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Asymptotic behaviour of the first and second moments for the number of steps in the Euclidean algorithm

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Abstract. We prove asymptotic formulae with two significant terms for the expectation and variance of the random variable $s(c/d)$ when the variables c and d range over the set $1 \leqslant c \leqslant d \leqslant R$ and $R \to \infty$, where $s(c,d) = s(c/d)$ is the number of steps in the Euclidean algorithm applied to the numbers c and d.

§ 1. Notation

The symbol $[x_0; x_1, \dots, x_s]$ stands for the continued fraction

$$x_0 + \cfrac{1}{x_1 + \ddots + \cfrac{1}{x_s}}$$

of length s with formal variables $x_0, x_1, \dots, x_s$.

For rational r we use (if not otherwise stated) the canonical continued fraction expansion, $r = [t_0; t_1, \dots, t_s]$, of length $s = s(r)$, where $t_0 = [r]$ (the integer part of r), $t_1, \dots, t_s$ are partial quotients (positive integers) and $t_s \geqslant 2$ for $s \geqslant 1$. We denote by $s_1(r)$ the sum of the partial quotients of r: $s_1(r) = t_0 + t_1 + \dots + t_s$. If r is written as an irreducible fraction, then $q(r)$ will stand for the denominator of this fraction.

If A is some assertion, then $[A]$ is equal to 1 if A is true; otherwise, it is equal to 0.

For every positive integer q we denote by $\delta_q(a)$ the characteristic function of divisibility by q:

$$\delta_q(a) = \bigl[a \equiv 0 \pmod q\bigr] = \begin{cases} 1 & \text{if} \quad a \equiv 0 \pmod q, \\ 0 & \text{if} \quad a \not\equiv 0 \pmod q. \end{cases}$$

The asterisk in a double sum

$$\sum_n \sum_m{}^{*} \dots$$

means that the variables over which the sum is taken are subject to the supplementary condition $(m,n) = 1$.

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The finite differences of a function $a(u,v)$ have the form

$$\Delta_{1,0}a(u,v)=a(u+1,v)-a(u,v),\qquad \Delta_{0,1}a(u,v)=a(u,v+1)-a(u,v),$$
$$\Delta_{1,1}a(u,v)=\Delta_{0,1}\bigl(\Delta_{1,0}a(u,v)\bigr)=\Delta_{1,0}\bigl(\Delta_{0,1}a(u,v)\bigr).$$

The sum of the powers of the divisors of a positive integer q will be denoted by

$$\sigma_\alpha(q)=\sum_{d\mid q}d^\alpha.$$

The Euler dilogarithm has the form

$$\mathrm{Li}_2(z)=\sum_{k=1}^\infty\frac{z^k}{k^2}=-\int_0^z\frac{\log(1-t)}{t}\,dt.$$

§ 2. Introduction

A detailed analysis of the Euclidean algorithm leads to various problems concerning the statistical properties of finite continued fractions (see. [1], § 4.5.3). If the input data of the algorithm are positive integers c and d, $c<d$, then the number of divisions performed, which coincides with the number $s(c/d)$ of partial quotients in the continued fraction

$$\frac{c}{d}=[0;t_1,\dots,t_s],$$

is our chief object of interest.

Heilbronn was the first to study the problem of the behaviour of $s(c/d)$ in the mean. In 1968 he proved [2] the asymptotic formula

$$\frac{1}{\varphi(d)}\sum_{\substack{1\leqslant c\leqslant d\\(c,d)=1}}s\biggl(\frac{c}{d}\biggr)=\frac{2\log 2}{\zeta(2)}\log d+O(\log^4\log d).$$

Later, Porter [3] obtained an asymptotic formula with two significant terms:

$$\frac{1}{\varphi(d)}\sum_{\substack{1\leqslant c\leqslant d\\(c,d)=1}}s\biggl(\frac{c}{d}\biggr)=\frac{2\log 2}{\zeta(2)}\log d+C_{\mathrm P}-1+O_\varepsilon(d^{-1/6+\varepsilon}),$$

where ε is any positive number and

$$C_{\mathrm P}=\frac{\log 2}{\zeta(2)}\biggl(3\log 2+4\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2\biggr)-\frac12$$

is a constant, which was called *Porter's constant* (its definitive form was found by J. W. Wrench [4]).

The methods of probability theory and ergodic theory made it possible to obtain the following results for the mean values with respect to the parameters c and d. Dixon showed in [5] that for any positive ε one can find a constant $c_0>0$ such that

$$\biggl|s\biggl(\frac{c}{d}\biggr)-\frac{12\log 2}{\pi^2}\log d\biggr|<(\log d)^{1/2+\varepsilon}$$

for all pairs (c, d) such that $1 \leqslant c \leqslant d \leqslant R$, with the possible exception of $R^2 \exp\bigl(-c_0(\log R)^{\varepsilon/2}\bigr)$ pairs. Hensley [6] refined Dixon's result and proved that the difference between $s(a/b)$ and its mean value has an asymptotically normal distribution whose parameters can be described explicitly. In particular, he proved an asymptotic formula for the second moment of $s(c/d)$. Later, Vallée [7] proved asymptotic formulae for the expectation, variance and higher moments with the remainder terms decreasing polynomically (see [8]).

For a fixed value of d, only the following estimate for the variance of $s(c/d)$ is known:

$$\frac{1}{d}\sum_{c=1}^{d}\biggl(s\biggl(\frac{c}{d}\biggr)-\frac{2\log 2}{\zeta(2)}\log d\biggr)^2 \ll \log d.$$

This estimate, which is exact to within a constant, is due to Bykovskii [9], who obtained it using methods of analytic number theory based on estimates of Kloosterman sums.

In this paper we use the approach suggested in [9] and study the mean value of $s(c/d)$:

$$\mathsf{E}(R)=\frac{2}{[R]\bigl([R]+1\bigr)}\sum_{d\leqslant R}\sum_{c\leqslant d}s\biggl(\frac{c}{d}\biggr) \tag{1}$$

for $R \geqslant 2$. We prove an asymptotic formula for it in which the rate of decrease in the remainder term is better than in Porter's formula. Namely, we prove the formula

$$\mathsf{E}(R)=\frac{2\log 2}{\zeta(2)}\log d + B + O(R^{-1}\log^5 R), \tag{2}$$

where

$$B = C_{\mathrm{P}} - 1 + \frac{\log 2}{\zeta(2)}\biggl(2\frac{\zeta'(2)}{\zeta(2)}-1\biggr).$$

Moreover, for the variance

$$\mathsf{D}(R)=\frac{2}{[R]\bigl([R]+1\bigr)}\sum_{d\leqslant R}\sum_{c\leqslant d}\biggl(s\biggl(\frac{c}{d}\biggr)-\mathsf{E}(R)\biggr)^2 \tag{3}$$

we prove the formula

$$\mathsf{D}(R)=\delta_1\log R+\delta_0+O_\varepsilon(R^{-1/4+\varepsilon}), \tag{4}$$

where $\delta_1 > 0$ and δ_0 are absolute constants and ε is a positive number as small as desired. Note that the corresponding result in [8] only contained a constant $\gamma > 0$ (instead of $1/4$) in the exponent of the remainder term.

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§ 3. Continued fractions

Following [9], we denote by $\mathcal{M}$ the set of all integer matrices

$$S=\begin{pmatrix}P & P'\\ Q & Q'\end{pmatrix}=\begin{pmatrix}P(S) & P'(S)\\ Q(S) & Q'(S)\end{pmatrix}$$

with determinant $\det S = \pm 1$ such that

$$1 \leqslant Q \leqslant Q', \qquad 0 \leqslant P \leqslant Q, \qquad 1 \leqslant P' \leqslant Q'.$$

For real $R > 0$ we denote by $\mathcal{M}(R)$ the finite subset of $\mathcal{M}$ consisting of all matrices S with $Q' \leqslant R$.

The following two properties of $\mathcal{M}$ (see [9]) are of interest.

1. To every finite (non-empty) tuple of positive integers $(q_1, \dots, q_s)$ one can assign a matrix $S \in \mathcal{M}$ by the rule

$$S = S(q_1, \dots, q_s) = \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & q_s \end{pmatrix}.$$

We have

$$S = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix},$$

where

$$\frac{P}{Q} = [0; q_1, \dots, q_{s-1}], \qquad \frac{P'}{Q'} = [0; q_1, \dots, q_s]$$

(the last partial quotient can be equal to 1).

The map $(q_1, \dots, q_s) \to S(q_1, \dots, q_s)$ is a bijection between the set of all finite tuples of positive integers and $\mathcal{M}$.

2. If $Q < Q'$ and $(Q, Q') = 1$, then there are precisely two pairs, (P, P') and $(Q - P,\, Q' - P')$, such that the matrix whose first row coincides with one of these pairs and whose second row coincides with (Q, Q') belongs to $\mathcal{M}$. Moreover, if

$$\frac{Q}{Q'} = [0; q_s, \dots, q_1] = [0; q_s, \dots, q_1 - 1, 1], \qquad q_1 \geqslant 2,$$

then the corresponding matrices have the form

$$\begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & q_s \end{pmatrix} = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & q_1 - 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & q_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & q_s \end{pmatrix} = \begin{pmatrix} Q - P & Q' - P' \\ Q & Q' \end{pmatrix}. \qquad (5)$$

When $Q = Q'$ there is only one matrix $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ belonging to $\mathcal{M}$.

In the following lemma, for a rational number $r \in (0, 1]$ we use the (unique) continued fraction expansion ending with 1:

$$r = [0; t_1, \dots, t_s, 1], \qquad s \geqslant 0.$$

This expansion is more convenient than the canonical one in that it describes uniformly all these numbers, including $r = 1$.

Lemma 1. *Let c and d be positive integers, $1 \leqslant c \leqslant d$, and let*

$$\frac{c}{d} = [0; t_1, \dots, t_{s-1}, t_s, 1], \qquad s \geqslant 0. \tag{6}$$

Then

1) *the equation*

$$S\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{7}$$

in $k, l \in \mathbb{N}$, $k \leqslant l$, and $S \in \mathcal{M}$, has s solutions,

2) *the equation*

$$S_1 S_2 \begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{8}$$

in $k, l \in \mathbb{N}$, $1 \leqslant k \leqslant l$, $S_1, S_2 \in \mathcal{M}$ has $s(s-1)/2$ solutions.

Proof. If $k/l = [0; q_1, \dots, q_m, 1]$, $m \geqslant 0$,

$$S = \begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & z_n \end{pmatrix}$$

and the numbers c, d are defined by equation (7), then $c/d = [0; z_1, \dots, z_n, q_1, \dots, q_m, 1]$. It follows from (6), (7) and property 1 of $\mathcal{M}$ that there is a j, $1 \leqslant j \leqslant s$, such that

$$S = \begin{pmatrix} 0 & 1 \\ 1 & t_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & t_j \end{pmatrix}, \qquad \frac{k}{l} = [0; t_{j+1}, \dots, t_s, 1].$$

Hence, the number of solutions of equation (7) coincides with the number of ways in which one can choose j in the range from 1 to s, and so is equal to s.

We likewise deduce from (6) and (8) that there are j and r, $1 \leqslant j < r \leqslant s$, such that

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & t_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & t_j \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & 1 \\ 1 & t_{j+1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & t_r \end{pmatrix}, \qquad \frac{k}{l} = [0; t_{r+1}, \dots, t_s, 1].$$

Hence, the number of solutions of equation (8) is equal to the number of pairs (j, r) such that $1 \leqslant j < r \leqslant s$, that is, to $s(s-1)/2$.

§ 4. Expectation and variance

For a real $R \geqslant 1$ we put

$$\mathcal{L}_1(R) = \sum_{d \leqslant R} \sum_{c \leqslant d} s\biggl(\frac{c}{d}\biggr), \qquad \mathcal{L}_2(R) = \sum_{d \leqslant R} \sum_{c \leqslant d} s^2\biggl(\frac{c}{d}\biggr).$$

By (1) and (3), we have

$$\mathsf{E}(R) = \frac{2}{[R]\bigl([R]+1\bigr)} \mathcal{L}_1(R), \tag{9}$$

$$\mathsf{D}(R) = \frac{2}{[R]\bigl([R]+1\bigr)} \mathcal{L}_2(R) - \mathsf{E}^2(R). \tag{10}$$

To obtain our main results (2) and (4), we have to obtain asymptotic formulae for $\mathcal{L}_1(R)$ and $\mathcal{L}_2(R)$ with two and three significant terms, respectively.

We denote by $\lambda(d)$ the number of solutions of the equation

$$kQ + lQ' = d$$

in k, l, Q and Q' such that

$$1 \leqslant k \leqslant l, \qquad 1 \leqslant Q \leqslant Q', \qquad (Q, Q') = 1. \tag{11}$$

We denote by $N^*(R)$ the number of solutions of the inequality

$$kQ + lQ' \leqslant R \tag{12}$$

in k, l, Q, Q' subject to the conditions (11). In other words,

$$N^*(R) = \sum_{d \leqslant R} \lambda(d).$$

We denote by $M^*(R)$ the number of solutions of the inequality

$$k(aQ + bQ') + l(mQ + nQ') \leqslant R \tag{13}$$

in which

$$1 \leqslant k \leqslant l, \qquad 1 \leqslant Q \leqslant Q', \qquad (Q, Q') = 1, \qquad \begin{pmatrix} a & m \\ b & n \end{pmatrix} \in \mathcal{M}. \tag{14}$$

The following lemma reduces the problem of computing $\mathsf{E}(R)$ and $\mathsf{D}(R)$ to the study of the inequalities (12) and (13).

Lemma 2. *Let $R \geqslant 1$. Then*

$$\mathcal{L}_1(R) = 2N^*(R) - \left[\frac{R}{2}\right]\left[\frac{R+1}{2}\right], \tag{15}$$

$$\mathcal{L}_2(R) = 4M^*(R) + \left[\frac{R}{2}\right]\left[\frac{R+1}{2}\right]. \tag{16}$$

Proof. Assertion 1) of Lemma 1 implies that the sum

$$\sum_{c \leqslant d} s\left(\frac{c}{d}\right)$$

is equal to the number of solutions of the equation

$$\begin{pmatrix} * & * \\ Q & Q' \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} * \\ d \end{pmatrix}, \tag{17}$$

where

$$\begin{pmatrix} * & * \\ Q & Q' \end{pmatrix} \in \mathcal{M}, \qquad 1 \leqslant k \leqslant l.$$

If $Q' \geqslant 2$, then property 2 of $\mathcal{M}$ implies that for a given pair (Q, Q'), there are precisely two pairs (P, P') such that the matrix whose first row coincides with one of these pairs and whose second row coincides with (Q, Q') belongs to $\mathcal{M}$. Hence, in this case the number of solutions of equation (17) is equal to $2\lambda(d)$. If, on the other hand, $Q' = 1$, then $Q = 1$ and the number of solutions of equation (17) coincides with the number of solutions of the equation $k + l = d$, where $1 \leqslant k \leqslant l$, that is, it is equal to $[d/2]$.

Hence,

$$\sum_{d\leqslant R}\sum_{c\leqslant d} s\left(\frac{c}{d}\right) = \sum_{d\leqslant R}\left(2\lambda(d) - \left[\frac{d}{2}\right]\right)$$
$$= 2\sum_{d\leqslant R}\lambda(d) - \left[\frac{R}{2}\right]\left[\frac{R+1}{2}\right] = 2N^*(R) - \left[\frac{R}{2}\right]\left[\frac{R+1}{2}\right],$$

which completes the proof of (15).

To prove (16), we observe that, by Lemma 1, the sum

$$\frac{1}{2}\sum_{c\leqslant d} s\left(\frac{c}{d}\right)\left(s\left(\frac{c}{d}\right) - 1\right)$$

coincides with the number of solutions of the equation

$$\begin{pmatrix} * & * \\ Q & Q' \end{pmatrix}\begin{pmatrix} a & m \\ b & n \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} * \\ d \end{pmatrix}, \tag{18}$$

where

$$\begin{pmatrix} * & * \\ Q & Q' \end{pmatrix}, \begin{pmatrix} a & m \\ b & n \end{pmatrix} \in \mathcal{M}, \qquad 1 \leqslant k \leqslant l.$$

If $Q' \geqslant 2$, then property 2 of $\mathcal{M}$ implies that the number of solutions of equation (18) is equal to twice the number of solutions of the equation

$$k(aQ + bQ') + l(mQ + nQ') = d$$

with the restrictions (14). If, on the other hand, $Q' = 1$, then $Q = 1$, $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, and equation (18) has the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & m \\ b & n \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} * \\ d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * \\ a+b & m+n \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} * \\ d \end{pmatrix}. \tag{19}$$

By property 2 (see (5)), the set of pairs $(a+b, m+n)$ coincides with the set of all pairs (Q, Q') such that $1 \leqslant Q < Q'$ and $(Q, Q') = 1$. Hence, equation (19) can be written as

$$kQ + lQ' = d,$$

where $1 \leqslant k \leqslant l$, $1 \leqslant Q < Q'$ and $(Q, Q') = 1$. The number of solutions of this equation is equal to $\lambda(d) - [d/2]$. Hence,

$$\frac{1}{2}\sum_{d\leqslant R}\sum_{c\leqslant d} s\left(\frac{c}{d}\right)\left(s\left(\frac{c}{d}\right) - 1\right) = 2M^*(R) - \sum_{d\leqslant R}\left(\lambda(d) - \left[\frac{d}{2}\right]\right)$$
$$= 2M^*(R) - N^*(R) + \left[\frac{R}{2}\right]\left[\frac{R+1}{2}\right].$$

Therefore,

$$\mathcal{L}_2(R) = 4M^*(R) + \mathcal{L}_1(R) - 2N^*(R) + 2\left[\frac{R}{2}\right]\left[\frac{R+1}{2}\right].$$

Substituting (15) into the last formula, we obtain the desired formula for $\mathcal{L}_2(R)$.

§ 5. Auxiliary assertions

Lemma 3. *Let $\alpha = p(\alpha)/q(\alpha)$ be a rational number, let β, a and b be real numbers, $a \leqslant b$, and let $f(x) = \alpha x + \beta$. Then*

$$\sum_{a<x\leqslant b} \{f(x)\} = \frac{b-a}{2} + O\biggl(\biggl(\frac{b-a}{q(\alpha)} + 1\biggr) s_1(\alpha)\biggr).$$

A proof can be found in [10], § 2, Theorem 2.

Lemma 4.

$$\sum_{a=1}^{b} s_1\left(\frac{a}{b}\right) \ll b\log^2(b+1)$$

for all positive integers b.

A proof can be found in [11].

Lemma 5. *Assume that the function $f(x)$ is twice continuously differentiable on $[a,b]$, and let $\rho(x)$ and $\sigma(x)$ be the functions defined by the equations*

$$\rho(x) = \frac{1}{2} - \{x\}, \qquad \sigma(x) = \int_0^x \rho(u)\,du.$$

Then

$$\sum_{a<x\leqslant b} f(x) = \int_a^b f(x)\,dx + \rho(b)f(b) - \rho(a)f(a) + \sigma(a)f'(a) - \sigma(b)f'(b) + \int_a^b \sigma(x)f''(x)\,dx.$$

A proof can be found in [12], Theorem I, 1.

The next lemma, of which special cases were proved in [13], is based on estimates for Kloosterman sums obtained by Estermann [14].

Lemma 6. *Let $q \geqslant 1$ be a positive integer and let $a(u,v)$ be a function given at the integer points (u,v), where $1 \leqslant u,v \leqslant q$. Assume that the inequalities*

$$a(u,v) \geqslant 0, \qquad \Delta_{1,0}a(u,v) \leqslant 0, \qquad \Delta_{0,1}a(u,v) \leqslant 0, \qquad \Delta_{1,1}a(u,v) \geqslant 0$$

hold for this function at all points at which they are defined. Then the sum

$$W = \sum_{u,v=1}^{q} \delta_q(uv \pm 1)a(u,v)$$

(*with either choice of sign in the symbol* $\pm$) *satisfies the following asymptotic formula*:

$$W = \frac{\varphi(q)}{q^2} \sum_{u,v=1}^{q} a(u,v) + O\big(A\psi(q)\sqrt{q}\,\big),$$

where $\psi(q) = \sigma_0(q)\sigma_{-1/2}(q)\log^2(q+1)$ *and* $A = a(1,1)$ *is the maximal value of* $a(u,v)$.

A proof can be found in [15], Lemma 5.

The following lemma is a special case of Theorem 1 in [16].

Lemma 7. *Let* $n \geqslant 1$, *let* $f(x) \geqslant 0$ *be a function twice continuously differentiable on* $[P_1, P_2] \subset [0,n]$, *and let*

$$\frac{1}{c} \leqslant \big|f''(x)\big| \leqslant \frac{w}{c}$$

for $x \in [P_1, P_2]$ *with some* w, $1 \leqslant w \leqslant c$. *Then*

$$\sum_{P_1 < x \leqslant P_2} \sum_{1 \leqslant y \leqslant f(x)} \delta_n(xy \pm 1) = \frac{1}{n} \sum_{\substack{P_1 < x \leqslant P_2 \\ (x,n)=1}} f(x) + O_{\varepsilon,w}\big((nc^{-1/3} + c^{2/3})n^{\varepsilon}\big).$$

Lemma 8. *For* $R \geqslant 1$, *the sum*

$$\Phi(R) = \sum_{Q' \leqslant R} \sum_{Q \leqslant Q'} \frac{1}{Q'(Q+Q')} \tag{20}$$

satisfies the following asymptotic formula:

$$\Phi(R) = \log 2(\log R + \log 2 + \gamma) - \frac{\zeta(2)}{2} + \frac{1}{R}\bigg(\log\big(2\rho(R)\big) + \frac{1}{4}\bigg) + O\bigg(\frac{1}{R^2}\bigg).$$

Proof. Note that

$$\Phi(R) = \log 2 \sum_{Q' \leqslant R} \frac{1}{Q'} + \sigma_0 - \sum_{Q' > R} \frac{1}{Q'}\bigg(\sum_{Q=1}^{Q'} \frac{1}{Q+Q'} - \log 2\bigg), \tag{21}$$

where

$$\sigma_0 = \sum_{Q'=1}^{\infty} \frac{1}{Q'}\bigg(\sum_{Q=1}^{Q'} \frac{1}{Q+Q'} - \log 2\bigg). \tag{22}$$

Using the method of generating functions, we obtain the exact value of σ_0 (see [4]):

$$\sigma_0 = \log^2 2 - \frac{\zeta(2)}{2}. \tag{23}$$

Moreover, Lemma 5 implies that

$$\sum_{Q' \leqslant R} \frac{1}{Q'} = \log R + \gamma + \frac{\rho(R)}{R} + O\bigg(\frac{1}{R^2}\bigg),$$

$$\sum_{Q=1}^{Q'} \frac{1}{Q+Q'} = \log 2 - \frac{1}{4Q'} + O\bigg(\frac{1}{(Q')^2}\bigg).$$

Substituting the last three equations into formula (21), we complete the proof of the lemma.

Lemma 9. *For $\xi \geqslant 2$, the sum*

$$F(\xi) = \sum_{n<\xi} \sum_{m\leqslant n} \frac{1}{m}\left(\frac{1}{n} - \frac{1}{m+n}\right) - \sum_{n<\xi} \sum_{\substack{m\leqslant n\\ m+n>\xi}} \frac{1}{m}\left(\frac{1}{\xi} - \frac{1}{m+n}\right) \tag{24}$$

satisfies the following asymptotic formula:

$$F(\xi) = \log 2\left(\log\xi + \frac{\log 2}{2} + \gamma - 1\right) + \frac{1}{2\xi}(1-\log 2) + \frac{2\log 2}{\xi}\,\rho\left(\frac{\xi}{2}\right) + O\left(\frac{\log\xi}{\xi^2}\right). \tag{25}$$

Proof. Note that

$$F(\xi) = F_1(\xi) - F_2(\xi) + O\left(\frac{\log\xi}{\xi^2}\right), \tag{26}$$

where

$$F_1(\xi) = \sum_{n\leqslant\xi-1} \sum_{m\leqslant nx} \frac{1}{m}\left(\frac{1}{n} - \frac{1}{m+n}\right) = \Phi(\xi-1),$$

$$F_2(\xi) = \sum_{n\leqslant\xi-1} \sum_{\substack{m\leqslant n\\ m+n>\xi}} \frac{1}{m}\left(\frac{1}{\xi} - \frac{1}{m+n}\right),$$

and $\Phi(R)$ is defined by equation (20). By Lemma 8, we have

$$F_1(\xi) = \log 2(\log\xi + \log 2 + \gamma) - \frac{\zeta(2)}{2} + \frac{1}{\xi}\left(\log\big(2\big(\rho(\xi)-1\big)\big) + \frac{1}{4}\right) + O\left(\frac{1}{\xi^2}\right).$$

Using Lemma 5, we obtain the equation

$$\sum_{\xi-n<m\leqslant n} \frac{1}{m}\left(\frac{1}{\xi} - \frac{1}{m+n}\right) = g(n) + \frac{1}{n}\left(\log 2 + \frac{1}{2\xi}\right) - \frac{1}{4n^2} + O\left(\frac{1}{\xi^2(\xi-n)}\right)$$

for $n > \xi/2$, where

$$g(n) = \frac{1}{\xi}\big(\log n - \log(\xi-n)\big) + \frac{1}{n}\big(\log(\xi-n) - \log\xi\big).$$

Hence,

$$F_2(\xi) = \sum_{\xi/2<n\leqslant\xi-1} g(n) + \left(\log 2 + \frac{1}{2\xi}\right) \sum_{\xi/2<n\leqslant\xi-1} \frac{1}{n} - \frac{1}{4\xi} + O\left(\frac{\log\xi}{\xi^2}\right).$$

Again using Lemma 5, we obtain that

$$\sum_{\xi/2<n\leqslant\xi-1} \frac{1}{n} = \log 2 + \frac{1}{\xi}\left(\rho(\xi) - 1 - 2\rho\left(\frac{\xi}{2}\right)\right) + O\left(\frac{1}{\xi^2}\right),$$

$$\sum_{\xi/2<n\leqslant\xi-1} g(n) = \int_{\xi/2}^{\xi} g(n)\,dn + O\left(\frac{\log\xi}{\xi^2}\right).$$

By making the change of variable $n = x\xi$ and taking into account the equations $\mathrm{Li}_2(1) = \zeta(2)$ and $\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\zeta(2)}{2} - \frac{\log^2 2}{2}$ (see [17]), we obtain that

$$\begin{aligned}\int_{\xi/2}^{\xi} g(n)\,dn &= \int_{1/2}^{1}\left(\log x - \log(1-x) + \frac{\log(1-x)}{x}\right)dx \\ &= \left(x\log x + (1-x)\log(1-x) - \mathrm{Li}_2(x)\right)\Big|_{x=1/2}^{1} = \log 2 - \frac{1}{2}\left(\zeta(2) + \log^2 2\right).\end{aligned}$$

Therefore,

$$F_2(\xi) = \log 2 + \frac{1}{2}\left(\log^2 2 - \zeta(2)\right) + \frac{\log 2}{\xi}\left(\rho(\xi) - 2\rho\left(\frac{\xi}{2}\right) - \frac{1}{2}\right) - \frac{1}{4\xi} + O\left(\frac{\log\xi}{\xi^2}\right).$$

Substituting the asymptotic formulae obtained for $F_1(\xi)$ and $F_2(\xi)$ into (26), we complete the proof of the lemma.

Consider the four sums

$$\sigma_1(\alpha,R) = \sum_{n\leqslant R}\sum_{m=1}^{n}\frac{[\alpha m + n \leqslant R]}{(\alpha m+n)^2}, \qquad \sigma_2(\alpha,R) = \sum_{n\leqslant R}\sum_{m=1}^{n}\frac{1}{(\alpha m+n)^2},$$

$$\sigma_3(\alpha,R) = \sum_{n\leqslant R}\sum_{m=1}^{n}[\alpha m + n \leqslant R], \qquad \sigma_4(\alpha,R) = \sum_{n\leqslant R}\sum_{m=1}^{n}\frac{[\alpha m + n \leqslant R]}{\alpha m+n}$$

and the function

$$h(x) = \sum_{n=1}^{\infty}\frac{\rho(nx)}{n^2}.$$

Lemma 10. *Let* $1 \leqslant U \leqslant R/2$. *Then*

$$\sigma_1(\alpha,R) = \frac{1}{\alpha+1}\left(\log R + c_1(\alpha) + \frac{\alpha}{R}\left(\rho(R) + \frac{1}{2}\right)\right) + O\left(\frac{1}{R^2}\right),$$

$$\sigma_2(\alpha,R) = \frac{1}{\alpha+1}\left(\log R + c_2(\alpha)\frac{1}{R} + \left(\rho(R) + \frac{\alpha^2+\alpha}{\alpha+1}\right)\right) + O\left(\frac{1}{R^2}\right),$$

where

$$c_1(\alpha) = \frac{\log(\alpha+1)}{\alpha} - \frac{\zeta(2)}{2}\frac{\alpha^2+2\alpha}{\alpha+1} + \gamma - 1 + 2\alpha(\alpha+1)\int_0^1 \frac{h(\xi)}{(\alpha\xi+1)^3}\,d\xi, \qquad (27)$$

$$c_2(\alpha) = c_1(\alpha) - \frac{\log(\alpha+1)}{\alpha} + 1.$$

Moreover, for rational $\alpha \in (0,1]$ *with denominator* $q(\alpha)$, *we have*

$$\sigma_3(\alpha,R) = \frac{R^2}{2(\alpha+1)} - \frac{\alpha R}{2(\alpha+1)} + O\left(\left(\frac{R}{q(\alpha)} + 1\right)s_1(\alpha)\right),$$

$$\sigma_4(\alpha,R) = \frac{R}{\alpha+1} - \frac{\alpha}{2(\alpha+1)}\log R + O\left(\left(\frac{\log R}{q(\alpha)} + 1\right)s_1(\alpha)\right).$$

Proof. We compute $\sigma_1(\alpha, R)$ using Lemma 5:

$$\sigma_1(\alpha, R) = \sum_{n \leqslant \frac{R}{\alpha+1}} \sum_{m \leqslant n} \frac{1}{(\alpha m + n)^2} + \sum_{\frac{R}{\alpha+1} < n \leqslant R} \sum_{m \leqslant \frac{R-n}{\alpha}} \frac{1}{(\alpha m + n)^2}$$

$$= \sum_{n \leqslant \frac{R}{\alpha+1}} \left(\int_0^n \frac{dm}{(\alpha m + n)^2} + \frac{1}{2n^2}\left(\frac{1}{(\alpha+1)^2} - 1\right) + 2\alpha \int_0^n \frac{\rho(m)\, dm}{(\alpha m + n)^3} \right)$$

$$+ \sum_{\frac{R}{\alpha+1} < n \leqslant R} \left(\int_0^{(R-n)/\alpha} \frac{dm}{(\alpha m + n)^2} + \frac{\rho(R)}{R} - \frac{1}{2n^2} \right) + O\left(\frac{1}{R^2}\right)$$

$$= \frac{1}{\alpha+1}\left(\log R + c_1(\alpha) + \frac{\alpha}{R}\left(\rho(R) + \frac{1}{2}\right)\right) + O\left(\frac{1}{R^2}\right).$$

The asymptotic formula for $\sigma_2(\alpha, R)$ can be verified likewise.

The sum $\sigma_3(\alpha, R)$ can be written as

$$\sigma_3(\alpha, R) = \sum_{n \leqslant \frac{R}{\alpha+1}} n + \sum_{\frac{R}{\alpha+1} < n \leqslant R} \left[\frac{R-n}{\alpha}\right].$$

Applying Lemma 3 to the second sum in this equation, we obtain that

$$\sigma_3(\alpha, R) = \sum_{n \leqslant R} f(n) - \frac{\alpha R}{2(\alpha+1)} + O\left(\left(\frac{R}{q(\alpha)} + 1\right) s_1(\alpha)\right),$$

where $f(n) = \min\{n, \frac{R-n}{\alpha}\}$. Applying Lemma 5 to the sum of the values of $f(n)$ on each interval of linearity, we obtain the desired formula for $\sigma_3(\alpha, R)$.

Let $1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_P < R$ be the values taken by the form $\alpha m + n$ when $1 \leqslant m \leqslant n$ and $\alpha m + n \leqslant R$. Then

$$\sigma_4(\alpha, R) = \sum_{j=1}^{P} \frac{1}{\lambda_j}.$$

We apply the Abel transformation in the integral form

$$\sum_{j=1}^{P} a_j g(\lambda_j) = A(\lambda_P) g(\lambda_P) - \int_{\lambda_1}^{\lambda_P} A(t) g'(t)\, dt \tag{28}$$

to this sum, where

$$A(t) = \sum_{\lambda_j \leqslant t} a_j.$$

To do this, we put $a_j = 1$, $j = 1, \dots, P$ and $g(t) = 1/t$. Then $A(t) = \sigma_3(\alpha, t)$ and

$$\sigma_4(\alpha, R) = \frac{\sigma_3(\alpha, \lambda_P)}{\lambda_P} + \int_{\lambda_1}^{\lambda_P} \frac{\sigma_3(\alpha, t)}{t^2}\, dt.$$

Since $\lambda_1 > 1$, $\lambda_P - R \ll 1$, $\sigma_3(\alpha, R) \ll R^2$ and $\sigma_3(\alpha, \lambda_P) = \sigma_3(\alpha, R)$, we have

$$\begin{aligned}\sigma_4(\alpha, R) &= \frac{\sigma_3(\alpha, R)}{R} + \int_1^R \frac{\sigma_3(\alpha, t)}{t^2}\, dt + O(1)\\ &= \frac{R}{\alpha+1} - \frac{\alpha}{2(\alpha+1)} \log R + O\biggl(\biggl(\frac{\log R}{q(\alpha)} + 1\biggr) s_1(\alpha)\biggr).\end{aligned}$$

The lemma is proved.

Lemma 11. *The function $c_1(\alpha)$ given by formula (27) satisfies the equation*

$$\int_0^1 \frac{c_1(\alpha)}{\alpha+1}\, d\alpha = \log 2\biggl(\gamma - 1 + \frac{\log 2}{2}\biggr). \tag{29}$$

Proof. If $F(x) = \int f(x)\, dx$, then

$$\int_0^1 \rho(nx) f(x)\, dx = n \int_0^1 F(x)\, dx - {\sum_{k=0}^{n}}' F\biggl(\frac{k}{n}\biggr)$$

(the prime in the sum means that for $k = 0$ and $k = n$, the terms are taken with coefficient $1/2$). Thus, the integral involving $h(\xi)$ can be expressed in terms of the sum (22), and its computation reduces to equation (23):

$$\begin{aligned}\int_0^1 \frac{h(x)}{(x+1)^2}\, dx &= \sum_{q=1}^{\infty} \frac{1}{q}\biggl(\frac{1}{q+1} + \dots + \frac{1}{2q} - \log 2 + \frac{1}{4q}\biggr)\\ &= \sigma_0 + \frac{\zeta(2)}{4} = \log^2 2 - \frac{\zeta(2)}{4}.\end{aligned} \tag{30}$$

The other integrals can be computed directly.

Given a matrix $S = \left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathcal{M}$, we denote by $f_S(t)$ the function

$$f_S(t) = \frac{1}{(mt+n)\bigl((a+m)t + (b+n)\bigr)}. \tag{31}$$

The prime in sums of the form ${\sum_{b,m=1}^{n}}' \delta_n(bm \pm 1) \dots$ means that (in accordance with property 2 of $\mathcal{M}$) for $n = 1$ we take the minus sign in the symbol $\pm$, while for $n > 1$ we consider both signs independently. By a in these sums we mean the fraction $\frac{bm \pm 1}{n}$.

For real $U \geqslant 1$ we consider the sums

$$A_0(U) = \sum_{S\in\mathcal{M}(U)} f_S(0), \qquad A_1(U) = \sum_{S\in\mathcal{M}(U)} f_S(1),$$

$$B(U,t) = \sum_{S\in\mathcal{M}(U)} f'_S(t), \qquad W_1(U) = \sum_{S\in\mathcal{M}(U)} \int_0^1 f_S(t)\,dt,$$

$$W_2(U) = \sum_{S\in\mathcal{M}(U)} \int_0^1 \log(mt+n) f_S(t)\,dt,$$

$$W_3(U) = \sum_{S\in\mathcal{M}(U)} \int_0^1 c_1\left(\frac{at+b}{mt+n}\right) f_S(t)\,dt.$$

Lemma 12. *For $U \geqslant 2$ we have*

$$A_0(U) = \frac{2\log 2}{\zeta(2)}\log U + C_{A_0} + O\left(\frac{\log U}{U}\right),$$

$$A_1(U) = \frac{\log 2}{\zeta(2)}\log U + C_{A_1} + O\left(\frac{\log^5 U}{U^{1/2}}\right),$$

$$B(U,t) = -\frac{2\log 2}{\zeta(2)(t+1)^2}\log U + C_B(t) + O\left(\frac{\log^5 U}{U^{1/2}}\right),$$

$$W_1(U) = \frac{2\log^2 2}{\zeta(2)}\left(\log U + \gamma - \frac{\zeta'(2)}{\zeta(2)}\right) + C_1 + O\left(\frac{\log^5 U}{U^{1/2}}\right),$$

$$W_2(U) = \frac{\log^2 2}{\zeta(2)}\log^2 U + \frac{\log^2 2}{\zeta(2)}\left(2 + \log 2 - \frac{\zeta(2)}{\log 2}\right)\log U + C_2 + O\left(\frac{\log^6 U}{U^{1/2}}\right),$$

$$W_3(U) = \frac{2\log^2 2}{\zeta(2)}\left(\gamma - 1 + \frac{\log 2}{2}\right)\log U + C_3 + O\left(\frac{\log^5 U}{U^{1/2}}\right),$$

where $C_B(t)$ is a continuous function of t and C_{A_0}, C_{A_1}, C_1, C_2, C_3 are absolute constants, and

$$C_1 = \sum_{n=1}^{\infty}\left(\sum_{b,m=1}^{n}{}' \delta_n(bm\pm 1)\int_0^1 f_S(\xi)\,d\xi - 2\log^2 2\cdot\frac{\varphi(n)}{n^2}\right). \tag{32}$$

Proof. The asymptotic formulae for $A_0(U)$, $A_1(U)$, $B(U,t)$, $W_1(U)$ and $W_2(U)$ were proved in [18], Lemma 6, Remark 2, Corollaries 1 and 2.

We write $W_3(U)$ as

$$W_3(U) = \sum_{n\leqslant U} w_3(n),$$

where

$$w_3(n) = \sum_{b,m=1}^{n}{}' \delta_n(bm\pm 1)\int_0^1 c_1\left(\frac{at+b}{mt+n}\right)\frac{dt}{(mt+n)\big((a+m)t+(b+n)\big)}$$

$$= \sum_{b,m=1}^{n}{}' \delta_n(bm\pm 1)c_1\left(\frac{b}{n}\right)\int_0^1 \frac{dt}{\left(\frac{b}{n}+1\right)(mt+n)^2} + O\left(\frac{1}{n^3}\right).$$

Lemma 6 cannot be applied to this double sum directly, but the assumptions of Lemma 6 hold for each of the sums

$$\sum_{b,m=1}^{n}{}' \delta_n(bm\pm 1)C_0\int_0^1 \frac{dt}{\left(\frac{b}{n}+1\right)(mt+n)^2},$$
$$\sum_{b,m=1}^{n}{}' \delta_n(bm\pm 1)\left(C_0+c_1\left(\frac{b}{n}\right)\right)\int_0^1 \frac{dt}{\left(\frac{b}{n}+1\right)(mt+n)^2}$$

for sufficiently large C_0. Therefore,

$$\begin{aligned} w_3(n) &= 2\frac{\varphi(n)}{n^2}\sum_{b,m=1}^{n} c_1\left(\frac{b}{n}\right)\int_0^1 \frac{dt}{\left(\frac{b}{n}+1\right)(mt+n)^2}+O\left(\frac{\psi(n)}{n^{3/2}}\right)\\ &= 2\log 2\frac{\varphi(n)}{n^2}\int_0^1 \frac{c_1(\alpha)}{\alpha+1}\,d\alpha+O\left(\frac{\psi(n)}{n^{3/2}}\right). \end{aligned}$$

Formula (29) implies that

$$w_3(n)=2\log^2 2\left(\gamma-1+\frac{\log 2}{2}\right)\frac{\varphi(n)}{n^2}+O\left(\frac{\psi(n)}{n^{3/2}}\right).$$

Hence,

$$W_3(U)=2\log^2 2\left(\gamma-1+\frac{\log 2}{2}\right)\sum_{n\leqslant U}\frac{\varphi(n)}{n^2}+C_3'+O\left(\frac{\log^5 U}{U^{1/2}}\right),$$

where

$$C_3'=\sum_{n=1}^{\infty}\left(w_3(n)-2\log^2 2\left(\gamma-1+\frac{\log 2}{2}\right)\frac{\varphi(n)}{n^2}\right).$$

Substituting the relations

$$\sum_{n\leqslant U}\frac{\varphi(n)}{n^2}=\frac{1}{\zeta(2)}\left(\log U+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\log U}{U}\right)$$

into this equation, we complete the proof of the lemma.

Lemma 13. *Let R, $U\geqslant 2$ and let R be a half-integer. Then the sum*

$$W_4(R,U)=\sum_{S\in\mathcal{M}(U)}\sum_{q\leqslant R}\sum_{x=1}^{q}\frac{1}{q^2}f_S\left(\frac{x}{q}\right),$$

where the function $f_S(t)$ *is defined by equation* (31), *satisfies the asymptotic formula*

$$W_4(R,U)=\frac{2\log^2 2}{\zeta(2)}\log R\log U+\left(\frac{2\log^2 2}{\zeta(2)}\left(\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+C_1\right)\log R+C_4$$
$$+\frac{2\log^2 2}{\zeta(2)}\left(\log 2+\gamma-\frac{\zeta(2)}{2\log 2}\right)\log U$$
$$+\frac{1}{2R}\left(\frac{\log 2}{\zeta(2)}\log U+C_{A_0}-C_{A_1}\right)$$
$$+O\left(\frac{(\log^5 U)\log R}{U^{1/2}}\right)+O\left(\frac{\log U}{R^2}\right),$$

where C_4 *is an absolute constant and* C_{A_0}, C_{A_1} *are the constants occurring in Lemma* 12.

Proof. Applying Lemma 5 to the function

$$g(x)=\frac{1}{(mx+nq)\big((a+m)x+(b+n)q\big)}=\frac{1}{q^2}f_S\left(\frac{x}{q}\right),$$

we obtain that

$$\sum_{x=1}^{q}\frac{1}{q^2}f_S\left(\frac{x}{q}\right)=\frac{1}{q}\int_0^1 f_S(t)\,dt+\frac{1}{2q^2}\big(f(1)-f(0)\big)-\frac{1}{q^2}\int_0^1\rho(qt)f_S'(t)\,dt.$$

Taking into account the relation

$$\int_0^1\rho(qt)f_S'(t)\,dt=-\frac{1}{q}\int_0^1\sigma(qt)f_S''(t)\,dt\ll\frac{1}{n^2q}$$

and using Lemma 5, we obtain that

$$W_4(R,U)=(\log R+\gamma)W_1(U)+\frac{1}{2}\left(\zeta(2)-\frac{1}{R}\right)\big(A_1(U)-A_0(U)\big)$$
$$-\int_0^1 h(t)B(U,t)\,dt+O\left(\frac{\log U}{R^2}\right).$$

Substituting the asymptotic formulae obtained in Lemma 12 into this equation and using formula (30), we complete the proof of the lemma.

Lemma 14. *Let* R, $U\geqslant 2$. *Then the sum*

$$W_5(R,U)=\sum_{\left(\begin{smallmatrix}a&m\\b&n\end{smallmatrix}\right)\in\mathcal{M}(U)}\sum_{l\leqslant R}\sum_{k\leqslant l}\left(\frac{R}{bk+nl}\right.$$
$$\left.-\max\left\{1,\frac{R}{(a+b)k+(m+n)l}\right\}\right)[bk+nl\leqslant R]$$

satisfies the following asymptotic formula:

$$W_5(R,U)=\frac{\log 2}{2\zeta(2)}R^2\log U+\frac{R^2}{2}(C_{A_0}-C_{A_1})+O(R^2U^{-1/2}\log^5 U)+O(RU\log^2 U),$$

where C_{A_0} *and* C_{A_1} *are the constants occurring in Lemma* 12.

Proof. We write $W_5(R,U)$ as

$$W_5(R,U) = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \left(\sigma_5\left(\frac{b}{n}, \frac{R}{n}\right) - \sigma_5\left(\frac{a+b}{m+n}, \frac{R}{m+n}\right)\right),$$

where

$$\sigma_5(\alpha, R) = R\sigma_4(\alpha, R) - \sigma_3(\alpha, R) = \frac{R^2}{2(\alpha+1)} + O\left(\left(\frac{R^{3/2}}{q(\alpha)} + R\right)s_1(\alpha)\right)$$

in the notation of Lemma 10. By Lemma 4, we have

$$\begin{aligned} W_5(R,U) &= \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \left(\frac{R^2}{2n(b+n)} - \frac{R^2}{2(m+n)(a+b+m+n)}\right. \\ &\qquad \left. + O\left(\left(\frac{R^{3/2}}{n^{5/2}} + \frac{R}{n}\right)s_1\left(\frac{b}{n}\right)\right)\right) \\ &= \frac{R^2}{2}(A_0(U) - A_1(U)) + O(R^{3/2}) + O(RU\log^2 U). \end{aligned}$$

Substituting the asymptotic formulae for $A_1(U)$ and $A_0(U)$ (see Lemma 12) into the last equation and observing that $R^{3/2} \ll RU + R^2U^{-1/2}$, we obtain the desired formula for $W_5(R,U)$.

Lemma 15. *Let R, $U \geqslant 2$. Then the sums*

$$W_6(R,U) = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \sum_{Q'\leqslant R} \sum_{Q\leqslant Q'} \frac{1}{mQ + nQ'},$$

$$W_7(R,U) = \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \sum_{Q'\leqslant R} \sum_{Q\leqslant Q'} \frac{1}{(a+m)Q + (b+n)Q'}$$

satisfy the following asymptotic formulae:

$$\begin{aligned} W_6(R,U) &= RU + O(U\log R) + O(RU^{1/2}\log^5 U), \\ W_7(R,U) &= RU\log 2 + O(U\log R) + O(RU^{1/2}\log^5 U). \end{aligned}$$

Proof. We prove the assertion of the lemma for $W_7(R,U)$. The formula for $W_6(R,U)$ can be verified likewise.

Using the equation

$$\sum_{Q'\leqslant R} \sum_{Q\leqslant Q'} \frac{1}{(a+m)Q + (b+n)Q'} = R\int_0^1 \frac{dt}{(mt+n)\left(1+\frac{b}{n}\right)} + O\left(\frac{\log R}{n}\right),$$

we obtain that

$$W_7(R,U) = R\sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \int_0^1 \frac{dt}{(mt+n)\left(1+\frac{b}{n}\right)} + O(U\log R).$$

By Lemma 6, we have

$$W_7(R,U) = 2\log 2\int_0^1 \frac{\log(1+t)}{t}\,dt\cdot R\sum_{n\leqslant U}\left(\frac{\varphi(n)}{n}+O\left(\frac{\psi(n)}{n^{1/2}}\right)\right)+O(U\log R).$$

Using the equations

$$\int_0^1 \frac{\log(1+t)}{t}\,dt = -\operatorname{Li}_2(-1) = \frac{\zeta(2)}{2}, \qquad \sum_{n\leqslant U}\frac{\varphi(n)}{n} = \frac{U}{\zeta(2)}+O(\log U),$$

we obtain the desired formula for $W_7(R,U)$.

Lemma 16. *Let* $2\leqslant U$, $R_1\leqslant R$ *and*

$$W_8(R_1,U) = \sum_{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathcal{M}(U)}\int_0^1 \sigma_3\left(\frac{at+b}{mt+n},\frac{R_1}{mt+n}\right)dt.$$

Then

$$W_8(R_1,U) = \frac{R_1^2}{2}W_1(U)-\frac{R_1U}{2}(1-\log 2)+O(R_1U^{1/2}\log^5 R)+O(U^2\log^2 R). \tag{33}$$

Proof. Let $p\geqslant 2$ be a prime number. By definition, $\sigma_3\left(\frac{at+b}{mt+n},\frac{R_1}{mt+n}\right)$ is a non-increasing function of t and $\sigma_3\left(\frac{at+b}{mt+n},\frac{R_1}{mt+n}\right)\ll R_1^2/n^2$. Therefore,

$$W_8(R_1,U) = \frac{1}{p}\sum_{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathcal{M}(U)}\sum_{x=1}^{p-1}\sigma_3\left(\frac{\frac{ax}{p}+b}{\frac{mx}{p}+n},\frac{R_1}{\frac{mx}{p}+n}\right)+O\left(\frac{R_1^2\log R}{p}\right).$$

Using Lemma 10 and observing that

$$q\left(\frac{ax+bp}{mx+np}\right) = mx+np\geqslant np, \qquad s_1\left(\frac{ax+bp}{mx+np}\right)\ll s_1\left(\frac{m}{n}\right)+s_1\left(\frac{x}{p}\right),$$

we obtain the following representation for $W_8(R_1,U)$:

$$\begin{aligned}W_8(R_1,U) = \frac{1}{2p}&\sum_{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathcal{M}(U)}\sum_{x=1}^{p-1}\left(R_1^2 f_S\left(\frac{x}{p}\right)-\left(\frac{R_1}{\frac{mx}{p}+n}-\frac{R_1}{(a+m)\frac{x}{p}+b+n}\right)\right)\\ &+O\left(\frac{1}{p}\sum_{n\leqslant U}\sideset{}{^*}\sum_{m\leqslant n}\sum_{x=1}^{p-1}\left(\frac{R_1}{n^2p}+1\right)\left(s_1\left(\frac{m}{n}\right)+s_1\left(\frac{x}{p}\right)\right)\right)\\ &+O\left(\frac{R_1^2\log R}{p}\right).\end{aligned}$$

By Lemma 4, we have

$$\frac{1}{p}\sum_{n\leqslant U}\sideset{}{^*}\sum_{m\leqslant n}\sum_{x=1}^{p-1}\left(\frac{R_1}{n^2p}+1\right)\left(s_1\left(\frac{m}{n}\right)+s_1\left(\frac{x}{p}\right)\right)\ll\left(\frac{R_1}{p}\log R+U^2\right)\log^2 R.$$

Hence, for all p in the range $R_1^2 \leqslant p \leqslant 2R_1^2$, we have

$$W_8(R_1,U) = \frac{1}{2} \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathcal{M}(U)} \int_0^1 \left(R_1^2 f_S(t) - \left(\frac{R_1}{mt+n} - \frac{R_1}{(a+m)t+b+n} \right) \right) dt + O(U^2 \log^2 R).$$

As in the proof of Lemma 15, we obtain that

$$\sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathcal{M}(U)} \int_0^1 \frac{dt}{mt+n} = U + O(U^{1/2} \log^5 U),$$

$$\sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right) \in \mathcal{M}(U)} \int_0^1 \frac{dt}{(a+m)t+b+n} = U \log 2 + O(U^{1/2} \log^5 U).$$

Hence,

$$W_8(R_1,U) = \frac{R_1^2}{2} W_1(U) - \frac{R_1 U}{2}(1-\log 2) + O(R_1 U^{1/2} \log^5 R) + O(U^2 \log^2 R).$$

§ 6. Asymptotic formula for the expectation

We denote by $N(R)$ the number of solutions of the inequality

$$kQ + lQ' \leqslant R \tag{34}$$

for which

$$1 \leqslant k \leqslant l, \qquad 1 \leqslant Q \leqslant Q'. \tag{35}$$

Theorem 1. *Let $R \geqslant 2$. Then*

$$N(R) = \frac{\log 2}{2} R^2 \log R + \frac{R^2}{4} \big(\log 2(3\log 2 + 4\gamma - 3) - \zeta(2)\big) + O(R \log^4 R).$$

Proof. Let U be a half-integer such that $1 \leqslant U \leqslant R$. We denote by $N_1(R,U)$ the number of solutions of the inequality (34) with the restrictions (35) for which the supplementary condition $Q' \leqslant U$ holds. The number of solutions such that $Q' > U$ will be denoted by $N_2(R,U)$. Hence,

$$N(R) = N_1(R,U) + N_2(R,U). \tag{36}$$

To compute $N_1(R,U)$, we observe that

$$N_1(R,U) = \sum_{d \leqslant U} N_1^*\left(\frac{R}{d}, \frac{U}{d}\right), \tag{37}$$

where

$$N_1^*(R,U) = \sum_{Q' \leqslant U} \sideset{}{^*}\sum_{Q \leqslant Q'} \sum_{l \leqslant R/Q'} \sum_{k \leqslant l} [kQ + lQ' \leqslant R].$$

By Lemma 3, we have

$$\sum_{l\leqslant R/Q'}\sum_{k\leqslant l}[kQ+lQ'\leqslant R]=\sum_{l\leqslant R/(Q+Q')}l+\sum_{R/(Q+Q')\leqslant l\leqslant R/Q'}\left[\frac{R-lQ'}{Q}\right]$$
$$=\sum_{l\leqslant R/Q'}\min\left\{l,\frac{R-lQ'}{Q}\right\}-\frac{1}{2}\left(\frac{R}{Q'}-\frac{R}{Q+Q'}\right)$$
$$+O\left(\left(\frac{R}{(Q')^2}+1\right)s_1\left(\frac{Q}{Q'}\right)\right).$$

It follows from Lemma 5 that

$$\sum_{l\leqslant R/Q'}\min\left\{l,\frac{R-lQ'}{Q}\right\}=\int_0^{R/Q'}\min\left\{l,\frac{R-lQ'}{Q}\right\}dl+O\left(\frac{Q'}{Q}\right)$$
$$=\frac{R}{2Q'(Q+Q')}+O\left(\frac{Q'}{Q}\right).$$

Lemma 4 implies that

$$N_1^*(R,U)=\frac{R^2}{2}\sum_{Q'\leqslant U}\sideset{}{^*}\sum_{Q\leqslant Q'}\frac{1}{Q'(Q+Q')}$$
$$-\frac{RU}{2\zeta(2)}(1-\log 2)+O(R\log^3 R)+O(U^2\log^2 R).$$

Formula (37) implies that

$$N_1(R,U)=\frac{R^2}{2}\Phi(U)-\frac{RU}{2}(1-\log 2)+O(R\log^4 R)+O(U^2\log^2 R),$$

where the function $\Phi(R)$ is given by (20). Using Lemma 8, we obtain the final formula for $N_1(R,U)$:

$$N_1(R,U)=\frac{R^2}{2}\left(\log 2(\log U+\log 2+\gamma)-\frac{\zeta(2)}{2}\right)+\frac{R^2}{8U}-\frac{RU}{2}(1-\log 2)$$
$$+O(R\log^4 R)+O(U^2\log^2 R)+O(R^2U^{-2}). \tag{38}$$

Let $R_1=RU^{-1}$. For $N_2(R,U)$ we likewise obtain

$$N_2(R,U)=\sum_{d\leqslant R_1}N_2^*\left(\frac{R}{d},U\right),$$

where

$$N_2^*(R,U)=\sum_{l\leqslant R_1}\sideset{}{^*}\sum_{k\leqslant l}\sum_{U<Q'\leqslant R/l}\sum_{Q\leqslant Q'}[kQ+lQ'\leqslant R]$$
$$=\sum_{l\leqslant R_1}\sideset{}{^*}\sum_{k\leqslant l}\left(\sum_{U<Q'\leqslant R/(k+l)}Q'+\sum_{\max\{U,R/(k+l)\}<Q'\leqslant R/l}\left[\frac{R-lQ'}{k}\right]\right).$$

Successively using Lemmas 3 and 4, we obtain that

$$N_2^*(R,U) = \sum_{l\leqslant R_1} \sideset{}{^*}\sum_{k\leqslant l} \sum_{U<Q'\leqslant R/l} \min\left\{Q', \frac{R-lQ'}{k}\right\} - \frac{1}{2}\sum_{l\leqslant R_1} \sideset{}{^*}\sum_{\substack{k\leqslant l\\ k+l\leqslant R_1}} \left(\frac{R}{l} - \frac{R}{k+l}\right)$$
$$- \frac{1}{2}\sum_{l\leqslant R_1} \sum_{\substack{k\leqslant l\\ k+l> R_1}} \left(\frac{R}{l} - U\right) + O(R\log^3 R) + O(R_1^2 \log^2 R).$$

Hence,

$$N_2(R,U) = \sum_{l\leqslant R_1} \sum_{k\leqslant l} \sum_{U<Q'\leqslant R/l} \min\left\{Q', \frac{R-lQ'}{k}\right\} - \frac{1}{2}\sum_{l\leqslant R_1} \sum_{\substack{k\leqslant l\\ k+l\leqslant R_1}} \left(\frac{R}{l} - \frac{R}{k+l}\right)$$
$$- \frac{1}{2}\sum_{l\leqslant R_1} \sum_{\substack{k\leqslant l\\ k+l> R_1}} \left(\frac{R}{l} - U\right) + O(R\log^4 R) + O(R^2U^{-2} \log^2 R)$$
$$= \sum_{l\leqslant R_1} \sum_{k\leqslant l} \sum_{U<Q'\leqslant R/l} \min\left\{Q', \frac{R-lQ'}{k}\right\} - \frac{R^2}{8U}$$
$$+ O(R\log^4 R) + O(R^2U^{-2} \log^2 R).$$

By Lemma 5, we have

$$N_2(R,U) = \sum_{l\leqslant R_1} \sum_{k\leqslant l} \int_U^R \int_0^{Q'} [kQ + lQ' \leqslant R]\, dQ\, dQ'$$
$$- \frac{R^2}{8U} + O(R\log^4 R) + O(R^2U^{-2} \log^2 R).$$

To compute the double integral, we successively make the changes of variables $w = kQ + lQ',\ \xi = wU^{-1},\ y = Q'U^{-1}$ and obtain that

$$\int_U^R \int_0^{Q'} [kQ + lQ' \leqslant R]\, dQ\, dQ' = \frac{1}{k}\int_U^R \int_0^R \left[\frac{w}{k+l} \leqslant Q' \leqslant \frac{w}{l}\right] dw\, dQ'$$
$$= \frac{U^2}{k}\int_1^{R_1}\int_0^{R_1} \left[\frac{\xi}{k+l} \leqslant y \leqslant \frac{\xi}{l}\right] d\xi\, dy$$
$$= \frac{U^2}{k}\int_0^{R_1} \xi\left(\frac{1}{l} - \max\left\{\frac{1}{k+l}, 1\right\}\right)[\xi \geqslant l]\, d\xi$$
$$= \frac{U^2}{k}\int_0^{R_1} \xi\left(\frac{1}{l} - \frac{1}{k+l}\right)[\xi \geqslant k+l]\, d\xi$$
$$+ \frac{U^2}{k}\int_0^{R_1} \xi\left(\frac{1}{l} - \frac{1}{\xi}\right)[l \leqslant \xi < k+l]\, d\xi.$$

Hence,

$$N_2(R,U) = U^2 \int_0^{R_1} \xi F(\xi)\,d\xi - \frac{R^2}{8U} + O(R\log^4 R) + O(R^2U^{-2}\log^2 R),$$

where the function $F(\xi)$ is defined by (24). By Lemma 9, we have

$$N_2(R,U) = \frac{\log 2}{2}R^2\biggl(\log\frac{R}{U} + \frac{\log 2}{2} + \gamma - \frac{3}{2}\biggr) + \frac{RU}{2}(1-\log 2) - \frac{R^2}{8U} + O(R\log^4 R) + O(R^2U^{-2}\log^2 R) + O(U^2\log^2 R). \qquad (39)$$

Substituting formulae (38) and (39) into (36) and putting $U = [R^{1/2}] + 1/2$, we complete the proof of the theorem.

Corollary 1. *Let $R \geqslant 2$. Then*

$$\mathsf{E}(R) = \frac{2\log 2}{\zeta(2)}\log R + \frac{\log 2}{\zeta(2)}\biggl(3\log 2 + 4\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - 3\biggr) - \frac{3}{2} + O(R^{-1}\log^5 R).$$

Proof. Substituting the result of Theorem 1 into the inversion formula

$$N^*(R) = \sum_{d\leqslant R}\mu(d)N\biggl(\frac{R}{d}\biggr),$$

we obtain that

$$N^*(R) = \frac{\log 2}{2\zeta(2)}R^2\log R + \frac{R^2}{4\zeta(2)}\biggl(\log 2\biggl(3\log 2 + 4\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - 3\biggr) - \zeta(2)\biggr) + O(R\log^5 R).$$

Substituting this result into the first equation in Lemma 2, we obtain the equation

$$\mathcal{L}_1(R) = \frac{\log 2}{\zeta(2)}R^2\log R + \frac{R^2}{2\zeta(2)}\biggl(\log 2\biggl(3\log 2 + 4\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - 3\biggr) - \frac{3}{2}\zeta(2)\biggr) + O(R\log^5 R).$$

Substituting this into (9), we complete the proof of the corollary.

§ 7. Computation of two auxiliary quantities

To compute the variance $\mathsf{D}(R)$, we have to know asymptotic formulae for two auxiliary quantities. They can be studied in the same way as $N(R)$.

Let $\alpha, \beta \in [0,1]$ be real numbers. We denote by $T(R) = T(\alpha,\beta;R)$ the sum

$$T(R) = \sum_{l,Q'\leqslant R}\sum_{k\leqslant l}\sum_{Q\leqslant Q'}\bigl[(\alpha k + l)(\beta Q + Q') \leqslant R\bigr],$$

which obviously coincides with the number of solutions of the inequality

$$(\alpha m+n)(\beta k+l)\leqslant R \tag{40}$$

in k, l, m and n for which

$$1\leqslant k\leqslant l,\qquad 1\leqslant m\leqslant n. \tag{41}$$

Lemma 17. *Let $\alpha,\beta\in(0,1]$ be rational numbers with denominators $q(\alpha)$ and $q(\beta)$, respectively. Then*

$$T(R)=\frac{R^2}{2(\alpha+1)(\beta+1)}\left(\log R+c_1(\alpha)+c_1(\beta)-\frac{1}{2}\right)\\ +O\left(R^{3/2}\left(\frac{s_1(\alpha)}{q(\alpha)}+\frac{s_1(\beta)}{q(\beta)}\right)+R\big(s_1(\alpha)+s_1(\beta)\big)\right)$$

for all $R\geqslant 1$, where $c_1(\alpha)$ and $c_1(\beta)$ are defined by (27).

Proof. We can assume without loss of generality that $\alpha\leqslant\beta$. Put $U=\left[\sqrt{R}\right]+1/2$. We denote by $T_1(R,U)$ the number of solutions of the inequality (40) with the restrictions (41) for which $l\leqslant U$. The number of solutions with $l>U$ will be denoted by $T_2(R,U)$. Thus, we have

$$T(R)=T_1(R,U)+T_2(R,U). \tag{42}$$

To determine $T_1(R,U)$, we observe that, by Lemma 3, for fixed k and l the number of solutions of the inequality (40) in m and n is equal to

$$\frac{1}{2(\alpha+1)}\left(\frac{R}{\beta k+l}\right)^2-\frac{1}{2}\frac{R}{\beta k+l}\left(1-\frac{1}{\alpha+1}\right)+O\left(\left(\frac{R}{q(\alpha)l}+1\right)s_1(\alpha)\right).$$

Therefore,

$$T_1(R,U)=\frac{R^2}{2(\alpha+1)}\sigma_2(\beta,U)-\frac{R\alpha}{2(\alpha+1)}\sum_{l\leqslant U}\sum_{k=1}^{l}\frac{1}{\beta k+l}+O\left(\left(\frac{R^{3/2}}{q(\alpha)}+R\right)s_1(\alpha)\right).$$

Using Lemma 10 and the equation

$$\sum_{l\leqslant U}\sum_{k=1}^{l}\frac{1}{\beta k+l}=\frac{\log(\beta+1)}{\beta}U+O(\log U),$$

which follows from Lemma 5, we obtain that

$$T_1(R,U)=\frac{R^2}{2(\alpha+1)(\beta+1)}\left(\log U+c_1(\beta)-\frac{\log(\beta+1)}{\beta}+1+\frac{1}{U}\frac{\beta^2+\beta}{2(\beta+1)}\right)\\ -\frac{R\alpha}{2(\alpha+1)}\frac{\log(\beta+1)}{\beta}+O\left(\left(\frac{R^{3/2}}{q(\alpha)}+R\right)s_1(\alpha)\right).$$

Putting $R_1 = RU^{-1}$, we likewise obtain the following formula for $T_2(R,U)$:

$$T_2(R,U) = \sum_{n\leqslant R_1}\sum_{m\leqslant n}\sum_{U<l\leqslant \frac{R}{\alpha m+n}} \min\left\{l, \left[\frac{1}{\beta}\left(\frac{R}{\alpha m+n}-l\right)\right]\right\}$$

$$= \sum_{n\leqslant R_1}\sum_{m\leqslant n}\sum_{U<l\leqslant \frac{R}{\alpha m+n}} \min\left\{l, \frac{1}{\beta}\left(\frac{R}{\alpha m+n}-l\right)\right\}$$

$$-\frac{1}{2}\sum_{n\leqslant R_1}\sum_{m\leqslant n}\left(\frac{R}{\alpha m+n} - \max\left\{U, \frac{R}{(\alpha m+n)(\beta+1)}\right\}\right)$$

$$+ O\left(\left(\frac{R}{q(\beta)n}+1\right)s_1(\beta)\right).$$

Using Lemma 5 and taking into account that U is a half-integer, we obtain that

$$T_2(R,U) = \sum_{n\leqslant R_1}\sum_{m\leqslant n}\int_U^R\int_0^l \big[(\alpha m+n)(\beta k+l)\leqslant R\big]\,dk\,dl$$

$$-\frac{U}{2}\left(\sigma_5(\alpha,R_1) - \sigma_5\left(\alpha, \frac{R_1}{\beta+1}\right)\right) + O\left(\left(\frac{R^{3/2}}{q(\beta)}+R\right)s_1(\beta)\right), \quad (43)$$

where, in the notation of Lemma 10,

$$\sigma_5(\alpha,R) = R\sigma_4(\alpha,R) - \sigma_3(\alpha,R) = \frac{R^2}{2(\alpha+1)} + O\left(\left(\frac{R^{3/2}}{q(\alpha)}+R\right)s_1(\alpha)\right),$$

whence

$$\frac{U}{2}\left(\sigma_5(\alpha,R_1) - \sigma_5\left(\alpha,\frac{R_1}{\beta+1}\right)\right) = \frac{\beta^2+2\beta}{4(\alpha+1)(\beta+1)^2}\frac{R^2}{U} + O\left(\left(\frac{R^{3/2}}{q(\alpha)}+R\right)s_1(\alpha)\right). \quad (44)$$

To compute the double integral in formula (43), we make the changes of variables $k = tl$, $l = U\xi$ and obtain (in the notation of Lemma 10) that

$$\sum_{n\leqslant R_1}\sum_{m\leqslant n}\int_U^R\int_0^l \big[(\alpha m+n)(\beta k+l)\leqslant R\big]\,dk\,dl$$

$$= U^2\int_0^1\sum_{n\leqslant R_1}\sum_{m\leqslant n}\int_1^{R_1}\xi\big[\xi(\alpha m+n)(\beta t+1)\leqslant R_1\big]\,d\xi\,dt$$

$$= \frac{U^2}{2}\int_0^1\sum_{n\leqslant R_1}\sum_{m\leqslant n}\left(\frac{R_1^2}{(\beta t+1)^2(\alpha m+n)^2}-1\right)\big[(\alpha m+n)(\beta t+1)\leqslant R_1\big]\,dt$$

$$= \frac{R^2}{2}\int_0^1\frac{dt}{(\beta t+1)^2}\cdot\sigma_1\left(\alpha,\frac{R_1}{\beta t+1}\right) - \frac{U^2}{2}\int_0^1\sigma_3\left(\alpha,\frac{R_1}{\beta t+1}\right)dt.$$

By Lemma 10, we have

$$\begin{aligned}
\int_0^1 & \frac{dt}{(\beta t+1)^2}\cdot\sigma_1\Bigl(\alpha,\frac{R_1}{\beta t+1}\Bigr) \\
&= \frac{1}{(\alpha+1)(\beta+1)}\Bigl(\log R_1 + c_1(\alpha) + \frac{\log(\beta+1)}{\beta} - 1\Bigr) \\
&\qquad + \frac{\alpha}{(\alpha+1)R_1}\Bigl(\frac{\log(\beta+1)}{2\beta} + \int_0^1 \frac{dt}{\beta t+1}\rho\Bigl(\frac{R_1}{\beta t+1}\Bigr)\Bigr) + O\Bigl(\frac{1}{R}\Bigr),
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \sigma_3\Bigl(\alpha,\frac{R_1}{\beta t+1}\Bigr)\,dt &= \frac{R_1^2}{2(\alpha+1)(\beta+1)} - \frac{\alpha R_1}{2(\alpha+1)}\frac{\log(\beta+1)}{\beta} \\
&\qquad + O\Bigl(\Bigl(\frac{R^{3/2}}{q(\alpha)}+R\Bigr)s_1(\alpha)\Bigr).
\end{aligned}$$

We have assumed that $\alpha \leqslant \beta$. Hence,

$$\alpha\int_0^1 \frac{dt}{\beta t+1}\cdot\rho\Bigl(\frac{R_1}{\beta t+1}\Bigr) \ll \frac{\alpha}{\beta}\frac{1}{R_1} \ll \frac{1}{R_1},$$

$$\begin{aligned}
\sum_{n\leqslant R_1}\sum_{m\leqslant n} & \int_U^R\int_0^l \bigl[(\alpha m+n)(\beta k+l)\leqslant R\bigr]\,dk\,dl \\
&= \frac{R^2}{2(\alpha+1)(\beta+1)}\Bigl(\log R_1 + c_1(\alpha) + \frac{\log(\beta+1)}{\beta} - \frac{3}{2}\Bigr) \\
&\qquad + \frac{\alpha}{(\alpha+1)RU}\frac{\log(\beta+1)}{2\beta} + O\Bigl(\Bigl(\frac{R^{3/2}}{q(\alpha)}+R\Bigr)s_1(\alpha)\Bigr). \qquad (45)
\end{aligned}$$

Combining equations (43)–(45), we obtain the following asymptotic formula for $T_2(R,U)$:

$$\begin{aligned}
T_2(R,U) &= \frac{R^2}{2(\alpha+1)(\beta+1)}\Bigl(\log\frac{R}{U} + c_1(\alpha) + \frac{\log(\beta+1)}{\beta} - \frac{3}{2}\Bigr) \\
&\qquad + \frac{\alpha}{(\alpha+1)RU}\frac{\log(\beta+1)}{2\beta} - \frac{R^2U^{-1}(\beta^2+2\beta)}{4(\alpha+1)(\beta+1)^2} \\
&\qquad + O\Bigl(R^{3/2}\Bigl(\frac{s_1(\alpha)}{q(\alpha)} + \frac{s_1(\beta)}{q(\beta)}\Bigr) + R\bigl(s_1(\alpha)+s_1(\beta)\bigr)\Bigr).
\end{aligned}$$

Substituting the asymptotic formulae obtained for $T_1(R,U)$ and $T_2(R,U)$ into (42), we complete the proof of the lemma.

For real α, $\beta\in(0,1]$ we denote by $L(R)=L(\alpha,\beta;R)$ the sum

$$L(R) = \sum_{l,n\leqslant R}\sum_{k\leqslant l}\sum_{m\leqslant n}\frac{\bigl[(\alpha m+n)(\beta k+l)\leqslant R\bigr]}{(\alpha m+n)^2(\beta k+l)^2}.$$

Corollary 2. *Let $\alpha, \beta \in (0,1]$ be rational numbers with denominators $q(\alpha)$ and $q(\beta)$, respectively. Then for every $R \geqslant 1$ the following asymptotic formula holds for $L(R)$:*

$$L(R) = \frac{1}{(\alpha+1)(\beta+1)}\left(\frac{\log^2 R}{2} + \log R\big(c_1(\alpha)+c_1(\beta)\big) + c(\alpha,\beta)\right) + O\left(R^{-1/2}\left(\frac{s_1(\alpha)}{q(\alpha)} + \frac{s_1(\beta)}{q(\beta)}\right) + R^{-1}\big(s_1(\alpha)+s_1(\beta)\big)\right),$$

where $c_1(\alpha)$ is the function occurring in Lemma 10 and $c(\alpha,\beta)$ does not depend on R.

Proof. We obtain the assertion of the corollary from Lemma 17 using the Abel integral transformation (28). It is sufficient to consider the sequence $\lambda_1, \dots, \lambda_P$ of values of the product $(\alpha m + n)(\beta k + l)$ when $1 \leqslant m \leqslant n$, $1 \leqslant k \leqslant l$, $(\alpha m + n)(\beta k + l) \leqslant R$, and put $a_j = 1$, $j = 1, \dots, P$, $g(t) = 1/t^2$.

Corollary 3. *Let $R \geqslant 2$. Then*

$$\int_0^1\int_0^1 T(\alpha,\beta;R)\,d\alpha\,d\beta = \frac{\log^2 2}{2}R^2\left(\log R + 2\gamma - \frac{5}{2} + \log 2\right) + O(R\log^2 R),$$

$$\int_0^1\int_0^1 L(\alpha,\beta;R)\,d\alpha\,d\beta = \frac{\log^2 2}{2}\log^2 R + 2\log^2 2(\log R)\left(\gamma - 1 + \frac{\log 2}{2}\right) + C_L + O(R^{-1}\log^2 R),$$

where C_L is an absolute constant.

Proof. Let $p \geqslant 2$ be a positive integer. Then

$$\int_0^1\int_0^1 T(\alpha,\beta;R)\,d\alpha\,d\beta = \frac{1}{p^2}\sum_{a,b=1}^{p-1} T\left(\frac{a}{p},\frac{b}{p};R\right) + O\left(\frac{R^2\log R}{p}\right).$$

For prime numbers p, Lemma 17 implies that

$$\int_0^1\int_0^1 T(\alpha,\beta;R)\,d\alpha\,d\beta = \frac{1}{p^2}\sum_{a,b=1}^{p-1}\frac{R^2}{2(1+\frac{a}{p})(1+\frac{b}{p})}\left(\log R + c_1\left(\frac{a}{p}\right) + c_1\left(\frac{b}{p}\right) - \frac{1}{2}\right) + O\left(\frac{R^2\log R}{p}\right) + O\left(\left(\frac{R^{3/2}}{p} + R\right)\frac{1}{p}\sum_{a=1}^{p-1} s_1\left(\frac{a}{p}\right)\right).$$

Since $c_1(\alpha)$ is a continuous function, Lemma 4 implies that

$$\int_0^1\int_0^1 T(\alpha,\beta;R)\,d\alpha\,d\beta = \int_0^1\int_0^1 \frac{R^2}{2(1+\alpha)(1+\beta)}\left(\log R + c_1(\alpha) + c_1(\beta) - \frac{1}{2}\right) + O\left(\left(\frac{R^{3/2}}{p} + R\right)\log^2 p\right) + O\left(\frac{R^2\log R}{p}\right).$$

By choosing p in the range $R \leqslant p \leqslant 2R$ and applying (29), we complete the proof of the first equation of the corollary.

The second equation can be proved likewise. The integrability of $c(\alpha,\beta)$ follows from that of $L(\alpha,\beta;R)$, $c_1(\alpha)$ and $c_1(\beta)$.

§ 8. Asymptotic formula for the variance

We denote by $M(R)$ the number of solutions of the inequality (13), where

$$1 \leqslant k \leqslant l, \qquad 1 \leqslant Q \leqslant Q', \qquad \begin{pmatrix} a & m \\ b & n \end{pmatrix} \in \mathcal{M}. \tag{46}$$

Let U and U_0 be real numbers in the segment $[1,R]$. We partition the solutions of (13) with the restrictions (46) into three groups for which

1) $n \leqslant U$, $Q' \leqslant U_0$,
2) $n \leqslant U$, $Q' > U_0$,
3) $n > U$.

In accordance with this partition, we write $M(R)$ as

$$M(R) = M_1(R,U,U_0) + M_2(R,U,U_0) + M_3(R,U).$$

Each summand will be studied separately.

We begin with the computation of $M_1(R,U,U_0)$.

Lemma 18. *Let $2 \leqslant U, U_0 \leqslant R$, and assume that U_0 is a half-integer. Then*

$$\begin{aligned} M_1(R,U,U_0) = {} & \frac{\log^2 2}{\zeta(2)} R^2 \log U \log U_0 \\ & + \biggl(\frac{C_1}{2} + \frac{\log^2 2}{\zeta(2)}\biggl(\gamma - \frac{\zeta'(2)}{\zeta(2)}\biggr)\biggr) R^2 \log U_0 + \frac{C_4}{2} R^2 \\ & + \frac{\log^2 2}{\zeta(2)}\biggl(\log 2 + \gamma - \frac{\zeta(2)}{2\log 2}\biggr) R^2 \log U \\ & + \frac{R^2}{4U_0}\biggl(\frac{\log 2}{\zeta(2)} \log U + C_{A_0} - C_{A_1}\biggr) - \frac{1-\log 2}{2} R U U_0 \\ & + O(R^2 U_0^{-2} \log R) + O(R^2 U^{-1/2} \log^6 R) \\ & + O(R U_0 U^{1/2} \log^5 R) + O(U^2 U_0^2 \log^2 R), \end{aligned}$$

where C_1, C_{A_0} and C_{A_1} are the constants occurring in Lemma 12 and C_4 is the constant occurring in Lemma 13.

Proof. Let a, b, m, n, Q and Q' be fixed and put

$$f(l) = \min\biggl\{l, \frac{R - l(mQ + nQ')}{aQ + bQ'}\biggr\}.$$

Then the number of solutions of the inequality

$$k(aQ+bQ')+l(mQ+nQ')\leqslant R$$

in k and l with the restrictions (46) is equal to

$$\sum_{l\leqslant R/(mQ+nQ')} f(l)-\sum_{l\leqslant R/(mQ+nQ')}\{f(l)\}.$$

Applying Lemma 3 to the first sum and Lemma 5 to the second, we obtain that

$$\frac{R^2}{2(mQ+nQ')\big((a+m)Q+(b+n)Q'\big)}-\frac{R}{2}\left(\frac{1}{mQ+nQ'}-\frac{1}{(a+m)Q+(b+n)Q'}\right)$$
$$+O\left(\left(\frac{R}{nQ'}q^{-1}\left(\frac{aQ+bQ'}{mQ+nQ'}\right)+1\right)s_1\left(\frac{aQ+bQ'}{mQ+nQ'}\right)\right). \tag{47}$$

Observing that

$$s_1\left(\frac{aQ+bQ'}{mQ+nQ'}\right)\ll s_1\left(\frac{m}{n}\right)+s_1\left(\frac{Q}{Q'}\right)$$

and using Lemma 4, we obtain the following estimate for the sum of the remainder terms:

$$\begin{aligned}
&\sum_{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathcal{M}(U)}\ \sum_{Q'\leqslant U_0}\ \sum_{Q\leqslant Q'}\left(\frac{R}{nQ'}q^{-1}\left(\frac{aQ+bQ'}{mQ+nQ'}\right)+1\right)s_1\left(\frac{aQ+bQ'}{mQ+nQ'}\right)\\
&\quad=\sum_{\left(\begin{smallmatrix} a & m\\ b & n\end{smallmatrix}\right)\in\mathcal{M}(U)}\ \sum_{\delta\leqslant U_0}\ \sum_{Q'\leqslant U_0/\delta}\ \sideset{}{^*}\sum_{Q\leqslant Q'}\left(\frac{R}{n\delta Q'(mQ+nQ')}+1\right)s_1\left(\frac{aQ+bQ'}{mQ+nQ'}\right)\\
&\quad\ll\sum_{n\leqslant U}\ \sideset{}{^*}\sum_{m\leqslant n}\ \sum_{\delta\leqslant U_0}\ \sum_{Q'\leqslant U_0/\delta}\ \sideset{}{^*}\sum_{Q\leqslant Q'}\left(\frac{R}{\delta n^2(Q')^2}+1\right)\left(s_1\left(\frac{m}{n}\right)+s_1\left(\frac{Q}{Q'}\right)\right)\\
&\quad\ll R\log^5 R+U^2U_0^2\log^2 R.
\end{aligned}$$

Adding the terms of (47) together, we obtain that

$$\begin{aligned}
M_1(R,U,U_0)=\frac{R^2}{2}W_4(U_0,U)-\frac{R}{2}(W_6(U_0,U)-W_7(U_0,U))\\
+O(R\log^5 R)+O(U^2U_0^2\log^3 R).
\end{aligned}$$

Using Lemmas 13, 15 and taking into account the estimate $RU\ll R^2U_0^{-2}+U^2U_0^2$, we obtain the desired formula for $M_1(R,U,U_0)$.

We now compute $M_2(R,U,U_0)$.

Lemma 19. *Let $2 \leqslant U, U_0 \leqslant R$, and assume that U_0 is a half-integer. Then*

$$\begin{aligned} M_2(R,U,U_0) = {} & \frac{\log^2 2}{\zeta(2)} R^2 \log U \log \frac{R}{U_0} \\ & + \left(\frac{C_1}{2} + \frac{\log^2 2}{\zeta(2)}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)}\right)\right) R^2 \log \frac{R}{U_0} - \frac{\log^2 2}{2\zeta(2)} R^2 \log^2 U \\ & + \frac{\log^2 2}{\zeta(2)}\left(\gamma - \frac{5}{2} + \frac{\zeta(2)}{2\log 2}\right) R^2 \log U \\ & + C_2' R^2 - \frac{R^2}{4U_0}\left(\frac{\log 2}{\zeta(2)} \log U + C_{A_0} - C_{A_1}\right) + \frac{1-\log 2}{2} RUU_0 \\ & + O(R^2 U_0^{-2} \log^2 R) + O(R^2 U^{-1/2} \log^6 R) \\ & + O(RU_0 U^{1/2} \log^5 R) + O(U^2 U_0^2 \log^2 R), \end{aligned}$$

where C_2' is an absolute constant and C_{A_0}, C_{A_1}, C_1 are the constants occurring in Lemma 12.

Proof. Put $R_1 = RU_0^{-1}$. As in the proof of Lemma 18, we assume that a, b, m, n, k and l are fixed and the conditions (46) hold for them. Consider the function

$$f(l) = \min\left\{Q', \frac{R - Q'(bk+nl)}{ak+ml}\right\}.$$

The number of solutions of the inequality

$$k(aQ + bQ') + l(mQ + nQ') \leqslant R$$

in Q and Q' such that $Q' > U_0$ and $1 \leqslant Q \leqslant Q'$ is equal to

$$\sum_{U_0 < Q' \leqslant R/(bk+nl)} f(Q') - \sum_{U_0 < Q' \leqslant R/(bk+nl)} \{f(l)\}.$$

We again apply Lemma 3 to the first sum and Lemma 5 to the second. Since U_0 is a half-integer and

$$q\left(\frac{ak+ml}{bk+nl}\right) = \frac{bk+nl}{(k,l)},$$

we obtain that

$$\begin{aligned} & \sum_{U_0 < Q' \leqslant R/(bk+nl)} f(Q') - \sum_{U_0 < Q' \leqslant R/(bk+nl)} \{f(l)\} \\ & = \int_{U_0}^{R} \int_0^{Q'} \big[Q(ak+ml) + Q'(bk+nl) \leqslant R\big] \, dQ \, dQ' \\ & \quad - \frac{U_0}{2}\left(\frac{R_1}{bk+nl} - \max\left\{1, \frac{R_1}{(a+b)k + (m+n)l}\right\}\right)[bk+nl \leqslant R_1] \\ & \quad + O\left(\left(\frac{(k,l)R}{l^2 n^2} + 1\right) s_1\left(\frac{ak+ml}{bk+nl}\right)\right). \end{aligned}$$

We estimate the sum of the remainders as we did in the proof of Lemma 18. In the notation of Lemma 14 we have

$$\begin{aligned} M_2(R,U,U_0) &= \\ &= \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \sum_{l\leqslant R}\sum_{k\leqslant l}\int_{U_0}^{R}\int_0^{Q'} [Q(ak+ml)+Q'(bk+nl)\leqslant R]\,dQ\,dQ' \\ &\quad - \frac{U_0}{2}W_5(R_1,U) + O(R\log^5 R) + O(R^2U_0^2\log^2 R). \end{aligned}$$

Using the change of variables $Q = tQ'$ and $Q' = \xi U_0$, we transform the sum thus obtained as follows:

$$\begin{aligned} &U_0^2 \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \sum_{l\leqslant R}\sum_{k\leqslant l}\int_0^1\int_1^{R_1} \xi[\xi(t(ak+ml)+bk+nl)\leqslant R_1]\,d\xi\,dt \\ &= \frac{U_0^2}{2} \sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \sum_{l\leqslant R}\sum_{k\leqslant l}\int_0^1\left(\left(\frac{R_1}{t(ak+ml)+bk+nl}\right)^2-1\right) \\ &\qquad \times \bigl[t(ak+ml)+bk+nl\leqslant R_1\bigr]\,dt \\ &= \frac{1}{2}\sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \int_0^1\left(\frac{R^2}{(mt+n)^2}\sigma_1\left(\frac{at+b}{mt+n},\frac{R_1}{mt+n}\right)\right. \\ &\qquad \left. - U_0^2\sigma_3\left(\frac{at+b}{mt+n},\frac{R_1}{mt+n}\right)\right)dt. \end{aligned}$$

Hence,

$$\begin{aligned} M_2(R,U,U_0) &= \frac{1}{2}\sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \int_0^1 \frac{R^2}{(mt+n)^2}\sigma_1\left(\frac{at+b}{mt+n},\frac{R_1}{mt+n}\right)dt \\ &\quad - \frac{U_0^2}{2}W_8(R_1,U) - \frac{U_0}{2}W_5(R_1,U) + O(R\log^5 R) + O(R^2U_0^{-2}\log^2 R). \end{aligned} \tag{48}$$

Using Lemma 10, we obtain, in the notation of Lemma 12, that

$$\begin{aligned} &\sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \int_0^1 \frac{1}{(mt+n)^2}\sigma_1\left(\frac{at+b}{mt+n},\frac{R_1}{mt+n}\right)dt \\ &\qquad = W_1(U)\log R_1 - W_2(U) + W_3(U) + \frac{U}{2R_1}(1-\log 2) \\ &\qquad\quad + I(U) + O(R_1^{-1}U^{1/2}\log^5 U) + O(R_1^{-2}U^2), \end{aligned} \tag{49}$$

where

$$I(U) = \frac{1}{R_1}\sum_{\left(\begin{smallmatrix} a & m \\ b & n \end{smallmatrix}\right)\in\mathcal{M}(U)} \int_0^1\left(\frac{1}{mt+n}-\frac{1}{(a+m)t+b+n}\right)\rho\left(\frac{R_1}{mt+n}\right)dt.$$

Integrating by parts, we obtain the estimate

$$I(U) \ll \frac{U \log U}{R_1^2} + \frac{\log U}{R_1}.$$

Hence, the order of $I(U)$ does not exceed the orders of the remainder terms.

Substituting (33) and (49) into (48), we obtain that

$$\begin{aligned} M_2(R,U,U_0) = \frac{R^2}{2}\biggl(\Bigl(\log R_1 - \frac{1}{2}\Bigr) W_1(U) - W_2(U) + W_3(U)\biggr) \\ - \frac{U_0}{2} W_5(R_1,U) + \frac{RUU_0}{2}(1-\log 2) \\ + O(R^2 U_0^{-2} \log^2 R) + O(U^2 U_0^2 \log^2 R) + O(RU_0 U^{1/2} \log^5 R). \end{aligned}$$

Using Lemma 12, we obtain the desired formula for $M_2(R,U,U_0)$.

Corollary 4. *Let* $1 \leqslant U \leqslant R$. *Then*

$$\begin{aligned} M_1(R,U,U_0) + M_2(R,U,U_0) = \frac{\log^2 2}{\zeta(2)} R^2 \log R \log U \\ + \biggl(\frac{C_1}{2} + \frac{\log^2 2}{\zeta(2)}\Bigl(\gamma - \frac{\zeta'(2)}{\zeta(2)}\Bigr)\biggr) R^2 \log R + C_0 R^2 \\ - \frac{\log^2 2}{2\zeta(2)} R^2 \log^2 U + \frac{\log^2 2}{\zeta(2)}\Bigl(\log 2 - \frac{5}{2} + 2\gamma\Bigr) R^2 \log U \\ + O(RU \log^2 R) + O(R^2 U^{-1/2} \log^6 R). \end{aligned}$$

Proof. We prove this by combining the results of Lemmas 18 and 19 and putting $U_0 = \bigl[R^{1/2} U^{-1/2}\bigr] + 1/2$.

We now compute $M_3(R,U)$.

Lemma 20. *Let* $8R^{1/2} \leqslant U \leqslant R/2$. *Then*

$$\begin{aligned} M_3(R,U) = \frac{\log^2 2}{2\zeta(2)} R^2 \Bigl(\log^2 \frac{R}{U} + 2 \log \frac{R}{U}\Bigl(2\gamma - \frac{5}{2} + \log 2\Bigr) + C_3'\Bigr) \\ + O(RU \log^2 R) + O(R^{2+\varepsilon} U^{-1/3}). \end{aligned}$$

Proof. Let $R_2 = R/U$. By the definition of $M_3(R,U)$, we have

$$M_3(R,U) = \sum_{lQ' \leqslant R_1} \sum_{k \leqslant l} \sum_{Q \leqslant Q'} \sum_{n > U} T_{\pm}(k,l,Q,Q',n), \tag{50}$$

where

$$T_{\pm}(k,l,Q,Q',n) = \sum_{b,m=1}^{n} \delta_n(bm \pm 1)\Bigl[k\Bigl(\frac{bm \pm 1}{n} Q + bQ'\Bigr) + l(mQ + nQ') \leqslant R\Bigr].$$

To compute $T_\pm(k,l,Q,Q',n)$, we assume for the moment that $l/k \leqslant Q'/Q$. Consider the function

$$m(b) = \min\left\{n, \frac{1}{Q}\left(\frac{R \mp \frac{k}{n}Q}{\frac{k}{n}b + l} - nQ'\right)\right\}$$

on the segment inside $[0,n]$ on which this function is non-negative. If $m(b) = \frac{1}{Q}\left(\frac{R\mp\frac{k}{n}Q}{\frac{k}{n}b+l} - nQ'\right)$, then

$$m''(b) = \frac{2}{Q}\left(R \mp \frac{k}{n}Q\right)\left(\frac{k}{n}\right)^2 \frac{1}{\left(\frac{k}{n}b + l\right)^3} \in \left[\frac{1}{c}, \frac{w}{c}\right],$$

where $c = \frac{Q}{2}\left(R \mp \frac{k}{n}Q\right)^{-1}\left(\frac{n}{k}\right)^2 l^3$ and $w = 8$. Since $m(b) \leqslant n$ and $lQ' \leqslant R_2$, we have $R \mp \frac{k}{n}Q \leqslant 4lnQ'$. For $U \geqslant 8R^{1/2}$ we have

$$c \geqslant \frac{Qnl^2}{8Q'k^2} \geqslant \frac{n}{8Q'} \geqslant \frac{U}{8R_2} = \frac{U^2}{8R} \geqslant 8 = w.$$

Hence, Lemma 7 can be applied to $m(b)$. We obtain that

$$n^\varepsilon(nc^{-1/3} + c^{2/3}) \ll n^{2/3+\varepsilon}\left(\left(\frac{Q'}{Q}\right)^{1/3}\left(\frac{k}{l}\right)^{2/3} + \left(\frac{Q}{Q'}\right)^{2/3}\left(\frac{l}{k}\right)^{4/3}\right)$$
$$\ll n^{2/3+\varepsilon}\left(\frac{lQ'}{kQ}\right)^{1/3}.$$

On the set where $m(b) = n$ we use the equation

$$\sum_{\substack{1\leqslant b\leqslant x\\(b,n)=1}} 1 = \frac{\varphi(n)}{n}x + O\big(\sigma_0(n)\big)$$

(see [19], Ch. II, problem 19). Hence,

$$T_\pm(k,l,Q,Q',n) = \frac{1}{n}\sum_{\substack{x=1\\(x,n)=1}}^{n} f(x) + O\left(n^{2/3+\varepsilon}\left(\frac{lQ'}{kQ}\right)^{1/3}\right).$$

Further, we have

$$\frac{1}{n}\sum_{\substack{x=1\\(x,n)=1}}^{n} f(x) = \frac{1}{n}\sum_{\delta|n}\mu(\delta)\sum_{x=1}^{n/\delta} f(\delta x).$$

Lemma 5 implies that

$$\sum_{x=1}^{n/\delta} f(\delta x) = \frac{1}{\delta}\int_0^n f(t)\,dt + O\left(\frac{1}{\delta}\frac{lQ'}{kQ}\right).$$

Hence,

$$\frac{1}{n}\sum_{\substack{x=1\\(x,n)=1}}^{n} f(x) = \frac{\varphi(n)}{n^2}\int_0^n f(t)\,dt + O\biggl(\frac{\log R}{n}\frac{lQ'}{kQ}\biggr),$$

$$T_\pm(k,l,Q,Q',n) = \frac{\varphi(n)}{n^2}V_\pm(k,l,Q,Q',n) + O\biggl(\frac{\log R}{n}\frac{lQ'}{kQ}\biggr) + O\biggl(n^{2/3+\varepsilon}\biggl(\frac{lQ'}{kQ}\biggr)^{1/3}\biggr), \qquad (51)$$

where $V_\pm(k,l,Q,Q',n)$ is the area of the domain $\Omega_\pm(k,l,Q,Q',n)$ in the plane Obm defined by the inequalities

$$0 \leqslant b, m \leqslant n, \qquad k\biggl(\frac{bm\pm 1}{n}Q + bQ'\biggr) + l(mQ+nQ') \leqslant R. \qquad (52)$$

If $l/k \geqslant Q'/Q$, then formula (51) can be proved by applying similar arguments to the function

$$b(m) = \min\biggl\{n, \frac{1}{k}\biggl(\frac{R\mp\frac{k}{n}}{\frac{Q}{n}m+Q'} - ln\biggr)\biggr\}.$$

Substituting (51) into (50), we obtain that

$$M_3(R,U) = \sum_{lQ'\leqslant R_2}\sum_{k\leqslant l}\sum_{Q\leqslant Q'}\sum_{U<n\leqslant R/(lQ')}\frac{\varphi(n)}{n^2}V_\pm(k,l,Q,Q',n) + O(R^{2+\varepsilon}U^{-1/3}).$$

We denote by $\Omega(k,l,Q,Q',n)$ the domain obtained by omitting ± 1 in the inequalities (52):

$$0 \leqslant b, m \leqslant n, \qquad \biggl(\frac{kb}{n}+l\biggr)(mQ+nQ') \leqslant R.$$

The area of this domain will be denoted by $V(k,l,Q,Q',n)$. Since

$$\Omega_+(k,l,Q,Q',n) \subset \Omega(k,l,Q,Q',n) \subset \Omega_-(k,l,Q,Q',n),$$

the error caused by the replacement of $V_\pm(k,l,Q,Q',n)$ by $V(k,l,Q,Q',n)$ does not exceed $V_-(k,l,Q,Q',n) - V_+(k,l,Q,Q',n)$. Since for fixed m the difference between b_- and b_+ such that

$$k\biggl(\frac{b_\pm m\pm 1}{n}Q + b_\pm Q'\biggr) + l(mQv+nQ') = R$$

is $O(1/n)$, we have $V_-(k,l,Q,Q',n) - V_+(k,l,Q,Q',n) = O(1)$ and

$$M_3(R,U) = 2\sum_{lQ'\leqslant R_2}\sum_{k\leqslant l}\sum_{Q\leqslant Q'}\sum_{U<n\leqslant R}\frac{\varphi(n)}{n^2}V(k,l,Q,Q',n) + O(R^{2+\varepsilon}U^{-1/3}). \qquad (53)$$

Further, we have

$$\sum_{U<n\leqslant R}\frac{\varphi(n)}{n^2}V(k,l,Q,Q',n)=\sum_{\delta\leqslant R}\frac{\mu(\delta)}{\delta^2}\sum_{\frac{U}{\delta}<n\leqslant\frac{R}{\delta}}\frac{V(k,l,Q,Q',\delta n)}{n}.$$

Therefore, the sum of the principal terms in formula (53) is transformed as follows:

$$\begin{aligned}
&2\sum_{\delta\leqslant R}\frac{\mu(\delta)}{\delta^2}\sum_{lQ'\leqslant R_2}\sum_{k\leqslant l}\sum_{Q\leqslant Q'}\sum_{\frac{U}{\delta}<n\leqslant\frac{R}{\delta}}\frac{1}{n}\\
&\qquad\times\int_0^{\delta n}\int_0^{\delta n}\left[\left(\frac{b}{\delta n}k+l\right)\left(\frac{m}{\delta n}Q+Q'\right)\leqslant\frac{R}{\delta n}\right]dm\,db\\
&\quad=2\sum_{\delta\leqslant R}\mu(\delta)\sum_{lQ'\leqslant R_2}\sum_{k\leqslant l}\sum_{Q\leqslant Q'}\sum_{\frac{U}{\delta}<n\leqslant\frac{R}{\delta}}n\\
&\qquad\times\int_0^1\int_0^1\left[(\alpha k+l)(\beta Q+Q')\leqslant\frac{R}{\delta n}\right]d\alpha\,d\beta=2\sum_{\delta\leqslant R}\frac{\mu(\delta)}{\delta^2}\\
&\qquad\times\int_0^1\int_0^1\sum_{lQ'\leqslant R_2}\sum_{k\leqslant l}\sum_{Q\leqslant Q'}\sum_{\frac{U}{\delta}<n\leqslant\frac{R}{\delta}}n\left[n\leqslant\frac{R}{\delta(\alpha k+l)(\beta Q+Q')}\right]d\alpha\,d\beta\\
&\quad=\sum_{\delta\leqslant R}\frac{\mu(\delta)}{\delta^2}\int_0^1\int_0^1\sum_{lQ'\leqslant R_2}\sum_{k\leqslant l}\sum_{Q\leqslant Q'}\left(\frac{R^2}{(\alpha k+l)^2(\beta Q+Q')^2}-U^2\right)\\
&\qquad\times\left[(\alpha k+l)(\beta Q+Q')\leqslant R_2\right]d\alpha\,d\beta+O(R^2U^{-1}\log R).
\end{aligned}$$

Therefore,

$$M_3(R,U)=\frac{U^2}{\zeta(2)}\int_0^1\int_0^1\left(R_2^2L(\alpha,\beta;R_2)-T(\alpha,\beta;R_2)\right)d\alpha\,d\beta+O(R^{2+\varepsilon}U^{-1/3}).$$

Substituting into this formula the equations in Corollary 3, we complete the proof of the lemma.

Theorem 2. *Let $R\geqslant 2$. Then*

$$\begin{aligned}M(R)=\frac{\log^2 2}{2\zeta(2)}R^2\log^2 R+\left(\frac{C_1}{2}+\frac{\log^2 2}{\zeta(2)}\left(3\gamma-\frac{5}{2}+\log 2-\frac{\zeta'(2)}{\zeta(2)}\right)\right)R^2\log R\\
+CR^2+O(R^{2-1/4+\varepsilon})\end{aligned}$$

with absolute constants C and C_1, where C_1 is the constant occurring in Lemma 12.

Proof. To prove the formula, it is sufficient to combine the equations in Corollary 4 and Lemma 20 and put $U=R^{3/4}$.

We now compute $\mathsf{D}(R)$.

Corollary 5. *Let $R \geqslant 2$. Then*

$$\mathsf{D}(R) = \delta_1 \log R + \delta_0 + O_\varepsilon(R^{-1/4+\varepsilon})$$

for all $\varepsilon > 0$, where δ_1 and δ_0 are absolute constants,

$$\delta_1 = \frac{8\log^2 2}{\zeta^2(2)}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{\log 2}{2} - 1\right) + \frac{4}{\zeta(2)}\left(C_1 + \frac{3\log 2}{2}\right),$$

and the constant C_1 is defined by (32).

Proof. Using Theorem 2, we obtain that

$$M^*(R) = \sum_{d \leqslant R} \mu(d) M\left(\frac{R}{d}\right) = \frac{\log^2 2}{2\zeta^2(2)} R^2 \log^2 R$$
$$+ \left(\frac{C_1}{2\zeta(2)} + \frac{\log^2 2}{\zeta^2(2)}\left(3\gamma - \frac{5}{2} + \log 2 - 2\frac{\zeta'(2)}{\zeta(2)}\right)\right) R^2 \log R$$
$$+ C' R^2 + O(R^{2-1/4+\varepsilon}).$$

By formula (16), we have

$$\mathcal{L}_2(R) = \frac{2\log^2 2}{\zeta^2(2)} \log^2 R + 4\left(\frac{C_1}{2\zeta(2)} + \frac{\log^2 2}{\zeta^2(2)}\left(3\gamma - \frac{5}{2} + \log 2 - 2\frac{\zeta'(2)}{\zeta(2)}\right)\right) \log R$$
$$+ C' + O(R^{-1/4+\varepsilon}).$$

Substituting this formula into (10), we complete the proof of the theorem.

Remark 1. For irrational numbers we can use the following analogue of $s(\alpha)$:

$$N(\alpha, R) = \#\{j \geqslant 1 \colon Q_j(\alpha) \leqslant R\},$$

where $Q_j(\alpha)$ is the denominator of the jth convergent of the continued fraction for α. It was proved in [18] that the mean value of $N(\alpha, R)$,

$$N(R) = \int_0^1 N(\alpha, R)\, d\alpha,$$

satisfies the following asymptotic formula with two significant terms:

$$N(R) = \frac{2\log 2}{\zeta(2)} \log R + \frac{2\log 2}{\zeta(2)}\left(\log 2 + \gamma - \frac{\zeta'(2)}{\zeta(2)}\right) - \frac{3}{2} + O\left(\frac{\log R}{R}\right),$$

and that the variance

$$\mathsf{D}(R) = \int_0^1 \big(N(\alpha, R) - N(R)\big)^2\, d\alpha = \int_0^1 N^2(\alpha, R)\, d\alpha - N^2(R)$$

satisfies the following asymptotic formula:

$$\mathsf{D}(R) = \delta_1 \log R + \delta_0' + O(R^{-1/3}\log^5 R),$$

where δ_1 and δ_0' are absolute constants (δ_1 is the same as in Corollary 5).

Computer computations give the following approximate value of δ_1:

$$\delta_1 = 0.51606\dots.$$

Remark 2. The Gauss–Kuz'min statistics $s_x(r)$, which are more general characteristics than the length of a continued fraction, are given for a fixed $x \in [0,1]$ and a rational $r = [t_0; t_1, \dots, t_s]$ by the equation

$$s_x(r) = \#\bigl\{j\colon 1 \leqslant j \leqslant s(r), [0; t_j, \dots, t_s] \leqslant x\bigr\}.$$

Using the ideas of [15], [20] and proceeding as in the proof of Corollary 1, we can prove the following asymptotic formula for the mean value of $s_x(c/d)$:

$$\frac{2}{[R]\bigl([R]+1\bigr)} \sum_{d\leqslant R}\sum_{c\leqslant d} s\biggl(\frac{c}{d}\biggr) = \frac{2\log(1+x)}{\zeta(2)}\log R + \frac{2}{\zeta(2)}C(x) + O(R^{-1}\log^5 R),$$

where

$$C(x) = \log(1+x)\biggl(2\gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{\log(1+x)}{2} - \log x - \frac{3}{2}\biggr) + f_1(x) + f_2(x) - \frac{x\zeta(2)}{2(x+1)} + \frac{x\zeta(2)}{2}[x<1],$$

$$f_1(x) = \sum_{Q'=1}^{\infty}\frac{1}{Q'}\biggl(\sum_{Q=1}^{Q'}\frac{x}{Q'+Qx} - \log(1+x)\biggr),$$

$$f_2(x) = \sum_{m=1}^{\infty}\frac{1}{m}\biggl(\sum_{\frac{m}{x}\leqslant n<\frac{m}{x}+m}\frac{1}{n} - \log(1+x)\biggr).$$

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