm(h) = \frac{\pi^2}{6} \int_0^{\alpha_0} \frac{dx}{1+x^2} \int_0^{t_0} dr \int_0^{u_0} \rho(r,u)\,du \cdot \sum_{d\leqslant h^{-1}} \frac{\mu(d)}{d^2} + O_\varepsilon(h^{1/2-\varepsilon}) =$$

$$= \int_0^{\alpha_0} \frac{dx}{1+x^2} \int_0^{t_0} dr \int_0^{u_0} \rho(r,u)\,du + O_\varepsilon(h^{1/2-\varepsilon}).$$

The proof of the theorem stated in the Introduction is complete.

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