Limiting Distribution of Frobenius Numbers for $n = 3$

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1 Introduction

The purpose of this paper is to give a complete derivation of the limiting distribution of large Frobenius numbers outlined in [1] and fill some gaps formulated there as hypotheses. We start with the basic definitions and descriptions of some results.

Consider n mutually coprime positive integers $a_1, a_2, \ldots, a_n$. This means that there is no $r > 1$ such that each a_j, $1 \leqslant j \leqslant n$, is divisible by r. Take N which later will tend to infinity and will be our main large parameter. Introduce the ensemble Q_N of mutually coprime $a = (a_1, \ldots, a_n)$, $1 \leqslant a_j \leqslant N$, $1 \leqslant j \leqslant n$, and P_N be the uniform probability distribution on Q_N. For each $a \in Q_N$ denote by $F(a)$ the largest integer number that is not representable in the form $x = x_1a_1 + \cdots + x_na_n$, where x_j are non-negative integers. $F(a)$ can be considered as a random variable defined on Q_N. The basic problem which will be discussed in this paper is the existence and the form of the limiting distribution for the normalized Frobenius number $f(a) = \dfrac{1}{N^{1+1/n}}F(a)$. The reason for this normalization will be explained below.

The case of $n = 2$ is simple in view of the classical result of Sylvester (see [7]) according to which $F(a_1, a_2) = a_1a_2 - a_1 - a_2$. It shows that in a typical situation F grows as N^2. The first non-trivial case is $n = 3$ where $F(a)$ grows as $N^{3/2}$. It is known (see [11]) that the numbers $F(a_1, a_2, a_3)$ have weak asymptotics:

$$\frac{1}{x_1x_2a_3^{7/2}} \sum_{a_1 \leqslant x_1a_3} \sum_{a_2 \leqslant x_2a_3} \left(F(a_1, a_2, a_3) - \frac{8}{\pi}\sqrt{a_1a_2a_3} \right) = O_{x_1,x_2,\varepsilon}\left(a_3^{-1/6+\varepsilon} \right)$$

(i.e. average value of $F(a_1, a_2, a_3)$ over small cube with the center (a, b, c) is equal to $\frac{8}{\pi}\sqrt{abc}$). For arbitrary n the following theorem was proven in [1].

Theorem 1. *Under some additional technical condition (see [1]) the family of probability distributions of* $f(a) = \frac{1}{N^{1+\frac{1}{n-1}}}F(a)$ *is weakly compact. This means that for every* $\varepsilon > 0$ *one can find* $\mathcal{D} = \mathcal{D}(\varepsilon)$ *such that*

$$P_N \left\{ \frac{1}{N^{1+\frac{1}{n-1}}} F(a) \leqslant \mathcal{D} \right\} \geqslant 1 - \varepsilon.$$

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In this theorem $\varepsilon, \mathcal{D}$ do not depend on N. It also implies the existence of the limiting points (in the sense of weak convergence) for the sequence of probability distributions of $f_N(a)$. As was already mentioned, in this paper we shall study the limiting distribution of $f_N(a) = \dfrac{1}{N^{3/2}} F(a)$, $a = (a_1, a_2, a_3)$ as $N \to \infty$. This distribution is not universal and will be described below.

Take any ρ, $0 < \rho < 1$, and consider its expansion into continued fraction

$$\rho = [0; h_1, h_2, \dots, h_s, \dots] \tag{1}$$

where $h_j \geqslant 1$ are integers. If ρ is rational then the continued fraction (1) is finite. The finite continued fractions $\rho_s = [0; h_1, \dots, h_s] = \dfrac{p_s}{q_s}$ are called the s-approximants of ρ. The numbers q_s satisfy initial conditions $q_0 = 1$, $q_1 = h_1$ and recurrent relations

$$q_s = h_s q_{s-1} + q_{s-2}, \quad s \geqslant 2\,. \tag{2}$$

Introduce the Gauss measure on $[0,1]$ given by the density $\pi(x) = \frac{1}{\ln 2(1+x)}$. Then the elements of the continued fraction (1) become random variables. It is well known that their probability distributions are stationary in the sense that the distribution of any $h_{m-k}, h_{m-k+1}, \dots, h_m, \dots, h_{m+k}$ does not depend on m. We shall need the values of $s = s_1$, such that q_{s_1} is the first q_s greater than $\sqrt{N}$. It was proven in [6] that $q_{s_1}/\sqrt{N}$ have a limiting distribution as $N \longrightarrow \infty$. More precisely, the following theorem holds true.

Theorem 2. *Let k be fixed and $s(R)$ be the first number for which $q_s \geqslant R$. As $R \to \infty$ there exists the joint limiting distribution of $\frac{q_{s(R)}}{R}$, $h_{s(R)-k}$, $\dots$, $h_{s(R)+k}$.*

In the paper [10] the analytic form of this distribution was given.

Consider the sub-ensemble $Q_N^{(0)} \subset Q_N$ for which a_1, a_3 are coprime. Then there exists $a_1^{-1} (\operatorname{mod} a_3)$, $1 \leq a_1^{-1} < a_3$. Denote $\rho = \frac{a_1^{-1} a_2 (\operatorname{mod}\, a_3)}{a_3}$. The expansion of ρ into continued fraction will be need below. Clearly, ρ is a rational number. However, the following theorem is valid.

Theorem 3. *As before, consider s_1 such that $q_{s_1 - 1} < \sqrt{N} < q_{s_1}$. Then in the sub-ensemble $Q_N^{(0)}$ equipped with the uniform measure and for any $k > 0$ in the limit $N \to \infty$ there exists the joint limiting probability distribution of $\frac{q_{s_1}}{\sqrt{N}}$, $h_{s_1 - k}$, $\dots$, $h_{s_1 + k}$ which coincides with the distribution in Theorem 2.*

A stronger version of theorem 3 is also valid.

Theorem 4. *Let the first elements of the continued fraction for ρ are given: $h_1, h_2, \dots, h_l$. Then as $N \to \infty$ the conditional distribution of $\frac{q_{s_1}}{\sqrt{N}}$, $h_{s_1 - k}$, $\dots$, $h_{s_1 + k}$ converges to the same limit as in Theorems 2 and 3.*

All these theorems are proven in section 3. Now we can formulate the main result of this paper.

Theorem 5. *There exists the limiting distribution of $f_N(a) = f_N((a_1, a_2, a_3))$, $(a_1, a_2, a_3) \in Q_N$ as $N \to \infty$.*

The proof of the main theorem is given in section 2. First we consider the sub-ensemble $Q_N^{(0)}$ and then explain how to extend the proof to Q_N.

Recently J. Marklof using different methods proved the existence of the limiting distribution of $\frac{1}{N^{1+\frac{1}{n-1}}} F(a)$ for any n (see [3]).

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2 The limiting Distribution of $f_N(a)$.

Return back to the case of arbitrary n. Introduce arithmetic progressions

$$\Pi_r = \{r + m a_n, m \geqslant 0\}, \quad 0 \leqslant r < a_n.$$

For non-negative integers $x_1, \ldots, x_{n-1}$ such that $x_1a_1 + x_2a_2 + \cdots + x_{n-1}a_{n-1} \in \Pi_r$ we write

$$x_1a_1 + \cdots + x_{n-1}a_{n-1} = r + m(x_1, \ldots, x_{n-1})a_n.$$

Define $\overline{m}(r) = \min\limits_{x_1\ldots,x_{n-1}} m(x_1, \ldots, x_{n-1})$ and put

$$\begin{aligned} F_1(a) &= \max_{0\leqslant r<a_n} \min_{\substack{x_1,\ldots,x_{n-1} \\ x_1a_1+\cdots+x_{n-1}a_{n-1}\in\Pi_r}} (r + m(x_1, \ldots, x_{n-1})a_n) \\ &= \max_{0\leq r<a_n} \min_{x_1a_1+\ldots+x_{n-1}\, a_{n-1} \equiv r(\bmod a_n)} (x_1a_1 + \ldots + x_{n-1}a_{n-1}). \end{aligned}$$

It was proven in [4] that $F(a) = F_1(a) - a_n$. A slightly weaker statement can be found in [1]. Since in a typical situation a_j grow as N while $F_1(a)$ grows as $N^{1+\frac{1}{n-1}}$ (see also [BS]) the limiting behavior of $\frac{F(a)}{N^{1+\frac{1}{n-1}}}$ and $\frac{F_1(a)}{N^{1+\frac{1}{n-1}}}$ is the same, but the analysis of $\frac{F_1(a)}{N^{1+\frac{1}{n-1}}}$ is slightly simpler. Let us write for $n = 3$

$$x_1a_1 + x_2a_2 = r + m(x_1, x_2)a_3$$

or

$$x_1a_1 + x_2a_2 \equiv r (\bmod a_3) \tag{3}$$

Assume that a_1, a_3 are coprime. Then there exists a_1^{-1}, $1 \leqslant a_1^{-1} < a_3$, such that $a_1 \cdot a_1^{-1} \equiv 1 (\bmod a_3)$. Choose a_1^{-1} so that $1 \leqslant a_1^{-1} < a_3$ and rewrite (3) as follows

$$x_1 + a_{12}x_2 \equiv r_1 (\bmod a_3) \tag{4}$$

where $a_{12} \equiv a_1^{-1}a_2 (\bmod a_3)$, $0 < a_{12} < a_3$ and $r_1 \equiv ra_1^{-1} (\bmod a_3)$, $0 \leqslant r_1 < a_3$. From (4)

$$a_{12}x_2 \equiv (r_1 - x_1)(\bmod a_3) \tag{5}$$

The expression (5) has a nice geometric interpretation. Consider $S=[0,1,\dots,a_3-1]$ as a "discrete circle". Let $\mathcal{R}$ be the rotation of this circle by a_{12}, i.e. $\mathcal{R}x=x+a_{12}(\mathrm{mod} a_3)$. Then $\mathcal{R}^p x=x+pa_{12}(\mathrm{mod} a_3)$ and (5) means that r_1-x_1 belongs to the orbit of 0 under the action of $\mathcal{R}$. From the definition of $F_1(a)$,

$$
\begin{aligned}
F_1(a) &= \max_{0\leqslant r<a_3} \min_{\substack{x_1a_1+x_2a_2\equiv r(\bmod a_3)\\ 0\leqslant x_1,x_2<a_3}} (x_1a_1+x_2a_2)= \\
&= N^{3/2}\max_{0\leqslant r_1<a_3} \min_{x_1+x_2a_{12}\equiv r_1 \pmod{a_3}} \left(\frac{x_1}{\sqrt{N}}\frac{a_1}{N}+\frac{x_2}{\sqrt{N}}\frac{a_2}{N}\right) \qquad (6)
\end{aligned}
$$

Choose $h^{(j)}=(h_1^{(j)},\dots,h_m^{(j)})$, $j=1,2,3$, and denote by $Q^{(0)}_{N,h^{(1)},h^{(2)},h^{(3)}}$ the ensemble of $a=(a_1,a_2,a_3)\in Q_N^{(0)}$ such that the first m elements of the continued fraction of $\frac{a_j}{N}$ are given by h^j, $j=1,2,3$. This step means the localization of the ensemble $Q_N^{(0)}$. It is easy to see that for every $\varepsilon>0$ one can find rational α_1, α_2, α_3 and N such that $\left|\frac{a_j}{N}-\alpha_j\right|\leqslant\varepsilon,\ 1\leqslant j\leqslant 3$. Then in (6) one can replace $\frac{a_j}{N}$ by α_j. Since $\frac{x_j}{\sqrt{N}}$ will take the values $O(1)$ the whole expression in (6) takes values $O(1)$ and instead of (6) we consider

$$
\max_{r_1} \min_{x_1+a_{12}x_2\equiv r_1 \pmod{a_3}} \left(\frac{x_1}{\sqrt{N}}\alpha_1+\frac{x_2}{\sqrt{N}}\alpha_2\right) \qquad (7)
$$

with the error $O(\varepsilon)$. We assume that in $Q^{(0)}_{N,h^{(1)},h^{(2)},h^{(3)}}$ we also have the uniform distribution.

We shall need some facts from the theory of rotations of the circle. According to our assumption a_{12} and a_3 are coprime. Therefore $\mathcal{R}$ is ergodic in the sense that $\mathcal{R}^{a_3}=Id$ and a_3 is the smallest number with this property. Put $\rho=\frac{a_{12}}{a_3}$ and write down the expansion of ρ into continued fraction: $\rho=[h_1,h_2,\dots,h_{s_0}]$. Also let $\rho_s=[h_1,h_2,\dots,h_s]=\frac{p_s}{q_s}$ and s_1 be such that $q_{s_1-1}<\sqrt{N}<q_{s_1}$.

It will be more convenient to consider the usual unit circle instead of S and use the same letter $\mathcal{R}$ for the rotation of the unit circle by ρ. Introduce the interval $\Delta_0^{(p)}$ bounded by 0 and $\{q_p\rho\}$ and $\Delta_j^{(p)}=\mathcal{R}^j\Delta_0^{(p)}$. Using the induction one can show that $\Delta_j^{(p)}$, $0\leqslant j<q_{p+1}$ and $\Delta_{j'}^{(p+1)}$, $0\leqslant j'<q_p$ are pair-wise disjoint and their union is the whole circle except the boundary points (see [5]). Denote by $\eta^{(p)}$ the partition of the unit circle onto $\Delta_j^{(p)}$, $\Delta_{j'}^{(p+1)}$. Then $\eta^{(p+1)}\geqslant\eta^{(p)}$ in the sense that each clement of $\eta^{(p)}$ consists of several elements of $\eta^{(p+1)}$. More precisely, $\Delta_0^{(p-1)}$ consists of h_p elements $\Delta_j^{(p)}$ and one element $\Delta_0^{(p+1)}$. The partitions $\eta^{(p)}$ show how the orbit of 0 fills the circle.

Return back to the discrete circle S. The partitions $\eta^{(p)}$ can be constructed in the same way as before. We have to analyze

$$
\max_{0\leqslant r_1<a_3} \min_{\substack{x_1,x_2\\ x_1+a_{12}x_2\equiv r_1(\bmod a_3)}} \left(\frac{x_1}{\sqrt{N}}\alpha_1+\frac{x_2}{\sqrt{N}}\alpha_2\right) \qquad (8)
$$

for given $\alpha_1,\alpha_2,\ 0<\alpha_1,\alpha_2<1$.

Lemma 1. *There exists some number $C_1(\alpha_1, \alpha_2) = C_1$ such that for any r_1 the point x_1 giving* $\min\left(\frac{x_1}{\sqrt{N}}\alpha_1 + \frac{x_2}{\sqrt{N}}\alpha_2\right)$ *under the condition $x_1 + a_{12}x_2 \equiv r_1 (\bmod\ a_3)$ is such that $r_1 - x_1$ is an end-point of some element $\eta^{(s_1+m_1)}$ where $m_1 \geqslant 0$ and $q_{s_1+m_1}/q_{s_1} \leq C_1(\alpha_1, \alpha_2)$.*

Proof. Choose y_1 so that $r_1 - y_1$ is an end-point of some element $\eta^{(s_1)}$ and find y_2 for which $r_1 - y_1 \equiv a_{12}y_2 (\bmod\ a_3)$. Then both y_1, y_2 satisfy the inequalities $|y_1| \leqslant C_2 \cdot q^{(s_1)}, |y_2| \leqslant C_2 \cdot q^{(s_1)}$ where C_2 is another constant depending on the elements of our continued fraction near s_1 and $\frac{y_1}{\sqrt{N}}\alpha_1 + \frac{y_2}{\sqrt{N}}\alpha_2 < 2C_2(\alpha_1, \alpha_2)$. If $r_1 - x_1$ is the end-point of some element of $\eta^{(s_1+m_1)}$ which is not the end-point of some element of $\eta^{(s_1+m_1-1)}$ then $\frac{x_1}{\sqrt{N}}\alpha_1 + \frac{x_2}{\sqrt{N}}\alpha_2 \geqslant 2C_2)(\alpha_1, \alpha_2)$ and the pair (x_1, x_2) cannot give the solution of our max-min problem. This completes the proof of the lemma. □

Its meaning is the following. If $r_1 - x_1$ is an end-point of $\eta^{(s_1+m_1)}$ with too big m_1 then x_2 is also too big. The next lemma shows that x_1 also cannot be too big.

Lemma 2. *There exists an integer $m_2 > 0$ depending on α_1, α_2, the ratio q_{s_1}/N and the elements of the continued fraction $h_{s_1}, h_{s_1+1}, \ldots, h_{s_1+m_2}$ of ρ such that for any r_1 the interval $[r_1 - x_1, r_1]$ corresponding to the minimum of*

$$\frac{x_1}{\sqrt{N}}\alpha_1 + \frac{x_2}{\sqrt{N}}\alpha_2$$

has not more than m_2 elements of $\eta^{(s_1)}$.

The proof is also simple. If x_1 is such that $[r_1 - x_1, r_1]$ is an element of $\eta^{(s_1)}$ then

$$\frac{x_1}{\sqrt{N}}\alpha_1 + \frac{x_2}{\sqrt{N}}\alpha_2 \leqslant C_3$$

where C_3 is a number of depending on the values of parameters given in the formulation of the lemma. On the other hand if $[r_1 - x_1, r_1]$ consists of m elements of $\eta^{(s_1)}$ then

$$\frac{x_1}{\sqrt{N}}\alpha_1 + \frac{x_2}{\sqrt{N}}\alpha_2 \geqslant \frac{x_1}{\sqrt{N}}\alpha_1 = \frac{ml}{\sqrt{N}}\alpha_1$$

where ℓ is the minimal length of the elements of $\eta^{(s_1)}$. Therefore

$$\frac{\ell}{\sqrt{N}} = \frac{q_s}{\sqrt{N}} \cdot \frac{\ell}{q_s} \geqslant C_4$$

where C_4 is another constant. If m is so large that $mC_4\alpha_1 > C_3$ then the corresponding x_1, x_2 cannot give the solution of the main max-min problem.

The values of $q_{s_1}/\sqrt{N}$ and $h_{s_1}, h_{s_1+1} \ldots, h_{s_1+m_2}$ determine the structure of the partitions $\eta^{(s_1)}, \ldots, \eta^{(s_1+m_2)}$. The conclusion which follows from both lemmas is that for each r_1 we check only finitely many x_1 and x_2 and find $\min(x_1\alpha_1 + x_2\alpha_2)$ among them. The number of points which have to be checked depends on α_1, α_2, $\frac{q_{s_1}}{\sqrt{N}}$ and $h_{s_1}, \ldots, h_{s_1+m_2}$.

Now we remark that r_1 must be also an end-point of $\eta^{(s_1)}$. Indeed, if r_1 increases within some element of $\eta^{(s_1)}$ then the set of values $r_1 - x_1$ which have to be checked remain the same. The maximum over r_1 is attained at the end-point of this element $\eta^{(s_1)}$ because $r_1 - x_1$ is a monotone increasing function of r_1.

The last step in the proof is the final choice of r_1. As was mentioned above r_1 must be an end-point of some element of $\eta^{(s_1)}$ and $\frac{x_1}{\sqrt{N}}$ takes finitely many values. Therefore r_1 should be chosen so that $x_2/\sqrt{N}$ takes the largest possible value. Take the last point $r_1' = \mathcal{R}^{q_{s_1}-1}0$ on the orbit of 0 of the length q_{s_1}. Assume for definiteness that r_1' lies to the left from 0. Consider m_2 elements of $\eta^{(s_1)}$ which start from r_1' and go left. Then r_1 must be one of the end-points of these elements. Indeed, if r_1 lies more to the left from 0 then the values x_1 take finitely many values and x_2 will be significantly smaller. Therefore it cannot give maximum over r of our basic linear form.

Thus we take m_2 elements of $\eta^{(s_1)}$, consider their end-points. Each end-point is a possible value of r. Taking finitely many x_1 (see Lemma 1 and 2) we find minimum of our basic linear form. After that we find r for which this minimum takes maximal value. In this way we get the solution of our max-min problem. It is clear that this solution is a function of $\frac{q_{s_1}}{\sqrt{N}}$ and the elements $h_j, s_1 \leq j \leq s_1 + m_1$ of the continued fraction of ρ near s_1. Since $\frac{q_{s_1}}{\sqrt{N}}$ and $h_j, s_1 \leq j \leq s_1 + m_1$ have limiting distribution as $N \to \infty$ the number $f_N(a) = \frac{1}{N^{3/2}} F_1(a)$ has also a limiting distribution.

It remains to extend our proof to the case when the pairs from a_1, a_2, a_3 have non-trivial common divisors, say k_1 is *gcd* of a_1, a_3 and k_2 is *gcd* of a_2, a_3. The same methods which are used in the proof of the existence of the limiting density of the ensemble Q_N allow to prove the existence of the limiting distribution of k_1 and k_2. Fixing k_1, k_2, we can write $a_1 = k_1 a_1'$, $a_2 = k_2 a_2'$, $a_3 = k_1 k_2 a_3'$ where a_1', a_3' are coprime, a_2', a_3 are coprime and k_1, k_2 are coprime. This implies that $(a_1')^{-1} (\bmod a_3')$ exists and we can multiply both sides of (3) by $(a_1')^{-1}$. This will give

$$k_1 x_1 + k_2 a_2' \cdot x_2 \equiv r_1 (\bmod a_3) \tag{9}$$

where $r_1 = r \cdot (a_1')^{-1} (\bmod a_3)$. Denote $b = a_2'(a_1')^{-1}$.

Then from (9) we have the linear form

$$k_1 x_1 + k_2 b x_2 \equiv r_1 (\bmod a_3) \tag{10}$$

which we can treat in the same way as before.

3 Statistical properties of continued fractions

Statistical properties of elements of continued fractions usually are identical for real numbers and for rationales with bounded denominators (see [8]–[10]).

Let $\mathcal{M}$ be a set of integer matrices $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right)$ with determinant $\det S = \pm 1$ such that $1 \leqslant Q \leqslant Q'$, $0 \leqslant P \leqslant Q$, $1 \leqslant P' \leqslant Q'$. For real $\alpha \in (0,1)$ the fractions P/Q and P'/Q' with $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$ will be consecutive convergents to α (distinct from α) if and only if

$$0 < \frac{Q'\alpha - P'}{-Q\alpha + P} = S^{-1}(\alpha) < 1$$

(see [8, lemma 1]). Moreover if $\alpha = [0; h_1, h_2, \ldots]$ then for some $s \geqslant 1$,

$$\begin{aligned} \frac{P}{Q} &= [0; h_1, \ldots, h_{s-1}], & \frac{P'}{Q'} &= [0; h_1, \ldots, h_s], \\ \frac{Q}{Q'} &= [0; h_s, \ldots, h_1], & \frac{Q'\alpha - P'}{-Q\alpha + P} &= [0; h_{s+1}, h_{s+2}, \ldots]. \end{aligned} \tag{11}$$

It means that distribution of partial quotients $h_{s-k}, \ldots, a_{h+k}$ depends on Gauss-Kuz'min statistics of fractions Q/Q' and $(Q'\alpha - P')/(-Q\alpha + P)$.

For real α, x_1, x_2, y_1, $y_2 \in (0,1)$ denote by $N_{x_1,x_2,y_1,y_2}(\alpha, R)$ the number of solutions of the following system of inequalities

$$0 < S^{-1}(\alpha) \leqslant x_1, \quad Q \leqslant x_2 Q', \quad Q \leqslant y_1 R, \quad R \leqslant y_2 Q', \tag{12}$$

with variables P, P', Q, Q' such that $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$. Let

$$N(R) = N_{x_1,x_2,y_1,y_2}(R) = \int_0^1 N_{x_1,x_2,y_1,y_2}(\alpha, R)\, d\alpha$$

and

$$F(x_1, x_2, y_1, y_2) = \begin{cases} \frac{2}{\zeta(2)} \left(\log(1 + x_1 x_2) \log \frac{y_1 y_2}{x_2} - \mathrm{Li}_2(-x_1 x_2) \right), & \text{if } x_2 \leqslant y_1 y_2; \\ -\frac{2}{\zeta(2)} \mathrm{Li}_2(-x_1 y_1 y_2), & \text{if } x_2 > y_1 y_2, \end{cases}$$

where $\mathrm{Li}_2(\cdot)$ is dilogarithm

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\log(1-t)}{t} dt.$$

The next statement implies Theorem 2.

Proposition 1. *For $R \geqslant 2$,*

$$N(R) = F(x_1, x_2, y_1, y_2) + O\left(\frac{x_1 \log R}{R} \right).$$

Proof. For every number $\alpha = [0; a_1, a_2, \ldots]$ we can find unique matrix $S \in \mathcal{M}$ with elements P, P', Q, Q' defined by (11) with additional restriction $Q \leqslant R < Q'$. Inequalities $0 < S^{-1}(\alpha) \leqslant x_1$ define interval $I_{x_1}(S) \subset (0,1)$ of the length

$$|I_{x_1}(S)| = \left| \frac{P' + x_1 P}{Q' + x_1 Q} - \frac{P'}{Q'} \right| = \frac{x_1}{Q'(Q' + x_1 Q)}.$$

Hence

$$N(R)=\sum_{\left(\begin{smallmatrix}P & P'\\ Q & Q'\end{smallmatrix}\right)\in\mathcal{M}}[Q\leqslant x_2Q', Q\leqslant y_1R, R\leqslant y_2Q']\frac{x_1}{Q'(Q'+x_1Q)},$$

where $[A]$ is equal to 1 if statement A is true, and it is equal to 0 otherwise. Second row (Q,Q') can be complemented to the matrix from $\mathcal{M}$ in two ways. That is why

$$N(R)=2\sum_{Q'\geqslant R/y_2}\sum_{(Q,Q')=1}[Q\leqslant x_2Q', Q\leqslant y_1R]\frac{x_1}{Q'(Q'+x_1Q)}. \tag{13}$$

In the first case $x_2\leqslant y_1y_2$ and the Möbius inversion formula gives

$$\begin{aligned}
N(R)=&2\sum_{d\leqslant R}\frac{\mu(d)}{d^2}\sum_{R/(y_2d)\leqslant Q'<y_1R/(x_2d)}\sum_{Q\leqslant x_2Q'}\frac{x_1}{Q'(Q'+x_1Q)}+\\
&+2\sum_{d\leqslant R}\frac{\mu(d)}{d^2}\sum_{Q'\geqslant y_1R/(x_2d)}\sum_{Q\leqslant y_1R/d}\frac{x_1}{Q'(Q'+x_1Q)}=\\
=&\frac{2}{\zeta(2)}\left(\log(1+x_1x_2)\log\frac{y_1y_2}{x_2}+\int_{1/(x_1x_2)}^{\infty}\log\left(1+\frac{1}{t}\right)\frac{dt}{t}\right)+O\left(\frac{x_1\log R}{R}\right)=\\
=&\frac{2}{\zeta(2)}\left(\log(1+x_1x_2)\log\frac{y_1y_2}{x_2}-\mathrm{Li}_2(-x_1x_2)\right)+O\left(\frac{x_1\log R}{R}\right).
\end{aligned}$$

Second case $x_2>y_1y_2$ can be treated in the same way. □

Let

$$L(R)=L_{x_1,x_2,y_1,y_2}(R)=\sum_{b\leqslant R^2}\sum_{\substack{a\leqslant b\\(a,b)=1}}N_{x_1,x_2,y_1,y_2}\left(\frac{a}{b},R\right).$$

Theorem 3 will be proved in the following form.

Proposition 2. *For $R\geqslant 2$,*

$$\frac{2\zeta(2)}{R^4}L(R)=F(x_1,x_2,y_1,y_2)+O\left(\frac{x_1\log^2 R}{R}\right).$$

Proof. Let $\alpha=a/b$ be a given number and $S=\left(\begin{smallmatrix}P & P'\\ Q & Q'\end{smallmatrix}\right)\in\mathcal{M}$ is a solution of the system (12). Define by m and n such integers that $mP+nP'=a, mQ+nQ'=b$. Then the system (12) can be written in the following way

$$\begin{gathered}
mP+nP'=a,\quad mQ+nQ'=b,\\
0<m/n\leqslant x_1,\quad 0<Q/Q'\leqslant x_2,\quad Q\leqslant y_1R,\quad R\leqslant y_2Q'.
\end{gathered}$$

Summing up solutions of this system over a and b we get that the sum $L(R)$ is equal to the number of solutions of the following system

$$mQ+nQ'\leqslant R^2,\quad 0<m/n\leqslant x_1,\quad 0<Q/Q'\leqslant x_2,\quad Q/y_1\leqslant R<y_2Q',$$

where $\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$, $0 \leqslant m \leqslant n$, $(m,n) = 1$. For known Q and Q' the values of P and P' can be founded in two ways. The number of solutions of the last system is equal to the area of corresponding domain multiplied by $1/\zeta(2)$ (see [13, Chapter II, Problems 21–22])

$$\frac{R^4}{2\zeta(2)} \cdot \frac{x_1}{Q'(Q'+x_1Q)} + O\left(\frac{x_1R^2\log R}{Q'}\right).$$

It leads to the sum similar to (13):

$$L(R) = \frac{R^4}{\zeta(2)} \sum_{R/y_2 \leqslant Q' \leqslant R^2} \sum_{\substack{Q \leqslant \min\{y_1R, x_2Q'\} \\ (Q,Q')=1}} \frac{x_1}{Q'(Q'+x_1Q)} + O(x_1R^3\log^2 R).$$

Therefore

$$L(R) = \frac{R^4}{\zeta(2)} N(R) + O(x_1R^3\log^2 R),$$

and Proposition 2 follows from Proposition 1. □

In order to prove Theorem 4 we have to use Kloosterman sums

$$K_q(m,n) = \sum_{x,y=1}^{q} \delta_q(xy-1)\, e^{2\pi i \frac{mx+ny}{q}}.$$

Using Estermann bound (see [2])

$$|K_q(m,n)| \leqslant \sigma_0(q) \cdot (m,n,q)^{1/2} \cdot q^{1/2}.$$

it is easy to prove the following statement (see [9] for details).

Lemma 3. *Let $q \geqslant 1$ be an integer, Q_1, Q_2, P_1, P_2 be real numbers and $0 \leqslant P_1, P_2 \leqslant q$. Then the sum*

$$\Phi_q(Q_1,Q_2;P_1,P_2) = \sum_{\substack{Q_1 < u \leqslant Q_1+P_1 \\ Q_2 < v \leqslant Q_2+P_2}} \delta_q(uv-1)$$

satisfies the asymptotic formula

$$\Phi_q(Q_1,Q_2;P_1,P_2) = \frac{\varphi(q)}{q^2} \cdot P_1P_2 + O\left(\psi(q)\right),$$

where

$$\psi(q) = \sigma_0(q)\log^2(q+1)q^{1/2}.$$

It implies more general result (see [8]).

Lemma 4. *Let $q \geqslant 1$ be an integer and let $a(u,v)$ be a function that is defined in integral points (u,v) such that $1 \leqslant u, v \leqslant q$. Assume that this function satisfies the inequalities*

$$a(u,v) \geqslant 0, \quad \Delta_{1,0}a(u,v) \leqslant 0, \quad \Delta_{0,1}a(u,v) \leqslant 0, \quad \Delta_{1,1}a(u,v) \geqslant 0 \tag{14}$$

at all points at which these conditions are meaningful. Then the sum

$$W = \sum_{u,v=1}^{q} \delta_q(uv-1)a(u,v)$$

satisfies the asymptotic relation

$$W = \frac{\varphi(q)}{q^2} \sum_{u,v=1}^{q} a(u,v) + O\left(A\psi(q)\sqrt{q}\right),$$

where $\psi(q)$ is the function from lemma 3 and $A = a(1,1)$ is the maximum of the function $a(u,v)$.

Let

$$\begin{aligned} N_z(R) =& N_{z,x_1,x_2,y_1,y_2}(R) = \int_0^z N_{x_1,x_2,y_1,y_2}(\alpha, R)\, d\alpha, \\ L_z(R) =& L_{z,x_1,x_2,y_1,y_2}(R) = \sum_{b \leqslant R^2} \sum_{\substack{a \leqslant zb \\ (a,b)=1}} N_{x_1,x_2,y_1,y_2}\left(\frac{a}{b}, R\right). \end{aligned}$$

The next statement implies Theorem 4.

Proposition 3. *For $R \geqslant 2$,*

$$\begin{aligned} N_z(R) =& z \cdot F(x_1,x_2,y_1,y_2) + O\left(\frac{x_1 \log^3 R}{R^{1/2}}\right), \\ \frac{2\zeta(2)}{R^4} L_z(R) =& z \cdot F(x_1,x_2,y_1,y_2) + O\left(\frac{x_1 \log^3 R}{R^{1/2}}\right). \end{aligned}$$

Proof. Let

$$\mathcal{M}_z = \left\{ \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{M} : \frac{P'}{Q'} \leqslant z \right\}.$$

For a given z there is at most one matrix $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$ such that $Q \leqslant R < Q'$ and $z \in I_{x_1}(S)$. Hence

$$N_z(R) = \sum_{\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}_z} [Q \leqslant x_2 Q', Q \leqslant y_1 R, R \leqslant y_2 Q'] \frac{x_1}{Q'(Q'+x_1 Q)} + O\left(\frac{x_1}{R^2}\right).$$

If Q' is fixed then P' and Q satisfy the congruence $P'Q \equiv \pm 1 \pmod{Q'}$. Therefore

$$N_z(R) = \sum_{Q' \geqslant R/y_2} \sum_{P',Q=1}^{Q'} \delta_{Q'}(P'Q \pm 1)[Q \leqslant \min\{x_2 Q', y_1 R\}, P' \leqslant zQ'] \frac{x_1}{Q'(Q' + x_1 Q)} + \\ + O\left(\frac{x_1}{R^2}\right).$$

Using Lemma 4 we obtain

$$\begin{aligned} N_z(R) = &\sum_{Q' \geqslant R/y_2} \frac{\varphi(Q')}{(Q')^2} \sum_{P',Q=1}^{Q'} [Q \leqslant \min\{x_2 Q', y_1 R\}, P' \leqslant zQ'] \frac{x_1}{Q'(Q' + x_1 Q)} + \\ &+ O\left(\frac{x_1 \log^3 R}{R^{1/2}}\right) = \\ =& z \sum_{Q' \geqslant R/y_2} \frac{\varphi(Q')}{Q'} \sum_{Q=1}^{Q'} [Q \leqslant \min\{x_2 Q', y_1 R\}] \frac{x_1}{Q'(Q' + x_1 Q)} + \\ &+ O\left(\frac{x_1 \log^3 R}{R^{1/2}}\right). \end{aligned}$$

Applying the formula

$$\frac{\varphi(Q')}{Q'} = \sum_{d \mid Q'} \frac{\mu(d)}{d} \tag{15}$$

we get the same sum as in the proof of Proposition 1.

As in Proposition 2 the sum $L_z(R)$ is equal to the number of solutions of the system

$$\begin{gathered} mQ + nQ' \leqslant R^2, \quad mP + nP' \leqslant z(mQ + nQ'), \\ 0 < m/n \leqslant x_1, \quad 0 < Q/Q' \leqslant x_2, \quad Q/y_1 \leqslant R < y_2 Q', \end{gathered}$$

where $\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$, $0 \leqslant m \leqslant n$, $(m,n) = 1$. Again, there is at most one matrix $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$ such that $Q \leqslant R < Q'$ and $z \in I_{x_1}(S)$. Also for $Q' \geqslant R$,

$$\sum_{n \geqslant 1} \sum_{m \leqslant x_1 n} [mQ + nQ' \leqslant R^2] \ll x_1 R^2.$$

This estimate implies that

$$\begin{aligned} L_z(R) =& \frac{R^4}{\zeta(2)} \sum_{\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}_z} [R/y_2 \leqslant Q' \leqslant R^2, Q \leqslant \min\{y_1 R, x_2 Q'\}] \frac{x_1}{Q'(Q' + x_1 Q)} + \\ &+ O(x_1 R^3 \log^2 R) = \\ =& \frac{R^4}{\zeta(2)} \sum_{R/y_2 \leqslant Q' \leqslant R^2} \sum_{P',Q=1}^{Q'} [Q \leqslant \min\{y_1 R, x_2 Q'\}, P' \leqslant zQ'] \frac{x_1 \delta_{Q'}(P'Q \pm 1)}{Q'(Q' + x_1 Q)} + \\ &+ O(x_1 R^3 \log^2 R). \end{aligned}$$

Using Lemma 4 one more time we obtain

$$\begin{aligned}L_z(R) =&\frac{R^4}{\zeta(2)}\sum_{Q'\geqslant R/y_2}\frac{\varphi(Q')}{(Q')^2}\sum_{P',Q=1}^{Q'}[Q\leqslant\min\{x_2Q',y_1R\},P'\leqslant zQ']\frac{x_1}{Q'(Q'+x_1Q)}+\\&+O\left(x_1R^{7/2}\log^3 R\right)=\\&=\frac{zR^4}{\zeta(2)}\sum_{Q'\geqslant R/y_2}\frac{\varphi(Q')}{Q'}\sum_{Q=1}^{Q'}[Q\leqslant\min\{x_2Q',y_1R\}]\frac{x_1}{Q'(Q'+x_1Q)}+\\&+O\left(x_1R^{7/2}\log^3 R\right).\end{aligned}$$

Applying formula (15) we get the same sum as in proof of Proposition 1. □

Remark 1. *In the simplest case $x_2=y_1=y_2=1$ we have cumulative distribution function*

$$F(x)=F(x,1,1,1)=-\frac{2}{\zeta(2)}\mathrm{Li}_2(-x),$$

which is not equal to the Gaussian function $\log_2(1+x)$. As $x\to 0$ the function $F(x)$ (with error terms in Propositions 1 and 2) decreases as a linear function $F(x)\sim 2x/\zeta(2)$. This fact implies that the expectation of the partial quotient a_s (defined by the inequalities $q_{s-1}\leqslant R<q_s$) equals to infinity.

4 Concluding remarks

Methods of the work [11] allow to prove that normalized Frobenius numbers $F(a,b,c)/\sqrt{abc}$ have the following limit density function (see [12])

$$p(t)=\begin{cases}0, & \text{if } t\in[0,\sqrt{3}];\\ \frac{12}{\pi}\left(\frac{t}{\sqrt{3}}-\sqrt{4-t^2}\right), & \text{if } t\in[\sqrt{3},2];\\ \frac{12}{\pi^2}\left(t\sqrt{3}\arccos\frac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}+\frac{3}{2}\sqrt{t^2-4}\log\frac{t^2-4}{t^2-3}\right), & \text{if } t\in[2,+\infty).\end{cases}$$

References

[1] Bourgain J., Sinai Ya. G. Limiting behavior of large Frobenius numbers. — Uspekhi Mat. Nauk **62** (2007), no. 4(376), 77–90; translation in Russian Math. Surveys **62** (2007), no. 4, 713–725.

[2] Estermann T. On Kloosterman's sum. — *Mathematika*, **8** (1961), 83–86.

[3] Marklof J. The Asymptotic Distribution of Frobenius Numbers. Preprint, Bristol University (2009).

[4] Selmer E.S., Beyer O. On the linear diophantine problem of Frobenius in three variables. — *J. Reine Angewandte Math.*, 301 (1978), 161–170.

[5] Sinai Ya.G. Topics in Ergodic Theory, Princeton University Press, Princeton, NJ (1994), 218.

[6] Sinai Ya. G., Ulcigrai C. Renewal-type limit theorem for Gauss map and continued fractions. — *Ergodic Theory & Dynam. Sys.*, **28** (2008), 643-655.

[7] Sylvester J.J. Problem 7382. — *Educational Times* 37 (1884), 26; reprinted in: Mathematical questions with their solution, *Educational Times* (with additional papers and solutions) 41 (1884), 21.

[8] Ustinov A. V. On the statistical properties of finite continued fractions — *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **322** (2005), Trudy po Teorii Chisel, 186–211; translation in *J. Math. Sci. (N. Y.)* **137** (2006), no. 2, 4722–4738.

[9] Ustinov A. V. *On the number of solutions of the congruence* $xy \equiv l \pmod q$ *under twice differentiabie curve* — Algebra and Analysis **20**: 5 (2008), 186–216.

[10] Ustinov A. V. *On the Statistical Properties of Elements of Continued Fractions* — Doklady Mathematics, **79**: 1 (2009), 87-89.

[11] Ustinov A. V. *The solution of Arnold's problem on weak asymptotic for Frobenius numbers with three arguments*, — Mat. Sb., **200**: 4 (2009), 131-160.

[12] Ustinov A. V. *On the distribution of Frobenius numbers with three arguments* — Izvestiya Rossiiskoi Akademii Nauk. Seriya Mathematicheskaya, (submitted).

[13] Vinogradov I. M. *Elements of number theory.* — Moscow: "Nauka", 1972.