ISSN 0001-4346, Mathematical Notes, 2009, Vol. 85, No. 1, pp. 142–145. © Pleiades Publishing, Ltd., 2009.
Original Russian Text © A. V. Ustinov, 2009, published in Matematicheskie Zametki, 2009, Vol. 85, No. 1, pp. 153–156.

SHORT COMMUNICATIONS

The Mean Number of Steps in the Euclidean Algorithm with Least Absolute Value Remainders

A. V. Ustinov*

Khabarovsk Branch of Institute of Applied Mathematics, Far East Division, Russian Academy of Sciences

Received January 29, 2008

DOI: 10.1134/S0001434609010179

Key words: *Euclidean algorithm, Euclidean algorithm with least absolute value remainders, continued fraction, Gauss–Kuz'min statistics.*

The classical Euclidean algorithm, in which the minimal nonnegative remainders

$$a = bq + r, \qquad q = \left[\frac{a}{b}\right], \quad 0 \le r < q$$

of division are taken, corresponds to the decomposition of a given number in a standard continued fraction of the form

$$\frac{a}{b} = t_0 + \cfrac{1}{t_1 + \ddots + \cfrac{1}{t_s}}$$

of length $s = s(a/b)$, in which t_0 is an integer, $t_1, \dots, t_s$ are positive integers, and $t_s \ge 2$ for $s \ge 1$. The Euclidean algorithm in which the remainders

$$a = bq + r, \qquad q = \left[\frac{a}{b} + \frac{1}{2}\right], \quad -\frac{q}{2} \le r < \frac{q}{2}$$

with least absolute values are chosen leads to the decomposition in a fraction

$$\frac{a}{b} = t_0 + \cfrac{\varepsilon_1}{t_1 + \cfrac{\varepsilon_2}{t_2 + \ddots + \cfrac{\varepsilon_l}{t_l}}} \tag{1}$$

of length $l = l(a/b)$, where t_0 is an integer, $t_1, \dots, t_l$ are positive integers, and

$$\varepsilon_k = \pm 1, \qquad t_k \ge 2, \quad k = 1, \dots, l, \qquad t_k + \varepsilon_{k+1} \ge 2, \quad k = 1, \dots, l-1.$$

There exists a simple algorithm transforming an ordinary continued fraction into a fraction of the form (1) (see [1, Sec. 39]). To the first partial quotient t_j ($j \ge 1$) equal to 1 the identity

$$t_{j-1} + \cfrac{1}{1 + \cfrac{1}{t_{j+1} + \ddots}} = t_{j-1} + 1 - \cfrac{1}{t_{j+1} + 1 + \ddots}$$

is applied. Thus, the first 1 in the decomposition is deleted, the neighboring partial quotients are increased by 1, and the negative sign between them is inserted. Then, the next 1 is found, and

*E-mail: ustinov@iam.khv.ru

the procedure is repeated. If the decomposition contains a chain of consecutive ones, then the transformation is applied to those occupying odd positions in this chain. For example,

$$[0;2,1,3,1,1,6] = \cfrac{1}{3-\cfrac{1}{5-\cfrac{1}{2+\cfrac{1}{6}}}}.$$

For the mean number of steps in the Euclidean algorithm, the following estimates are known (see [2], [3]):

$$\frac{1}{\varphi(b)} \sum_{\substack{1\le a\le b\\(a,b)=1}} s\left(\frac{a}{b}\right) = \frac{2\log 2}{\zeta(2)} \cdot \log b + C_1 + O_\varepsilon(b^{-1/6+\varepsilon}), \tag{2}$$

$$\frac{2}{R(R+1)} \sum_{b\le R} \sum_{a\le b} s\left(\frac{a}{b}\right) = \frac{2\log 2}{\zeta(2)} \cdot \log R + C_2 + O(R^{-1}\log^4(R+1)), \tag{3}$$

where the absolute constants are

$$C_1 = \frac{2\log 2}{\zeta(2)}\left(\frac{3\log 2}{2} + 2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - 1\right) - \frac{3}{2}, \qquad C_2 = C_1 + \frac{2\log 2}{\zeta(2)}\left(\frac{\zeta'(2)}{\zeta(2)} - \frac{1}{2}\right);$$

γ is the Euler constant. At the same time, for the mean number of steps in the Euclidean algorithm with the choice of least absolute value remainders, the only known estimate is

$$\frac{2}{R(R+1)} \sum_{b\le R} \sum_{a\le b} l\left(\frac{a}{b}\right) = \frac{2\log\varphi}{\zeta(2)} \cdot \log R + C_3 + O(R^{-\beta}),$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio, C_3 is an absolute constant, and $\beta > 0$ (see [4]). It turns out that estimates similar to (2) and (3) are also valid for $l(a/b)$.

Given $x \in (0,1]$ and a rational number $r = [t_0;t_1,\dots,t_s]$, let $s_x(r)$ denote the Gauss–Kuz'min statistics

$$s_x(r) = \#\{j = 1,\dots,s : [0;t_j,\dots,t_s] \le x\}.$$

In particular, $s_1(r) = s(r)$ is the length of the continued fraction for r.

Lemma. *For any rational number a/b,*

$$l\left(\frac{a}{b}\right) = s_{\varphi-1}\left(\frac{a}{b}\right).$$

Proof. Let $a/b = [t_0;t_1,\dots,t_s]$, and let $s'(a/b)$ denote the number of remainders

$$r_j = [0;t_j,\dots,t_s], \qquad \text{where} \quad 1 \le j \le s,$$

whose decompositions begin with an odd number of ones. Accordingly, let $s''(a/b)$ denote the number of remainders beginning with an even (possibly, zero) number of ones. Obviously, we have

$$s\left(\frac{a}{b}\right) = s'\left(\frac{a}{b}\right) + s''\left(\frac{a}{b}\right).$$

According to the algorithm described above, when an ordinary continued fraction is transformed into a fraction of the form (1), each sequence of k ones is replaced by $[k/2]$ partial quotients. Thus, precisely $s'(a/b)$ partial quotients disappear:

$$l\left(\frac{a}{b}\right) = s\left(\frac{a}{b}\right) - s'\left(\frac{a}{b}\right) = s''\left(\frac{a}{b}\right).$$

But the continued fraction for $r \in (0,1)$ begins with an even number of ones if and only if $r \in (0,\varphi-1)$. Therefore, $s''(a/b) = s_{\varphi-1}(a/b)$. □

Estimates (2) and (3) can be generalized to the case of Gauss–Kuz'min statistics (see [3], [5]). The bound for the remainder term in (7) is uniform in x provided that $x \in [x_0, 1]$ for some fixed $x_0 > 0$.

Substituting these estimates into the relation in the lemma, we obtain the following result.

Theorem 1. *For any $R \geq 2$,*

$$\frac{2}{R(R+1)} \sum_{b \leq R} \sum_{a \leq b} l\left(\frac{a}{b}\right) = \frac{2 \log \varphi}{\zeta(2)} \cdot \log R + C_3 + O(R^{-1} \log^4 R).$$

Proof. Estimate (3) is generalized to Gauss–Kuz'min statistics as (see [3])

$$\frac{2}{R(R+1)} \sum_{b \leq R} \sum_{a \leq b} s_x\left(\frac{a}{b}\right) = \frac{2}{\zeta(2)} (\log(1+x) \log R + C_2(x)) + O(R^{-1} \log^4 R).$$

Substituting the equality from the lemma into this relation, we obtain the required estimate with the constant

$$C_3 = \frac{2}{\zeta(2)} C_2(\varphi) = \frac{2 \log \varphi}{\zeta(2)} \left(-\frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} \log \varphi - \frac{3}{2}\right) + \frac{2}{\zeta(2)} (h_1(\varphi) + h_2(\varphi)) + \frac{1}{\varphi^3},$$

where the functions $h_1(x)$ and $h_2(x)$ are defined as the absolutely convergent series

$$h_1(x) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{n} \frac{x}{n+mx} - \log(1+x) \right), \tag{4}$$

$$h_2(x) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{n/x \leq m < n/x+n} \frac{1}{m} - \log(1+x) \right). \quad \square \tag{5}$$

Theorem 2. *For any $b \geq 2$,*

$$\frac{1}{\varphi(b)} \sum_{\substack{1 \leq a \leq b \\ (a,b)=1}} l\left(\frac{a}{b}\right) = \frac{2 \log \varphi}{\zeta(2)} \cdot \log b + C_4 + O_\varepsilon(b^{5/6} \log^{7/6+\varepsilon} b). \tag{6}$$

Proof. Substituting the equality from the lemma into the generalization

$$\frac{1}{\varphi(b)} \sum_{\substack{1 \leq a \leq b \\ (a,b)=1}} s_x\left(\frac{a}{b}\right) = \frac{2}{\zeta(2)} (\log(x+1) \log b + C_1(x)) + O_{\varepsilon,x}(b^{5/6} \log^{7/6+\varepsilon} b) \tag{7}$$

of Porter's estimate (2) (see [5]), we obtain the required relation with the constant

$$C_4 = \frac{2}{\zeta(2)} C_1(\varphi) = \frac{2 \log \varphi}{\zeta(2)} \left(2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} \log \varphi - 1\right) + \frac{2}{\zeta(2)} (h_1(\varphi) + h_2(\varphi)) - \frac{1}{\varphi^2},$$

where $h_1(x)$ and $h_2(x)$ are given by (4) and (5). $\square$

Remark. Consider the variance of $l(a/b)$, which is

$$D_l(R) = \frac{2}{R(R+1)} \sum_{d \leq R} \sum_{c \leq d} \left(l\left(\frac{c}{d}\right) - E_l(R) \right)^2, \tag{8}$$

where $E_l(R)$ is the expectation on the left-hand side of (3). One of the results of [4] is the asymptotic formula

$$D_l(R) = D_1 \cdot \log R + D_0 + O(R^{-\beta}),$$

where D_1 and D_0 are absolute constants and $\beta > 0$. The variance $D_s(R)$ of $s(a/b)$, which is defined in a similar way, satisfies the more accurate relation (see [3])

$$D_s(R) = D_1' \cdot \log R + D_0' + O(R^{-1/4+\varepsilon}).$$

The proof of this estimate remains valid when the length of the continued fraction $s(a/b)$ is replaced by the Gauss–Kuz'min statistics. This, together with the lemma, implies (8) for any $\beta < 1/4$.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (grant no. 07-01-00306) and Far East Division, Russian Academy of Sciences (grant no. 06-III-A-01-017).

REFERENCES

1. O. Perron, *Die Lehre von den Kettenbrüchen*, Vol. I: *Elementare Kettenbrüche* (B. G. Teubner, Stuttgart, 1954).
2. J. W. Porter, Mathematika **22** (1), 20 (1975).
3. A. V. Ustinov, Izv. Izv. Ross. Akad. Nauk Ser. Mat. Ser. Mat. **72** (5), 189 (2008) [Russian Acad. Sci. Izv. Math. **72** (5), 1023 (2008)].
4. V. Baladi and B. Vallée, J. Number Theory **110** (2), 331 (2005).
5. A. V. Ustinov, Algebra Anal. **20** (5), 186 (2008). под графиком