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The solution of Arnold's problem on the weak asymptotics of Frobenius numbers with three arguments

A. V. Ustinov

Abstract. It is shown that on the average the Frobenius numbers $f(a,b,c)$ behave like $\frac{8}{\pi}\sqrt{abc}$.

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§ 1. Introduction

Let $a_1,\dots,a_n$ be jointly coprime positive integers, which means that their greatest common divisor $(a_1,\dots,a_n)$ is 1. The *Frobenius number* $g(a_1,\dots,a_n)$ *of* $a_1,\dots,a_n$ is the largest integer m that cannot be represented as

$$x_1a_1+\dots+x_na_n=m, \tag{1}$$

where $x_1,\dots,x_n$ are non-negative integers. Often it is more convenient to consider the function

$$f(a_1,\dots,a_n)=g(a_1,\dots,a_n)+a_1+\dots+a_n,$$

which is equal to the largest integer m that cannot be represented in the form (1) with positive integer coefficients $x_1,\dots,x_n$ (see, for example, Johnson's identity in the proof of Lemma 3). The problem of finding $g(a_1,\dots,a_n)$ is called *Frobenius's problem*. The most comprehensive review of problems and results in this area is presented in [1].

For $n=2$ we have Sylvester's formula $f(a,b)=ab$ (see [2]). If $n=3$, then the problem of finding $f(a,b,c)$ reduces to the case of pairwise coprime arguments, and for $b\equiv lc \pmod a$, $1\leqslant l\leqslant a$, the value of $f(a,b,c)$ can be expressed in terms of the partial quotients of the continued fraction for l/a (see the results due to Selmer and Beyer, and Rødseth in [3] and [4]; as concerns other formulae for calculating $f(a,b,c)$, see [1], Ch. 2 and [5], [6]). For $n\geqslant 4$ no formulae for $f(a_1,\dots,a_n)$ are known. It has been proved that for fixed n the Frobenius number can be calculated in polynomial time (see [7]), while finding $f(a_1,\dots,a_n)$ for arbitrary n is an NP-complete problem (see [8]).

In the case $(a,b,c)=1$ Davison [9] proved the estimate $f(a,b,c)\geqslant\sqrt{3abc}$; the constant $\sqrt{3}$ here is sharp. He also conjectured in the same paper that for a 'random' set (a,b,c) the function $f(a,b,c)$ has order $\sqrt{abc}$. This was stated as two conjectures. Consider the set $X_N=\bigl\{(a,b,c):1\leqslant a,b,c\leqslant N,\ (a,b,c)=1\bigr\}$.

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Conjecture 1. The following inequality holds:

$$\sup_N \frac{1}{|X_N|} \sum_{(a,b,c)\in X_N} \frac{f(a,b,c)}{\sqrt{abc}} < \infty.$$

Conjecture 2. There exists a finite limit

$$\lim_{N\to\infty} \frac{1}{|X_N|} \sum_{(a,b,c)\in X_N} \frac{f(a,b,c)}{\sqrt{abc}}\,.$$

Arnold formulated a stronger conjecture (see [10], problems 1999-8, 2003-5; see also [11]).

Conjecture 3. For each $n \geqslant 2$ the distribution of the values of $f(a_1,\dots,a_n)$ is determined by a density proportional to $\sqrt[n-1]{a_1\cdots a_n}$. In other words, if

$$\begin{aligned} Q_{N,r} &= Q_{N,r}(\alpha_1,\dots,\alpha_n) \\ &= \biggl\{(a_1,\dots,a_n) : \left|\frac{a_j}{N} - \alpha_j\right| < r,\ j=1,\dots,n,\ (a_1,\dots,a_n)=1\biggr\}, \end{aligned}$$

then for some constant c_n, as $N\to\infty$ and $r=r(N)\to 0$, the normalized sum

$$\frac{1}{|Q_{N,r}|} \sum_{(a_1,\dots,a_n)\in Q_{N,r}} f(a_1,\dots,a_n),$$

behaves asymptotically like

$$c_n N^{1-1/n}\sqrt[n-1]{\alpha_1\cdots\alpha_n}\,.$$

The results of the corresponding numerical experiments were presented in [11]–[13].

Burgain and Sinai [14] investigated the limiting behaviour of the quantities $f(a,b,c)N^{-3/2}$ for $1\leqslant a,b,c\leqslant N$. Imposing a natural assumption, which was subsequently justified in [15], they proved by probabilistic methods the existence of a limiting distribution for $f(a,b,c)N^{-3/2}$.

It turns out that for $n=3$ the required density can be obtained by averaging with respect to two (of the three) parameters, and it can be explicitly described. The constant $c_3 = 8/\pi = 2.546\dots$ is important in this analysis.

Consider the set

$$M_a(x_1,x_2) = \bigl\{(b,c) : 1\leqslant b\leqslant x_1a,\ 1\leqslant c\leqslant x_2a,\ (a,b,c)=1\bigr\}.$$

Theorem 1. *Let a be a positive integer and x_1, x_2 and ε be positive real numbers. Then*

$$\frac{1}{a^{3/2}|M_a(x_1,x_2)|} \sum_{(b,c)\in M_a(x_1,x_2)} \biggl(f(a,b,c) - \frac{8}{\pi}\sqrt{abc}\biggr) = O_\varepsilon(R_\varepsilon(a;x_1,x_2)),$$

where

$$\begin{aligned} R_\varepsilon(a;x_1,x_2) &= \bigl(a^{-1/6}(x_1+x_2) + a^{-1/4}(x_1^{3/2}+x_2^{3/2})(x_1x_2)^{-1/4} + a^{-1/2}\bigr)a^\varepsilon \\ &\ll_{x_1,x_2} a^{-1/6+\varepsilon}. \end{aligned}$$

The proof of this theorem is based on Rødseth's formula for Frobenius numbers in [3], continued fraction theory and estimates for Kloosterman sums. It also uses ideas that we used earlier to investigate the statistical properties of continued fractions (see [16]–[18]).

The square

$$\left|\frac{b}{a}-\beta\right|<r, \qquad \left|\frac{c}{a}-\gamma\right|<r$$

in the (b,c)-plane can be expressed as combinations of rectangles of the form $[0,x_1a]\times[0,x_2a]$, where $x_1=\beta\pm r$, $x_2=\gamma\pm r$. Hence from Theorem 1 we obtain a stronger form of Arnold's conjecture for $n=3$ with constant $c_3=8/\pi$: if

$$Q'_{N,r}=Q'_{N,r}(\alpha,\beta,\gamma)=\left\{(a,b,c):a=\alpha N,\ \left|\frac{b}{N}-\beta\right|<r,\ \left|\frac{c}{N}-\gamma\right|<r,\ (a,b,c)=1\right\},$$

then

$$\frac{1}{|Q'_{N,r}|}\sum_{(a,b,c)\in Q'_{N,r}}f(a,b,c)=\frac{8}{\pi}\sqrt{\alpha\beta\gamma}\,N^{3/2}\bigl(1+O_{\alpha,\beta,\gamma,\varepsilon}(r^{-2}N^{-1/6+\varepsilon}+r)\bigr).$$

This is a nontrivial result for $N^{-1/12+\varepsilon}\ll r\ll N^{-\varepsilon}$.

It also follows from Theorem 1 that Conjecture 2 holds in a stronger form.

Theorem 2. *Let* $Y_N=\bigl\{(a,b,c):a=N,\ 1\leqslant b,\ c\leqslant N,\ (a,b,c)=1\bigr\}$. *The for each* $\varepsilon>0$,

$$\frac{1}{|Y_N|}\sum_{(a,b,c)\in Y_N}\frac{f(a,b,c)}{\sqrt{abc}}=\frac{8}{\pi}+O_\varepsilon(N^{-1/12+\varepsilon}).$$

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§ 2. Continuants

Let a, b and c be positive integers, $(a,b)=(a,c)=(b,c)=1$, and let l be an integer such that $bl\equiv c\pmod a$, $1\leqslant l\leqslant a$. Rødseth's formula for $f(a,b,c)$ is based on the expansion of a/l in a reduced regular continued fraction (see [19], §§ 42, 43):

$$\frac{a}{l}=\langle b_0;b_1,\dots,b_m\rangle=b_0-\cfrac{1}{b_1-\ddots-\cfrac{1}{b_m}}, \tag{2}$$

where $b_0=\lceil a/l\rceil=-\lfloor -a/l\rfloor$ (the integer closest from above) and $b_1,\dots,b_m\geqslant 2$. We denote by $m=m(a/l)$ the length of the fraction (2). To work with such fractions it is convenient to modify the standard definition of continuants as follows (see [20], § 6.7):

$$K_0(\,)=1, \qquad K_1(x_1)=x_1,$$
$$K_n(x_1,\dots,x_n)=x_nK_{n-1}(x_1,\dots,x_{n-1})-K_{n-2}(x_1,\dots,x_{n-2}), \qquad n\geqslant 2.$$

It is also natural to set $K_{-1}=0$. Due to the recurrence relations for the numerators and denominators of continued fractions, for $m\geqslant 0$ we have

$$\langle x_0;x_1,\dots,x_m\rangle=\frac{K_{m+1}(x_0,x_1,\dots,x_m)}{K_m(x_1,\dots,x_m)}\,.$$

We reformulate Euler's rule (see [20]): the polynomial $K_n(x_1,\dots,x_n)$ can be obtained by starting from the product $x_1\cdots x_n$, removing pairs of the form x_kx_{k+1} from it in all possible ways, and adding all the results together, taking the coefficients to be $(-1)^j$, where j is the total number of pairs deleted. For example,

$$\begin{aligned}K_4(x_1,x_2,x_3,x_4)&=x_1x_2x_3x_4-\cancel{x_1x_2}x_3x_4-x_1\cancel{x_2x_3}x_4-x_1x_2\cancel{x_3x_4}+\cancel{x_1x_2}\cancel{x_3x_4}\\&=x_1x_2x_3x_4-x_3x_4-x_1x_4-x_1x_2+1.\end{aligned}$$

From Euler's rule we obtain the symmetry

$$K_n(x_1,\dots,x_n)=K_n(x_n,\dots,x_1),$$

the left-hand recurrence relation

$$K_n(x_1,\dots,x_n)=x_1K_{n-1}(x_2,\dots,x_n)-K_{n-2}(x_3,\dots,x_n),\qquad n\geqslant 2,$$

and the more general formula

$$\begin{aligned}K_{m+n}(x_1,\dots,x_m,x_{m+1},\dots,x_{m+n})&=K_m(x_1,\dots,x_m)K_n(x_{m+1},\dots,x_{m+n})\\&\quad-K_{m-1}(x_1,\dots,x_{m-1})K_{n-1}(x_{m+2},\dots,x_{m+n})\end{aligned}$$

(this corresponds to (6.133) in [20]). All these relations are special cases of Euler's identity

$$\begin{aligned}&K_{m+n}(x_1,\dots,x_{m+n})K_l(x_{m+1},\dots,x_{m+l})\\&\qquad-K_{m+l}(x_1,\dots,x_{m+l})K_n(x_{m+1},\dots,x_{m+n})\\&\qquad+K_{m-1}(x_1,\dots,x_{m-1})K_{n-l-1}(x_{m+l+2},\dots,x_{m+n})=0\end{aligned}$$

($m\geqslant 1$, $l\geqslant 0$, $n\geqslant l+1$), which can be interpreted as the vanishing Pfaffian of a singular 4×4 matrix (see [21]).

Below we use the simple notation $K(x_1,\dots,x_n)$ without subscripts because the number of arguments of a continuant will always be clear from the context.

§ 3. The Rødseth function

Let l be a fixed integer, $1\leqslant l\leqslant a$, $(l,a)=1$, and let $\bar l$ be the solution of the congruence $\bar l\cdot l\equiv 1 \pmod a$, $1\leqslant \bar l\leqslant a$. In accordance with [3] consider the expansion of a/l as a continued fraction 'with minus signs'

$$\frac{a}{l}=\langle a_1;\dots,a_m\rangle$$

and consider the sequences $\{s_j\}$ and $\{q_j\}$, $-1\leqslant j\leqslant m$, defined by the equalities

$$s_j=K(a_{j+2},\dots,a_m),\qquad q_j=K(a_1,\dots,a_j).$$

The following properties of $\{s_j\}$ and $\{q_j\}$ are easy to prove.

1°. The sequences $\{s_j\}$ and $\{q_j\}$ are uniquely determined by the initial conditions

$$s_m = 0, \quad s_{m-1} = 1, \qquad q_{-1} = 0, \quad q_0 = 1$$

and the recurrence relations

$$s_{j-1} = a_{j+1}s_j - s_{j+1}, \quad q_{j+1} = a_{j+1}q_j - q_{j-1}, \qquad 0 \leqslant j \leqslant m-1.$$

Furthermore,

$$s_{-1} = q_m = K(a_1, \dots, a_m) = a, \qquad s_0 = K(a_2, \dots, a_m) = l,$$
$$q_{m-1} = K(a_1, \dots, a_{m-1}) = \bar{l}.$$

2°. The sequence $\{s_j\}$ is monotonically decreasing and $\{q_j\}$ is monotonically increasing, and we have

$$0 = \frac{s_m}{q_m} < \frac{s_{m-1}}{q_{m-1}} < \dots < \frac{s_0}{q_0} < \frac{s_{-1}}{q_{-1}} = \infty.$$

3°. For each n, $0 \leqslant n \leqslant m$, the vectors $e_n = (q_n, s_n)$ and $e_{n-1} = (q_{n-1}, s_{n-1})$ form a basis of the lattice

$$\Lambda_l = \bigl\{(x, y) \in \mathbb{Z}^2 : xl \equiv y \pmod{a}\bigr\}.$$

Furthermore,

$$\begin{vmatrix} q_n & s_n \\ q_{n-1} & s_{n-1} \end{vmatrix} = \det \Lambda_l = a.$$

4°. The points (q_n, s_n), $-1 \leqslant n \leqslant m$, are the vertices of the convex hull of the points in Λ_l distinct from the origin that lie in the first quadrant.

5°. For $1 \leqslant l < a$, $(l, a) = 1$ the quadruples $(q_n, s_{n-1}, q_{n-1}, s_n)$, $0 \leqslant n \leqslant m(l/a)$, are in one-to-one correspondence with the solutions (u_1, u_2, v_1, v_2) of the equation

$$u_1u_2 - v_1v_2 = a$$

such that

$$0 \leqslant v_1 < u_1 \leqslant a, \quad (u_1, v_1) = 1, \qquad 0 \leqslant v_2 < u_2 \leqslant a, \quad (u_2, v_2) = 1.$$

6°. For $0 \leqslant n \leqslant m$,

$$s_{n-1} - s_n \leqslant \frac{a}{q_n}\,, \qquad q_n - q_{n-1} \leqslant \frac{q}{s_{n-1}}\,.$$

Properties 1° and 2° are an immediate consequence of the definitions.

To prove 3° we observe that the vector pairs (e_{n-1}, e_n) and (e_n, e_{n+1}) are related by means of a unimodular transformation:

$$\begin{pmatrix} e_n \\ e_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_{n+1} \end{pmatrix} \begin{pmatrix} e_{n+1} \\ e_n \end{pmatrix}, \qquad 1 \leqslant n < m.$$

Furthermore, $e_{-1} = (0, a)$ and $e_0 = (1, l)$ form a basis of Λ_l, and we have

$$\begin{vmatrix} q_0 & s_0 \\ q_{-1} & s_{-1} \end{vmatrix} = \begin{vmatrix} 1 & l \\ 0 & a \end{vmatrix} = a.$$

Property 4° follows from the monotonicity of the sequences $\{s_j\}$, $\{q_j\}$ and property 3°.

Now we prove 5°. By property 3° each quadruple $(q_n, s_{n-1}, q_{n-1}, s_n)$ satisfies the equation $u_1u_2 - v_1v_2 = a$. To construct the inverse map we consider the expansions

$$\frac{u_1}{v_1} = \langle a_n; \dots, a_1 \rangle, \qquad \frac{u_2}{v_2} = \langle a_{n+1}; \dots, a_m \rangle$$

and take $l = K(a_2, \dots, a_m)$.

Property 6° holds because the equality $q_r s_{r-1} - q_{r-1} s_r = a$ (see property 3°) can be written as

$$q_n(s_{n-1} - s_n) + s_n(q_n - q_{n-1}) = a \quad \text{or} \quad (q_n - q_{n-1})s_{n-1} + q_{n-1}(s_{n-1} - s_n) = a.$$

Hence $s_{n-1} - s_n \leqslant a/q_n$ and $q_n - q_{n-1} \leqslant a/s_{n-1}$.

Consider the *Rødseth function* $\rho_{l,a}(t_1, t_2)$ that is defined by the equality

$$\rho_{l,a}(t_1, t_2) = t_1 s_{n-1} + t_2 q_n - \min\{t_1 s_n, t_2 q_{n-1}\} \tag{3}$$

for $t_1 \geqslant 0$ and $t_2 \geqslant 0$ such that

$$\frac{s_n}{q_n} \leqslant \frac{t_2}{t_1} < \frac{s_{n-1}}{q_{n-1}}$$

(in view of property 2°, in this way $\rho_{l,a}(t_1, t_2)$ is well defined for all $t_1 \geqslant 0$ and $t_2 \geqslant 0$). Then it was shown in [3] that for $(b, a) = 1$ and $c \equiv bl \pmod{a}$ the Frobenius number can be found by the formula

$$f(a, b, c) = \rho_{l,a}(b, c). \tag{4}$$

Remark 1. The function $\rho_{l,a}(t_1, t_2)$ is continuous and satisfies equality (3) for

$$\frac{s_n}{q_n} \leqslant \frac{t_2}{t_1} \leqslant \frac{s_{n-1}}{q_{n-1}}.$$

§ 4. Integer points in domains

Let Ω be a simply connected plane domain with rectifiable boundary. Let V be the area of Ω, P its perimeter, and N the number of points in the lattice $\mathbb{Z}^2$ lying in the interior of Ω. For convex domains we have Jarnik's inequality

$$|V - N| < P + 1$$

(see [22]). However, we also need to use the estimate

$$V - N = O(P + 1)$$

in a more general situation (see [23]).

Lemma 1. *For a simply-connected plane domain with rectifiable boundary*

$$|V - N| < 4(P + 1).$$

Proof. Let N_1 be the number of squares of the form $[a, a+1) \times [b, b+1)$, $a, b \in \mathbb{Z}$, in the interior of Ω and N_2 the number of squares intersecting Ω (maybe in just a single point). Then

$$N_1 \leqslant V, N \leqslant N_2,$$

so that $|V - N| \leqslant M = N_2 - N_1$, where M is the number of squares intersecting the boundary of Ω. In each of these squares we pick a point A_k from the boundary of Ω, $0 \leqslant k < M$ (we number these points in accordance with their order on the boundary). From any system of five squares intersecting the boundary of Ω we can select two with disjoint closures. Hence for each k the piece of the boundary between A_k and A_{k+4} has length $l(A_k, A_{k+4}) > 1$. Consequently,

$$P \geqslant l(A_0, A_4) + l(A_4, A_8) + \cdots + l(A_{4\lfloor M/4\rfloor - 4}, A_{4\lfloor M/4\rfloor}) > \left\lfloor \frac{M}{4} \right\rfloor > \frac{M}{4} - 1.$$

Hence $M < 4(P + 1)$ and $|V - N| < 4(P + 1)$.

We introduce the following notation (see [20]): if A is a proposition, then $[A] = 1$ if A is true and $[A] = 0$ otherwise.

Corollary 1. *Let $G(x, y)$ be a continuous real function defined in the interior of a simply connected domain Ω with perimeter $P \geqslant 1$ and such that $0 \leqslant G(x, y) \leqslant B$, $(x, y) \in \Omega$. Assume further that $G(x, y)$ is monotonic in each argument and for every $z \in [0, B]$ the inequality $G(x, y) \leqslant z$ defines a simply connected domain of perimeter $O(P)$ in Ω. Then*

$$\sum_{(x,y) \in \Omega \cap \mathbb{Z}^2} G(x, y) = \iint_\Omega G(x, y)\, dx\, dy + O(BP).$$

Proof. It is sufficient to approximate $G(x, y)$ by a linear combination of the functions

$$G_k(x, y) = \left[G(x, y) \leqslant \frac{k}{n} B \right], \qquad 0 \leqslant k \leqslant n,$$

to apply Lemma 1 to each of them and to pass to the limit as $n \to \infty$.

Remark 2. Let Λ be a sublattice of $\mathbb{Z}^2$ of index d. Let $N(\Lambda)$ be the number of points in Λ lying in Ω. Then

$$|V - dN(\Lambda)| \leqslant 4d(P + 1).$$

To prove this inequality we can repeat the arguments in the proof of Lemma 1, replacing the unit squares by fundamental parallelograms of Λ spanned by the reduced basis.

As a consequence of this inequality (under the same constraints as Corollary 1 and in a similar way), we obtain

$$\sum_{(x,y) \in \Omega \cap \Lambda} G(x, y) = \frac{1}{d} \iint_\Omega G(x, y)\, dx\, dy + O(BP). \tag{5}$$

§ 5. Distinguishing the density

For rational r we denote by square brackets the canonical expansion of r in a continued fraction of length $s = s(r)$,

$$r = [a_0; a_1, \dots, a_s] = a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_s}},$$

where $a_0 = \lfloor r \rfloor$ (the integer part of r), $a_1, \dots, a_s$, which are positive integers, are the partial quotients, $a_s \geqslant 2$, $s \geqslant 1$. We denote by $s_1(r)$ the sum of partial quotients of r: $s_1(r) = a_0 + a_1 + \dots + a_s$. For a positive integer q let $\delta_q(a)$ be the characteristic function of divisibility by q:

$$\delta_q(a) = [a \equiv 0 \pmod q] = \begin{cases} 1 & \text{if } a \equiv 0 \pmod q, \\ 0 & \text{if } a \not\equiv 0 \pmod q. \end{cases}$$

Lemma 2. *Assume that* $1 \leqslant l < a$, $(l, a) = 1$, *let* δ_1, δ_2 *be positive integers and let* x_1, x_2 *be positive real numbers. Then the sum*

$$S_{l,a}(\delta_1, \delta_2; x_1, x_2) = \sum_{\substack{b \leqslant x_1 a \\ \delta_1 | b}} \sum_{\substack{c \leqslant x_2 a \\ \delta_2 | c}} \delta_a(bl - c) \rho_{l,a}(b, c)$$

has the asymptotic representation

$$S_{l,a}(\delta_1, \delta_2; x_1, x_2) = a^2 \frac{(a, \delta_1, \delta_2)}{\delta_1 \delta_2} \int_0^{x_1} \int_0^{x_2} \rho_{l,a}(t_1, t_2)\, dt_1\, dt_2 + O\biggl(x_1 x_2 a^2 s_1\biggl(\frac{l}{a}\biggr)\biggr).$$

Proof. Consider the lattice

$$\Lambda_l(\delta_1, \delta_2) = \{(x, y) \in \Lambda_l : \delta_1 \,|\, x,\ \delta_2 \,|\, y\}.$$

Any point (x, y) in Λ_l has the form $(x, y) = u e_{-1} + v e_0$, where u and v are integers, $e_{-1} = (0, a)$, $e_0 = (1, l)$ (see property 3°). This point belongs to the sublattice $\Lambda_l(\delta_1, \delta_2)$ only when

$$v \equiv 0 \pmod{\delta_1}, \qquad au + lv \equiv 0 \pmod{\delta_2}.$$

Hence $\Lambda_l(\delta_1, \delta_2)$ is a sublattice of Λ_l of index $\delta_1 \delta_2 / (a, \delta_1, \delta_2)$.

Consider the sum

$$\begin{aligned} S_n &= S_{l,a,n}(\delta_1, \delta_2; x_1, x_2) \\ &= \sum_{\substack{b \leqslant x_1 a \\ \delta_1 | b}} \sum_{\substack{c \leqslant x_2 a \\ \delta_2 | c}} \biggl[(b, c) \in \Lambda_l(\delta_1, \delta_2),\ \frac{s_n}{q_n} \leqslant \frac{c}{b} < \frac{s_{n-1}}{q_{n-1}}\biggr] \rho_{l,a}(b, c). \end{aligned}$$

As the sequences $\{s_j\}$ and $\{q_j\}$ satisfy property 3°, all solutions of the congruence $bl \equiv c \pmod a$ for which $s_n / q_n \leqslant c/b < s_{n-1}/q_{n-1}$ have the form

$$b(u, v) = u q_n + v q_{n-1}, \qquad c(u, v) = u s_n + v s_{n-1}$$

with integer $u>0$ and $v\geqslant 0$. Hence

$$S_n=\sum_{u>0}\sum_{v\geqslant 0}\bigl[b(u,v)\leqslant x_1a,\ \delta_1\,|\,b(u,v),\ c(u,v)\leqslant x_2a,\ \delta_2\,|\,c(u,v)\bigr]h_{l,a}(u,v),$$

where $h_{l,a}(u,v)=\rho_{l,a}(uq_n+vq_{n-1},us_n+vs_{n-1})$.

Consider $r=r(l,a)$ defined by the inequalities

$$\frac{s_r}{q_r}\leqslant\frac{x_2}{x_1}<\frac{s_{r-1}}{q_{r-1}}\,.$$

For $n>r$ only the first of the constraints $b\leqslant x_1a$ and $c\leqslant x_2a$ is essential. Hence

$$S_n=\sum_{u>0}\sum_{v\geqslant 0}\bigl[uq_n+vq_{n-1}\leqslant x_1a,\ \delta_1\,|\,uq_n+vq_{n-1},\ \delta_2\,|\,us_n+vs_{n-1}\bigr]h_{l,a}(u,v).$$

The variables u and v range over a domain with perimeter $O(x_1a/q_{n-1})$. It follows from the estimate

$$\rho_{l,a}(t_1,t_2)\leqslant t_1s_{n-1}+t_2q_n \tag{6}$$

that the maximum of $h_l(u,v)$ it this domain does not exceed $2x_1as_{n-1}q_n/q_{n-1}$. Moreover, as we pointed out above, $\Lambda_l(\delta_1,\delta_2)$ is a sublattice of Λ_l of index $\delta_1\delta_2/(a,\delta_1,\delta_2)$. Hence from (5) we obtain

$$\begin{aligned}S_n&=\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^a\int_0^a[uq_n+vq_{n-1}\leqslant x_1a]h_{l,a}(u,v)\,du\,dv+O\biggl(\frac{x_1^2a^2}{q_{n-1}^2}\,s_{n-1}q_n\biggr)\\&=\frac{(a,\delta_1,\delta_2)}{a\delta_1\delta_2}\int_0^{x_1a}\int_0^{x_2a}\biggl[\frac{s_n}{q_n}\leqslant\frac{c}{b}<\frac{s_{n-1}}{q_{n-1}}\biggr]\rho_{l,a}(b,c)\,db\,dc+O\biggl(x_1x_2\frac{a^2q_n}{q_{n-1}}\biggr).\end{aligned}$$

Observing now that

$$\frac{q_n}{q_{n-1}}=\frac{K(a_1,\dots,a_n)}{K(a_1,\dots,a_{n-1})}=\langle a_n;a_{n-1},\dots,a_1\rangle\leqslant a_n,$$

we see that

$$S_n=a^2\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^{x_1}\int_0^{x_2}\biggl[\frac{s_n}{q_n}\leqslant\frac{t_2}{t_1}<\frac{s_{n-1}}{q_{n-1}}\biggr]\rho_{l,a}(t_1,t_2)\,dt_1\,dt_2+O(x_1x_2a^2a_n). \tag{7}$$

In a similar way, if $n<r$, then from the constraints $b\leqslant x_1a$ and $c\leqslant x_2a$ only the second remains. Taking account of the relations

$$\frac{s_{n-1}}{s_n}=\frac{K(a_{n+1},\dots,a_m)}{K(a_{n+2},\dots,a_m)}=\langle a_{n+1};\dots,a_m\rangle\leqslant a_{n+1},$$

we arrive at the equality

$$S_n=a^2\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^{x_1}\int_0^{x_2}\biggl[\frac{s_n}{q_n}\leqslant\frac{t_2}{t_1}<\frac{s_{n-1}}{q_{n-1}}\biggr]\rho_{l,a}(t_1,t_2)\,dt_1\,dt_2+O(x_1x_2a^2a_{n+1}). \tag{8}$$

On the other hand, if $n = r$, then the line $c/b = x_2/x_1$ in the plane Obc partitions the sector $s_n/q_n \leqslant c/b < s_{n-1}/q_{n-1}$ into two parts; in the first (where $s_n/q_n \leqslant c/b < x_2/x_1$) we must bear in mind that $b \leqslant x_1 a$, while in the second (where $x_2/x_1 \leqslant c/b < s_{n-1}/q_{n-1}$) we have $c \leqslant x_2 a$. Hence

$$S_n = \sum_{u>0}\sum_{v\geqslant 0}\bigl[b(u,v) \leqslant x_1 a,\ x_2 b(u,v) > x_1 c(u,v),\ \delta_1 \,|\, b(u,v),\ \delta_2 \,|\, c(u,v)\bigr] h_{l,a}(u,v)$$
$$+\sum_{u>0}\sum_{v\geqslant 0}\bigl[c(u,v) \leqslant x_2 a,\ x_2 b(u,v) \leqslant x_1 c(u,v),\ \delta_1 \,|\, b(u,v),\ \delta_2 \,|\, c(u,v)\bigr] h_{l,a}(u,v).$$

The variables u and v range over a domain whose perimeter is

$$O\biggl(\frac{x_1 a}{q_r} + \frac{x_2 a}{s_{r-1}} + x_1(s_{r-1} - s_r) + x_2(q_r - q_{r-1})\biggr) = O\biggl(\frac{x_1 a}{q_r} + \frac{x_2 a}{s_{r-1}}\biggr)$$

(see property 6°). The maximum of $h_{l,a}(u,v)$, in accordance with (6), is $O(ax_1 s_{r-1} + ax_2 q_r)$. Furthermore,

$$\biggl(\frac{x_1 a}{q_r} + \frac{x_2 a}{s_{r-1}}\biggr)(ax_1 s_{r-1} + ax_2 q_r) \ll x_1 x_2 a^2 (a_r + a_{r+1}).$$

Hence from Remark 2 we obtain

$$S_r = a^2 \frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2} \int_0^{x_1}\int_0^{x_2} \biggl[\frac{s_r}{q_r} \leqslant \frac{t_2}{t_1} < \frac{s_{r-1}}{q_{r-1}}\biggr] \rho_{l,a}(t_1,t_2)\,dt_1\,dt_2$$
$$+ O(x_1 x_2 a^2 (a_r + a_{r+1})). \tag{9}$$

Thus, in view of (7)–(9), for the sum

$$S_{l,a}(\delta_1,\delta_2; x_1, x_2) = \sum_{n=o}^{m} S_n,$$

we can deduce the asymptotic formula

$$S_{l,a}(\delta_1,\delta_2; x_1, x_2) = a^2 \frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2} \int_0^{x_1}\int_0^{x_2} \rho_{l,a}(t_1,t_2)\,dt_1\,dt_2$$
$$+ O(x_1 x_2 a^2 (a_1 + \dots + a_m)).$$

The continued fraction 'with minus signs' $a/l = \langle a_1; a_2, \dots, a_m\rangle$ can be obtained from an ordinary continued fraction $a/l = [b_1; b_2, \dots, b_s]$ by transforming the partial quotients with even indices as follows:

$$[t_{2j-1}; t_{2j}, t_{2j+1} + \alpha] = \langle t_{2j-1} + 1;\ \underbrace{2, \dots, 2,}_{t_{2j}-1 \text{ terms}}\ t_{2j+1} + 1 + \alpha\rangle.$$

However, the last partial quotient (if it has even index) is transformed by the formula

$$[t_{2j-1}; t_{2j}] = \langle t_{2j-1} + 1;\ \underbrace{2, \dots, 2}_{t_{2j}-1 \text{ terms}}\rangle.$$

Thus,

$$a_1 + \dots + a_m \leqslant 2(b_1 + \dots + b_s) = 2s_1\left(\frac{l}{a}\right),$$

which yields the required asymptotic formula.

In what follows, an asterisk on a summation sign of the form

$$\sum_{x=1}^{a}{}^{*}, \qquad \sum_{x=0}^{a-1}{}^{*}$$

means that the variable of summation x is coprime with a: $(a,x)=1$.

Corollary 2. *Under the assumptions of Lemma 2 the sum*

$$S_a(\delta_1,\delta_2;x_1,x_2) = \sum_{l=1}^{a}{}^{*} S_{l,a}(\delta_1,\delta_2;x_1,x_2)$$

has the asymptotic representation

$$S_a(\delta_1,\delta_2;x_1,x_2) = a^2\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^{x_1}\int_0^{x_2}\rho_a(t_1,t_2)\,dt_1\,dt_2 + O(x_1x_2a^3\log^2 a),$$

where

$$\rho_a(t_1,t_2) = \sum_{l=1}^{a}{}^{*}\rho_{l,a}(t_1,t_2).$$

Proof. It is sufficient to sum all the equalities in Lemma 2 and to use the estimate

$$\sum_{p=1}^{q} s_1\left(\frac{p}{q}\right) \ll q\log^2(q+1)$$

(see [24]).

Lemma 3. *Let x_1 and x_2 be positive real numbers. Then the sum*

$$F_a(x_1,x_2) = \sum_{(b,c)\in M_a(x_1,x_2)} f(a,b,c)$$

has the following asymptotic representation:

$$F_a(x_1,x_2) = a^2\sum_{\substack{d_1d_2\mid a\\(d_1,d_2)=1}}\frac{1}{d_1d_2}\sum_{\delta_1\mid d_2a_1}\frac{\mu(\delta_1)}{\delta_1}\sum_{\delta_2\mid d_1a_1}\frac{\mu(\delta_2)}{\delta_2}(a,\delta_1,\delta_2)$$
$$\times\int_0^{x_1d_2}\int_0^{x_2d_1}\rho_{a_1}(t_1,t_2)\,dt_1\,dt_2 + O(x_1x_2a^{3+\varepsilon}),$$

where $a_1 = a/(d_1d_2)$.

Proof. To find the sum $F_a(x_1,x_2)$ we introduce the parameters $d_1=(a,b)$, $d_2=(a,c)$ and set $b_1=b/d_1$, $c_1=c/d_2$, $a_1=a/(d_1d_2)$. For terms distinct from zero $(d_1,d_2)=1$. Hence

$$F_a(x_1,x_2)=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}\sum_{\substack{b\leqslant x_1a\\(b,a)=d_1}}\sum_{\substack{c\leqslant x_2a\\(c,a)=d_2}}f(a,b,c)$$
$$=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}\sum_{\substack{b_1\leqslant x_1d_2a_1\\(b_1,d_2a_1)=1}}\sum_{\substack{c_1\leqslant x_2d_1a_1\\(c_1,d_1a_1)=1}}f(d_1d_2a_1,d_1b_1,d_2c_1).$$

From Johnson's identity

$$f(a,b,c)=df\left(\frac{a}{d},\frac{b}{d},c\right)$$

(see [25]) we obtain

$$F_a(x_1,x_2)=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}d_1d_2\sum_{\substack{b_1\leqslant x_1d_2a_1\\(b_1,d_2a_1)=1}}\sum_{\substack{c_1\leqslant x_2d_1a_1\\(c_1,d_1a_1)=1}}f(a_1,b_1,c_1).$$

Now we can express the Frobenius number in terms of the Rødseth function by formula (4). Hence

$$F_a(x_1,x_2)=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}d_1d_2\sideset{}{^*}\sum_{l=1}^{a_1}\sum_{\substack{b_1\leqslant x_1d_2a_1\\(b_1,d_2a_1)=1}}\sum_{\substack{c_1\leqslant x_2d_1a_1\\(c_1,d_1a_1)=1}}\delta_{a_1}(b_1l-c_1)\rho_{l,a_1}(b_1,c_1)$$

$$=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}d_1d_2\sum_{\delta_1|d_2a_1}\mu(\delta_1)\sum_{\delta_2|d_1a_1}\mu(\delta_2)\sideset{}{^*}\sum_{l=1}^{a_1}\sum_{\substack{b_1\leqslant x_1d_2a_1\\\delta_1|b_1}}\sum_{\substack{c_1\leqslant x_2d_1a_1\\\delta_2|c_1}}\delta_{a_1}(b_1l-c_1)\rho_{l,a_1}(b_1,c_1).$$

Next by Corollary 2,

$$F_a(x_1,x_2)=a^2\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}\frac{1}{d_1d_2}\sum_{\delta_1|d_2a_1}\frac{\mu(\delta_1)}{\delta_1}\sum_{\delta_2|d_1a_1}\frac{\mu(\delta_2)}{\delta_2}(a,\delta_1,\delta_2)$$
$$\times\int_0^{x_1d_2}\int_0^{x_2d_1}\rho_{a_1}(t_1,t_2)\,dt_1\,dt_2+O(x_1x_2a^{3+\varepsilon}).$$

Remark 3. Applying the same arguments to the sum

$$G_a(x_1,x_2)=\sum_{(b,c)\in M_a(x_1,x_2)}\sqrt{abc}$$

we obtain the formula

$$G_a(x_1,x_2)=a^2\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}\frac{1}{d_1d_2}\sum_{\delta_1|d_2a_1}\frac{\mu(\delta_1)}{\delta_1}\sum_{\delta_2|d_1a_1}\frac{\mu(\delta_2)}{\delta_2}(a,\delta_1,\delta_2)\varphi(a_1)a_1^{1/2}$$
$$\times\int_0^{x_1d_2}\int_0^{x_2d_1}\sqrt{t_1t_2}\,dt_1\,dt_2+O(x_1x_2a^{3+\varepsilon}).$$

Remark 4. By homogeneity

$$\rho_{l,a}(t_1,t_2)=t_1\rho_{l,a}\left(\frac{t_2}{t_1}\right),$$

where

$$\rho_{l,a}(\xi)=s_{n-1}+\xi q_n-\min\{s_n,\xi q_{n-1}\}$$

for $s_n/q_n \leqslant \xi < s_{n-1}/q_{n-1}$. Hence if we know the function

$$\rho_a(\xi)=\sum_{l=1}^{a}{}^{*}\,\rho_{l,a}(\xi),$$

then it is easy to find the required density

$$\rho_a(t_1,t_2)=t_1\rho_a\left(\frac{t_2}{t_1}\right). \tag{10}$$

§ 6. The density transformation

In accordance with property 5° of the sequences $\{s_j\}$ and $\{q_j\}$,

$$\rho_a^*(\xi)=\sum_{u_1=1}^{a}\sum_{v_1=0}^{u_1-1}{}^{*}\sum_{u_2=1}^{a}\sum_{v_2=0}^{u_2-1}{}^{*}\left[u_1u_2-v_1v_2=a,\ \frac{v_2}{u_1}\leqslant\xi<\frac{u_2}{v_1}\right]h(u_1,u_2,v_1,v_2;\xi),$$

where

$$h(u_1,u_2,v_1,v_2;\xi)=u_2+\xi u_1-\min\{v_2,\xi v_1\}.$$

Considering the cases $v_2>\xi v_1$ and $v_2\leqslant\xi v_1$ separately, we express the required density in the following form:

$$\rho_a^*(\xi)=\lambda^*(a;\xi)+\eta^*(a;\xi),$$

where

$$\lambda^*(a;\xi)=\sum_{u_1=1}^{a}\sum_{v_1=0}^{u_1-1}{}^{*}\sum_{u_2=1}^{a}\sum_{v_2=0}^{u_2-1}{}^{*}\left[u_1u_2-v_1v_2=a,\ \frac{v_2}{u_1}\leqslant\xi<\frac{v_2}{v_1}\right]h_1(u_1,u_2,v_1,v_2;\xi),$$

$$\eta^*(a;\xi)=\sum_{u_1=1}^{a}\sum_{v_1=0}^{u_1-1}{}^{*}\sum_{u_2=1}^{a}\sum_{v_2=0}^{u_2-1}{}^{*}\left[u_1u_2-v_1v_2=a,\ \frac{v_2}{v_1}\leqslant\xi<\frac{u_2}{v_1}\right]h_2(u_1,u_2,v_1,v_2;\xi),$$

$$h_1(u_1,u_2,v_1,v_2;\xi)=u_2+\xi(u_1-v_1),\qquad h_2(u_1,u_2,v_1,v_2;\xi)=u_2-v_2+\xi u_1.$$

In view of Remark 1, the change of variables $u_1\leftrightarrow u_2$ and $v_1\leftrightarrow v_2$ leads to the equality $\eta^*(a;\xi)=\xi\lambda^*(a;1/\xi)$. Hence

$$\rho_a^*(\xi)=\lambda^*(a;\xi)+\xi\lambda^*\left(a;\frac{1}{\xi}\right). \tag{11}$$

To calculate $\lambda^*(a;\xi)$ we write the equation $u_1u_2-v_1v_2=a$ as

$$u_1(u_2-v_2)+v_2(u_1-v_1)=a$$

and introduce the variables $x = u_1$, $y = u_2 - v_2$, $z = u_1 - v_1$, $w = v_2$. Then we can write the sum $\lambda^*(a;\xi)$ in the following form:

$$\lambda^*(a;\xi) = \sum_{x=1}^{a} \sum_{z=1}^{x}{}^{*} \sum_{y=1}^{a} \sum_{w=0}^{a-1}{}^{*} \biggl[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z}\biggr](y + w + \xi z).$$

Eliminating the conditions on being coprime we obtain

$$\lambda^*(a;\xi) = \sum_{d_1 d_2 | a} \mu(d_1)\mu(d_2) d_2 \lambda\biggl(\frac{a}{d_1 d_2}, \frac{d_1\xi}{d_2}\biggr), \tag{12}$$

where

$$\lambda(a;\xi) = \sum_{x\geqslant 1} \sum_{z=1}^{x} \sum_{y\geqslant 1} \sum_{w\geqslant 0} \biggl[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z}\biggr](y + w + \xi z).$$

Taking into account the terms in the last pair of parentheses we express $\lambda(a;\xi)$ as

$$\lambda(a;\xi) = Y(a;\xi) + W(a;\xi) + \xi Z(a;\xi) \tag{13}$$

and find each of the three sums separately.

§ 7. Using estimates for Kloosterman sums

Let q be a positive integer, a an integer, and f a non-negative function. Let $T[f]$ be the number of solutions of the congruence $xy \equiv a \pmod q$ in the domain $P_1 < x \leqslant P_2$, $0 < y \leqslant f(x)$:

$$T[f] = \sum_{P_1 < x \leqslant P_2} \sum_{0 < y \leqslant f(x)} \delta_q(xy - a).$$

Bykovskiĭ [26] showed that calculating $T[f]$ reduces to finding the sum

$$S[f] = \frac{1}{q} \sum_{P_1 < x \leqslant P_2} \mu_{q,a}(x) f(x),$$

where $\mu_{q,a}(x)$ is the number of solutions of the congruence $xy \equiv a \pmod q$ with respect to y in the interval $1 \leqslant y \leqslant q$.

Now we present a simplified version of a result from [18], which refines the corresponding theorem in [26]. It is based on estimates for the Kloosterman sums

$$K_q(l,m,n) = \sum_{x,y=1}^{q} \delta_q(xy - l) \exp\biggl(2\pi i \frac{mx + ny}{q}\biggr) \tag{14}$$

and van der Corput's method for estimating trigonometric sums. The proof uses the inequality

$$|K_q(l,m,n)| \leqslant \sigma_0(q)\sigma_0((l,m,n,q))(lm,ln,mn,q)^{1/2} q^{1/2},$$

which generalizes the following result due to Estermann [27]:

$$|K_q(\pm 1, m, n)| \leqslant \sigma_0(q)(m, n, q)^{1/2} q^{1/2}.$$

Here and in what follows

$$\sigma_\alpha(q) = \sum_{d|q} d^\alpha$$

is the sum of powers of the divisors of the positive integer q. Throughout, $\varepsilon > 0$ will be arbitrarily small. We shall replace 2ε by ε in exponents.

Lemma 4. *Let P_1 and P_2 be real numbers, $P = P_2 - P_1 \geqslant 2$, and assume that a real function $f(x) \geqslant 0$ has two continuous derivatives on $[P_1, P_2]$ and that for some $A > 0$ and $w \geqslant 1$,*

$$\frac{1}{A} \leqslant |f''(x)| \leqslant \frac{w}{A}\,.$$

Then the asymptotic formula

$$T[f] = S[f] - \frac{P\delta_q(a)}{2} + R[f]$$

holds, where

$$R[f] \ll_{w,\varepsilon} (PA^{-1/3} + A^{1/2}D + q^{1/2})P^\varepsilon, \qquad D = (a, q).$$

Remark 5. It follows from Lemma 4 that the asymptotic formula for $T[f]$ does not change if the inequality $y \leqslant f(x)$ is replaced by the strict inequality $y < f(x)$ in the definition of $T[f]$.

Lemma 5. *Let P_1 and P_2 be real numbers, $P = P_2 - P_1 > 0$, and let $f(x) \geqslant 0$ be a real function that is constant on $[P_1, P_2]$. Then*

$$T[f] = S[f] + O\biggl(\biggl(q^{1/2} + \biggl(\frac{P}{q} + 1\biggr)D\biggr)q^\varepsilon\biggr),$$

where $D = (a, q)$.

Proof. By the definition of $\mu_{k,a}(x)$, for any Y we have

$$\sum_{P_1 < x \leqslant P_2} \sum_{Y < y \leqslant Y+q} \delta_q(xy - a) = \sum_{P_1 < x \leqslant P_2} \mu_{k,a}(x). \tag{15}$$

Next (see [18], Remark 2) for any X and $0 < Y \leqslant q$ we have the estimate

$$\sum_{X < x \leqslant X+q} \sum_{0 < y \leqslant Y} \delta_q(xy - a) = \frac{Y}{q} K_q(a, 0, 0) + O(Dq^\varepsilon), \tag{16}$$

where $K_q(l, m, n)$ is defined by (14). Moreover, for $X_2 - X_1 = X \leqslant q$ and $Y_2 - Y_1 = Y \leqslant q$ we have

$$\sum_{X_1 < x \leqslant X_2} \sum_{Y_1 < y \leqslant Y_2} \delta_q(xy - a) = \frac{XY}{q^2} K_q(a, 0, 0) + O((q^{1/2} + D)q^\varepsilon) \tag{17}$$

(see [18], Lemma 3). Bearing in mind the relations

$$\sum_{X_1<x\leqslant X_2}\mu_{k,a}(x)=\sum_{k|(a,q)}k\sum_{\substack{X_1<x\leqslant X_2\\(x,q)=k}}1=\sum_{k|(a,q)}k\biggl(\frac{\varphi(q/k)}{q/k}\,\frac{X}{k}+O\biggl(\sigma_0\biggl(\frac{q}{k}\biggr)\biggr)\biggr)$$
$$=\frac{X}{q}\sum_{k|(a,q)}k\varphi\biggl(\frac{q}{k}\biggr)+O(Dq^{\varepsilon})=\frac{X}{q}\,K_q(a,0,0)+O(Dq^{\varepsilon})$$

and asymptotic formulae (15)–(17) we arrive at the statement of the lemma.

Lemma 6. *Let f be a decreasing function on $[P_1,P_2]$ and let $f(P_1)-f(P_2)=Q$. Then*

$$S[f]=\psi(a,q)\int_{P_1}^{P_2}f(x)\,dx+O(DQq^{-1+\varepsilon}),$$

where

$$\psi(a,q)=\frac{1}{q}\sum_{k|(a,q)}\sum_{\delta|q/k}\frac{\mu(\delta)}{\delta}=\frac{K_q(a,0,0)}{q^2}\,. \tag{18}$$

Proof. In fact, $\mu_{q,a}(x)=k\delta_k(a)$, where $k=(q,x)$. Hence

$$S[f]=\frac{1}{q}\sum_{P_1<x\leqslant P_2}k\delta_k(a)f(x)=\frac{1}{q}\sum_{k|(a,q)}k\sum_{\substack{P_1/k<x\leqslant P_2/k\\(x,q/k)=1}}f(kx)$$
$$=\frac{1}{q}\sum_{k|(a,q)}k\sum_{\delta|q/k}\mu(\delta)\sum_{P_1/(k\delta)<x\leqslant P_2/(k\delta)}f(k\delta x).$$

Replacing the inner sum by an integral we arrive at the required asymptotic formula:

$$S[f]=\frac{1}{q}\sum_{k|(a,q)}k\sum_{\delta|q/k}\mu(\delta)\biggl(\frac{1}{k\delta}\int_{P_1}^{P_2}f(x)\,dx+O(Q)\biggr)$$
$$=\frac{1}{q}\sum_{k|(a,q)}\sum_{\delta|q/k}\frac{\mu(\delta)}{\delta}\int_{P_1}^{P_2}f(x)\,dx+O(DQq^{-1+\varepsilon}).$$

Lemma 7. *Assume that the function $I(r)/r\in C[0,1]$ has finitely many intervals of monotonicity and that $|I(r)/r|\leqslant B$ for all $r\in[0,1]$, let $\psi(a,q)$ be defined by equality (18), and assume that $1\leqslant U\leqslant a$ and $0\leqslant\theta\leqslant 1$. Then*

$$\sum_{q\leqslant\theta U}\psi(a,q)I\biggl(\frac{q}{U}\biggr)=\frac{\sigma_{-1}(a)}{\zeta(2)}\int_0^{\theta}\frac{I(r)}{r}\,dr+O(BU^{-1}a^{\varepsilon}).$$

Proof. From the definition of $\psi(a,q)$ we obtain

$$\sum_{q\leqslant\theta U}\psi(a,q)\biggl(\frac{q}{U}\biggr)=\sum_{k|a}\sum_{\delta<U/k}\frac{\mu(\delta)}{\delta}\sum_{\substack{q<\theta U\\ \delta k|q}}\frac{I(q/U)}{q}\,.$$

Replacing the inner sum by an integral we obtain the desired result:

$$\begin{aligned}\sum_{q\leqslant\theta U}\psi(a,q)\biggl(\frac{q}{U}\biggr)&=\sum_{k\mid a}\sum_{\delta<U/k}\frac{\mu(\delta)}{\delta}\biggl(\frac{1}{\delta k}\int_0^\theta\frac{I(r)}{r}\,dr+O(BU^{-1})\biggr)\\&=\sum_{k\mid a}\frac{1}{k}\biggl(\frac{1}{\zeta(2)}+O\biggl(\frac{k}{U}\biggr)\biggr)\int_0^\theta\frac{I(r)}{r}\,dr+O(BU^{-1}a^{\varepsilon})\\&=\frac{\sigma_{-1}(a)}{\zeta(2)}\int_0^\theta\frac{I(r)}{r}\,dr+O(BU^{-1}a^{\varepsilon}).\end{aligned}$$

§ 8. Calculating three auxiliary sums

To calculate the sums $Y(a;\xi)$, $W(a;\xi)$ and $Z(a;\xi)$ we introduce the parameters $U_1\asymp\sqrt{a\xi}$ and $U_2=aU_1^{-1}\asymp\sqrt{a/\xi}$. We shall assume that $a\geqslant 9$ and $9/a\leqslant\xi\leqslant a/9$, since otherwise the results that follow are trivial. For $\xi\geqslant 1$ we set $n=\lfloor\sqrt{a\xi}\,\rfloor-2\geqslant 1$. Then for $U_1\in[n+1/4,n+3/4]$ the parameter $U_2=a/U_1$ ranges over the interval $\bigl[\frac{a}{n+3/4},\frac{a}{n+1/4}\bigr]$, which has length greater than 1/2. Hence we can select U_1 and $U_2\geqslant 1$ such that

$$U_1U_2=a,\quad \|U_1\|,\|U_2\|\geqslant\frac14,\quad \frac{1}{12}\sqrt{a\xi}\leqslant U_1\leqslant\sqrt{a\xi}\,,\quad \sqrt{\frac{a}{\xi}}\leqslant U_2\leqslant 12\sqrt{\frac{a}{\xi}}$$

($\|x\|$ is the distance from the real number x to the closest integer). For $\xi>1$ we set $n=\lfloor\sqrt{a/\xi}\,\rfloor-2\geqslant 1$. Looking at U_2 over $[n+1/4,n+3/4]$ we see in a similar way that we can select U_1, $U_2\geqslant 1$ so that

$$U_1U_2=a,\qquad \|U_1\|,\|U_2\|\geqslant\frac14,\qquad \sqrt{a\xi}\leqslant U_1\leqslant 12\sqrt{a\xi}\,,\qquad \frac{1}{12}\sqrt{\frac{a}{\xi}}\leqslant U_2\leqslant\sqrt{\frac{a}{\xi}}\,.$$

Lemma 8. *The sum*

$$Y(a;\xi)=\sum_{x\geqslant1}\sum_{z=1}^{x}\sum_{y\geqslant1}\sum_{w\geqslant1}\biggl[xy+wz=a,\ \frac{w}{x}\leqslant\xi<\frac{w}{x-z}\biggr]y$$

has the asymptotic representation

$$Y(a;\xi)=\frac{2(4-\pi)}{3\zeta(2)}\,\sigma_{-1}(a)a^{3/2}\xi^{1/2}+O(R(a,\xi)),$$

where

$$R(a,\xi)=\bigl(a^{4/3}(1+\xi)+a^{5/4}(\xi^{5/4}+\xi^{-1/4})\bigr)a^{\varepsilon}. \tag{19}$$

Proof. We can write $Y(a;\xi)$ in the form

$$Y(a;\xi)=\sum_{t=1}^{a}Y(a,t;\xi),$$

where

$$Y(a,t;\xi)=\sum_{x\geqslant1}\sum_{z=1}^{x}\sum_{y\geqslant t}\sum_{w\geqslant1}\biggl[xy+wz=a,\ \frac{w}{x}\leqslant\xi<\frac{w}{x-z}\biggr].$$

We split $Y(a,t;\xi)$ into two sums:

$$Y(a,t;\xi) = Y_1(a,t;\xi) + Y_2(a,t;\xi),$$

where the first sum contains the terms for which $w \leqslant U_1$, and the second sum contains all other terms. For this decomposition, in the second sum we always have $z \leqslant U_2$.

In the sum

$$Y_1(a,t;\xi) = \sum_{w\leqslant U_1}\sum_{x\geqslant 1}\sum_{y\geqslant t}\sum_{z=1}^{x}\left[xy+wz=a,\ \frac{w}{x}\leqslant \xi < \frac{w}{x-z}\right],$$

for fixed $w \geqslant 1$ the variables x and y are related by $xy \equiv a \pmod{w}$. If w, x and y are known, then we find z is unique:

$$z = \frac{a-xy}{w}\,.$$

Hence, in view of the constraint $z \leqslant x$, we can express the sum $Y_1(a,t;\xi)$ in the following form:

$$\begin{aligned}
Y_1(a,t;\xi) &= \sum_{w\leqslant U_1}\ \sum_{w/\xi\leqslant x\leqslant a}\ \sum_{y\geqslant t}\delta_w(xy-a)\big[y_1(x)\leqslant y < y_2(x)\big]\\
&= \sum_{w\leqslant U_1}\sum_{y\geqslant t}\sum_{x\geqslant 1}\delta_w(xy-a)\big[x_1(y)\leqslant x < x_2(y)\big]\\
&= \sum_{w\leqslant U_1}\ \sum_{(x,y)\in\Omega}[y\geqslant t]\delta_w(xy-a),
\end{aligned}$$

where

$$y_1(x) = \frac{a}{x} - w, \qquad y_2(x) = \frac{a}{x} - w + \frac{w^2}{\xi x}\,, \tag{20}$$

$$x_1(y) = \frac{a}{w+y}\,, \qquad x_2(y) = \frac{1}{w+y}\left(a + \frac{w^2}{\xi}\right), \tag{21}$$

and the domain Ω is defined by

$$x \geqslant \frac{w}{\xi}\,, \qquad y > 0, \qquad y_1(x)\leqslant y < y_2(x)$$

or by the equivalent conditions

$$x \geqslant \frac{w}{\xi}\,, \qquad y > 0, \qquad x_1(y)\leqslant x < x_2(y).$$

We take $U = \sqrt{a + w^2/\xi}$ and represent Ω as

$$\Omega = (\Omega_1\setminus\Omega_2)\cup(\Omega_3\setminus(\Omega_4\cup\Omega_5)),$$

where

$$\Omega_1 = \biggl\{(x,y) : t \leqslant y \leqslant U-w,\ \frac{w}{\xi} < x < x_2(y)\biggr\},$$
$$\Omega_2 = \biggl\{(x,y) : t \leqslant y \leqslant U-w,\ \frac{w}{\xi} < x < x_1(y)\biggr\},$$
$$\Omega_3 = \biggl\{(x,y) : \frac{w}{\xi} < x \leqslant U,\ t \leqslant y < y_2(x)\biggr\},$$
$$\Omega_4 = \biggl\{(x,y) : \frac{w}{\xi} < x \leqslant \frac{a}{U}\,,\ t \leqslant y < y_1(x)\biggr\},$$
$$\Omega_5 = \biggl\{(x,y) : \max\biggl\{\frac{w}{\xi}\,,\ \frac{a}{U}\biggr\} < x \leqslant U,\ t \leqslant U-w\biggr\}.$$

Thus, the line $y=U-w$ partitions Ω into two pieces $\Omega_1\setminus\Omega_2$ and $\Omega_3\setminus(\Omega_4\cup\Omega_5)$, where Ω_1, Ω_2, Ω_3, and Ω_4 are curvilinear trapezia and Ω_5 is a rectangle.

In Ω_1 and Ω_2 we apply Lemmas 4 and 6 to the functions $x_1(y)$ and $x_2(y)$. To this end we partition the range of y into intervals of the form $(Y,2Y]=(P_1,P_2]$, where $Y=(U-w)/2,(U-w)/4,(U-w)/8,\dots$; on each of these intervals we have

$$x_1''(y)\asymp x_2''(y)\asymp \frac{a}{(w+Y)^3}\,,\qquad A\asymp\frac{(w+Y)^3}{a}\,.$$

Bearing in mind that

$$S[x_2]-S[x_1]=S[x_2-x_1],\qquad x_2(y)-x_1(y)\leqslant\frac{w^2}{(w+t)\xi}=Q,$$

after integrating over these intervals we obtain the leading term

$$\psi(a,w)\iint_{\Omega_1\setminus\Omega_2}[y\geqslant t]\,dx\,dy+O\bigl(\delta_w(a)U[t\leqslant U]\bigr)+O\biggl(\frac{D_w w a^{\varepsilon}}{\xi(w+t)}\biggr),$$

where $D_w=(a,w)$, and also the remainder

$$O\bigl((a^{1/3}+a^{1/4}D_w+w^{1/2})a^{\varepsilon}\bigr). \tag{22}$$

(We have added the condition $t\leqslant U$ because for $t>U$ the domain $\Omega_1\setminus\Omega_2$ is empty.)

In a similar way, in Ω_3 and Ω_4 we apply Lemmas 4 and 6 to the functions $y_1(x)$ and $y_2(x)$. Then for $x\in(X,2X]$ we obtain

$$y_1''(x)\asymp y_2''(x)\asymp\frac{a}{X^3}\,,\qquad y_2(x)-y_1(x)\leqslant\frac{w^2}{\xi x}\leqslant w.$$

Hence after integrating over intervals of the form $(X,2X]$, where $X=U/2,U/4,U/8,\dots$, we obtain the leading term

$$\psi(a,w)\iint_{\Omega_3\setminus\Omega_4}[y\geqslant t]\,dx\,dy+O\biggl(\delta_w(a)\min\biggl\{U,\frac{a}{w+t}\biggr\}\biggr)+O(D_w a^{\varepsilon})$$

and remainder (22).

Using Lemmas 5 and 6 in the domain Ω_5 we obtain

$$\psi(a,w)\iint_{\Omega_5}[y\geqslant t]\,dx\,dy+O\biggl(\frac{D_w}{w}\min\biggl\{U,\frac{a}{w+t}\biggr\}\biggr)+O((w^{1/2}+D_w)a^{\varepsilon}).$$

Combining the above, the sum $Y_1(a,t;\xi)$ has the estimate

$$Y_1(a,t;\xi)=\sum_{w\leqslant U_1}\biggl(\psi(a,w)\int_{w/\xi}^{a}dx\int_t^a\bigl[y_1(x)\leqslant y<y_2(x)\bigr]\,dy+O(R_1(a,t,w;\xi))\biggr),$$

where

$$R_1(a,t,w;\xi)=\biggl(a^{1/3}+a^{1/4}D_w+w^{1/2}+\frac{D_w w a^{\varepsilon}}{\xi(w+t)}\biggr)a^{\varepsilon}+\delta_w(a)U[t\leqslant U]
+\delta_w(a)\min\biggl\{U,\frac{a}{w+t}\biggr\}.$$

The terms in $Y_1(a,t;\xi)$ are distinct from zero only for $tw\leqslant a\xi$. Hence from the inequalities

$$\sum_{w\leqslant N}D_w\leqslant\sum_{D\mid a}\sum_{\substack{w\leqslant N\\ D\mid w}}1\leqslant N\sigma_0(a)$$

we obtain the following estimate for the remainder term:

$$\sum_{\substack{tw\leqslant a\xi\\ w\leqslant U_1}}R_1(a,t,w;\xi)\ll R_1(a,\xi),$$

where

$$R_1(a,\xi)=(a^{4/3}\xi+a^{5/4}(\xi^{5/4}+\xi)+a)a^{\varepsilon}.$$

Thus, for the sum

$$Y_1(a;\xi)=\sum_{t=1}^{a}Y_1(a,t;\xi)$$

we have the formula

$$\begin{aligned}Y_1(a;\xi)&=\sum_{t=1}^{a}\sum_{w\leqslant U_1}\psi(a,w)\int_{w/\xi}^{a}dx\int_t^a\bigl[y_1(x)\leqslant y<y_2(x)\bigr]\,dy+O(R_1(a;\xi))\\
&=\sum_{w\leqslant U_1}\psi(a,w)\int_{w/\xi}^{a}dx\int_t^a\bigl[y_1(x)\leqslant y<y_2(x)\bigr]y\,dy+O(R_1(a;\xi))\\
&=\frac12\sum_{w\leqslant U_1}\psi(a,w)\biggl(\int_{w/\xi}^{a/w+w/\xi}y_2^2(x)\,dx-\int_{w/\xi}^{a/w}y_1^2(x)\,dx\biggr)+O(R_1(a;\xi)).\end{aligned}$$

The integrals in it can be calculated directly, so we obtain

$$Y_1(a;\xi)=\frac{a^{3/2}\xi^{1/2}}{2}\sum_{1\leqslant w\leqslant U_1}\psi(a,w)I_1\biggl(\frac{w}{U_1}\biggr)+O(R_1(a;\xi)),$$

where

$$I_1(r) = r^3 - 2r^3 \log\left(1 + \frac{1}{r^2}\right) + 2r - 2r\log(1+r^2).$$

By Lemma 7,

$$\begin{aligned} Y_1(a;\xi) &= \frac{a^{3/2}\xi^{1/2}\sigma_{-1}(a)}{2\zeta(2)} \int_0^1 \frac{I_1(r)}{r}\,dr + O(R_1(a;\xi)) \\ &= \frac{a^{3/2}\xi^{1/2}\sigma_{-1}(a)}{\zeta(2)} \left(\frac{5}{2} - \frac{\pi}{3} - \frac{4}{3}\log 2\right) + O(R_1(a;\xi)). \end{aligned} \tag{23}$$

Now consider the sum

$$Y_2(a,t;\xi) = \sum_{z\geqslant 1}\sum_{x\geqslant z}\sum_{y\geqslant t}\sum_{w>U_1}\left[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z}\right].$$

For fixed $z \geqslant 1$ the variables x and y satisfy the relation $xy \equiv a \pmod z$, and if z, x and y are known, then w is determined uniquely:

$$w = \frac{a - xy}{z}\,.$$

The constraint $w > U_1$ means that $z < a/U_1 = U_2$, therefore we can express the sum $Y_2(a,t;\xi)$ in the following form:

$$\begin{aligned} Y_2(a,t;\xi) &= \sum_{z\leqslant U_2}\sum_{x\geqslant U_2}\sum_{y\geqslant t}\delta_z(xy-a)\big[y_3(x) \leqslant y < y_4(x)\big] \\ &= \sum_{z\leqslant U_2}\sum_{y\geqslant t}\sum_{x\geqslant U_2}\delta_z(xy-a)\big[x_3(y) \leqslant x < x_4(y)\big], \end{aligned}$$

where

$$y_3(x) = \frac{a}{x} - \xi z, \qquad y_4(x) = \min\left\{\frac{a - U_1 z}{x}\,,\ \frac{a}{x} - \xi z + \frac{\xi z^2}{x}\right\}, \tag{24}$$

$$x_3(y) = \frac{a}{\xi z + y}\,, \qquad x_4(y) = \min\left\{\frac{a - U_1 z}{y}\,,\ \frac{a + \xi z^2}{\xi z + y}\right\}. \tag{25}$$

Now we select $U = \sqrt{a + \xi z^2}\,$. As in the case of the sum $Y_1(a,t;\xi)$, for $y \leqslant U$ we apply Lemmas 4 and 6 to the functions $x_3(y)$ and $x_4(y)$ and for $y > U$ we apply them to the functions $y_3(x)$ and $y_4(x)$. Partitioning the ranges of the variables x and y in a similar fashion, into intervals of the form $(X,2X]$ and $(Y,2Y]$, we obtain

$$Y_2(a,t;\xi) = \sum_{z\leqslant U_2}\left(\psi(a,z)\int_z^a dx \int_t^a \big[y_3(x) \leqslant y < y_4(x)\big]\,dy + O(R_2(a,t,z;\xi))\right),$$

where

$$\begin{aligned} R_2(a,t,z;\xi) = (a^{1/3} + a^{1/4}D_z + z^{1/2})a^{\varepsilon} + \delta_z(a)\left(U[t \leqslant U] + \frac{a}{t}\right) \\ + D_z(\xi+1) + D_z\frac{a^{3/4}}{(a-U_1 z)^{1/2}}\,. \end{aligned}$$

In the sum $Y_2(a,t;\xi)$ the terms are distinct from zero only for $t \leqslant U_1$. Therefore, taking account of the estimate

$$\sum_{\substack{tz\leqslant a\\ z\leqslant U_2}} \frac{a^{3/4}}{(a-U_1z)^{1/2}} \ll \frac{a^{7/4}}{U_1^{1/2}} \sum_{z\leqslant U_2} \frac{1}{z(U_2-z)^{1/2}} \ll a^{5/4+\varepsilon},$$

we find the remainder term

$$R_2(a,\xi) = \sum_{\substack{tz\leqslant a\\ z\leqslant U_2}} R_2(a,t,z;\xi)$$

satisfies

$$R_2(a,\xi) \ll (a^{4/3} + a^{5/4}\xi^{-1/4} + a\xi)a^{\varepsilon}.$$

Hence the sum

$$Y_2(a;\xi) = \sum_{t=1}^{a} Y_2(a,t;\xi)$$

satisfies the relations

$$\begin{aligned} Y_2(a;\xi) &= \sum_{t=1}^{a} \sum_{z\leqslant U_2} \psi(a,z) \int_0^a dx \int_t^a \bigl[y_3(x) \leqslant y < y_4(x)\bigr]\, dy + O(R_2(a;\xi)) \\ &= \sum_{z\leqslant U_2} \psi(a,z) \int_0^a dx \int_t^a \bigl[y_3(x) \leqslant y < y_4(x)\bigr] y\, dy + O(R_2(a;\xi)) \\ &= \frac{1}{2} \sum_{z\leqslant U_2} \psi(a,z) \biggl(\int_{U_2}^{a/(\xi z)+z} y_4^2(x)\, dx - \int_{U_2}^{a/(\xi z)} y_3^2(x)\, dx \biggr) + O(R_2(a;\xi)). \end{aligned}$$

Calculating the integrals we arrive at the equality

$$Y_2(a;\xi) = \frac{a^{3/2}\xi^{1/2}}{2} \sum_{1\leqslant z\leqslant U_2} \psi(a,z) I_2\biggl(\frac{z}{U_2}\biggr) + O(R_2(a;\xi)),$$

where

$$I_2(r) = 2r\biggl(r^2 \log r - (1+r^2)\log\frac{1+r^2}{1+r}\biggr).$$

By Lemma 7,

$$\sum_{z\leqslant U_2} \psi(a,z) I_2\biggl(\frac{z}{U_2}\biggr) = \frac{\sigma_{-1}(a)}{\zeta(2)} \int_0^1 \frac{I_2(r)}{r}\, dr + O(a^{1+\varepsilon}\xi).$$

Hence

$$Y_2(a;\xi) = \frac{a^{3/2}\xi^{1/2}\sigma_{-1}(a)}{\zeta(2)} \biggl(\frac{1}{6} - \frac{\pi}{3} + \frac{4}{3}\log 2\biggr) + O(R_2(a;\xi)). \tag{26}$$

Substituting equalities (23) and (26) into the formula

$$Y(a;\xi) = Y_1(a;\xi) + Y_2(a;\xi)$$

we obtain the required result.

Lemma 9. *The sum*

$$W(a;\xi)=\sum_{x\geqslant 1}\sum_{z=1}^{x}\sum_{y\geqslant 1}\sum_{w\geqslant 0}\biggl[xy+wz=a,\ \frac{w}{x}\leqslant\xi<\frac{w}{x-z}\biggr]w$$

has the asymptotic representation

$$W(a;\xi)=\frac{2(\pi-2)}{3\zeta(2)}\,\sigma_{-1}(a)a^{3/2}\xi^{1/2}+O(R(a,\xi))$$

with its remainder term $R(a,\xi)$ defined by equality (19).

Proof. We write the sum under consideration as

$$W(a;\xi)=\sum_{t=1}^{a}W(a,t;\xi),$$

where

$$\begin{aligned}
W(a,t;\xi)&=W_1(a,t;\xi)+W_2(a,t;\xi),\\
W_1(a,t;\xi)&=\sum_{t\leqslant w<U_1}\ \sum_{x,y\geqslant 1}\ \sum_{z=1}^{x}\biggl[xy+wz=a,\ \frac{w}{x}\leqslant\xi<\frac{w}{x-z}\biggr],\\
W_2(a,t;\xi)&=\sum_{z\leqslant U_2}\ \sum_{x,y\geqslant 1}\ \sum_{w\geqslant\max\{t,U_1\}}\biggl[xy+wz=a,\ \frac{w}{x}\leqslant\xi<\frac{w}{x-z}\biggr].
\end{aligned}$$

In $W_1(a,t;\xi)$ we pass from the equality $xy+wz=a$ to the congruence $xy\equiv a \pmod{w}$. The constraints on the variables $z\leqslant x$, $w\leqslant \xi x$, and $\xi(x-z)<w$ are the same as in the sum $Y_1(a,t;\xi)$. Hence

$$\begin{aligned}
W_1(a,t;\xi)&=\sum_{t\leqslant w\leqslant U_1}\ \sum_{x\geqslant w/\xi}\ \sum_{y\geqslant 1}\delta_w(xy-a)\bigl[x_1(y)\leqslant x<x_2(y)\bigr]\\
&=\sum_{t\leqslant w<U_1}\ \sum_{y\geqslant 1}\ \sum_{x\geqslant w/\xi}\delta_w(xy-a)\bigl[y_1(x)\leqslant y<y_2(x)\bigr],
\end{aligned}$$

where $x_1(y)$, $x_2(y)$, $y_1(x)$ and $y_2(x)$ are defined by equalities (20) and (21). Lemmas 4 and 6 lead to the asymptotic formula

$$W_1(a,t;\xi)=\sum_{w\leqslant U_1}\biggl(\psi(a,w)\int_{w/\xi}^{a}dx\int_0^{a}\bigl[y_1(x)\leqslant y<y_2(x)\bigr]\,dy+O(R_3(a,t,w;\xi))\biggr),$$

where

$$R_3(a,t,w;\xi)=(a^{1/3}+a^{1/4}D_w+w^{1/2})a^{\varepsilon}+\delta_w(a)U+D_w\xi^{-1}.$$

As for the sum $Y_1(a;\xi)$, we have here

$$\sum_{t\leqslant a}\ \sum_{t\leqslant w<U_1}R_3(a,t,w;\xi)\ll R_1(a;\xi).$$

Hence

$$W_1(a;\xi) = \sum_{w\leqslant U_1} w\psi(a,w)\int_{w/\xi}^{a} dx \int_0^a \bigl[y_1(x)\leqslant y < y_2(x)\bigr]\,dy + O(R_1(a;\xi))$$
$$= a\sum_{w\leqslant U_1} w\psi(a,w) I_3\biggl(\frac{w}{U_1}\biggr) + O(R_1(a;\xi)),$$

where

$$I_3(r) = (1+r^2)\log(1+r^2) - r^2\log r - r^2. \tag{27}$$

By Lemma 7 we obtain

$$W_1(a;\xi) = \frac{a^{3/2}\xi^{1/2}\sigma_{-1}(a)}{\zeta(2)}\int_0^1 I_3(r)\,dr + O(R_1(a;\xi))$$
$$= \frac{a^{3/2}\xi^{1/2}\sigma_{-1}(a)}{\zeta(2)}\biggl(-\frac{5}{3}+\frac{\pi}{3}+\frac{4}{3}\log 2\biggr) + O(R_1(a;\xi)). \tag{28}$$

In $W_2(a,t;\xi)$ we pass from the equality $xy+wz=a$ to the congruence $xy\equiv a\ (\mathrm{mod}\, z)$. Then

$$W_2(a,t;\xi) = \sum_{z\leqslant U_2}\sum_{x\geqslant U_2}\sum_{y>0}\delta_z(xy-a)\bigl[y_3(x)\leqslant y < y_4(x,t)\bigr]$$
$$= \sum_{z\leqslant U_2}\sum_{x\geqslant U_2}\sum_{y>0}\delta_z(xy-a)\bigl[x_3(y)\leqslant x < x_4(y,t)\bigr],$$

where the functions $x_3(y)$ and $y_3(x)$ are defined in (24) and (25),

$$y_4(x,t) = \min\biggl\{\frac{a-\max\{U_1,t\}z}{x}\,,\ \frac{a}{x}-\xi z+\frac{\xi z^2}{x}\biggr\}$$

and

$$x_4(y,t) = \min\biggl\{\frac{a-\max\{U_1,t\}z}{y}\,,\ \frac{a+\xi z^2}{\xi z+y}\biggr\}.$$

Applying Lemmas 4 and 6 to the functions $x_3(y)$, $x_4(y,t)$, $y_3(x)$ and $y_4(x,t)$ we arrive at the equality

$$W_2(a,t;\xi) = \sum_{z\leqslant U_2}\biggl(\psi(a,z)\int_{U_2}^{a} dx\int_0^a \bigl[y_3(x)\leqslant y < y_4(x,t)\bigr]\,dy + O(R_4(a,t,z;\xi))\biggr),$$

where

$$R_3(a,t,z;\xi) = (a^{1/3}+a^{1/4}D_z+z^{1/2})a^{\varepsilon} + \delta_z(a)\min\biggl\{U_1,\ \frac{a\xi}{t}\biggr\}$$
$$+ D_z(1+\xi) + D_z\frac{a^{3/4}}{(a-zU_1)^{1/2}}\,.$$

Here the terms will be distinct from zero only for $tz \leqslant a$ and

$$\sum_{t=1}^{a} \sum_{z \leqslant \min\{U_2, a/t\}} R_4(a,t,z;\xi) \ll R_2(a,\xi).$$

Thus,

$$\begin{aligned} W_2(a;\xi) &= \sum_{t \leqslant U_1} \sum_{z \leqslant U_2} \psi(a,z) \int_{U_2}^{a} dx \int_0^a \big[y_3(x) \leqslant y < y_4(x)\big]\, dy \\ &\quad + \sum_{U_1 < t \leqslant a} \sum_{z \leqslant a/t} \psi(a,z) \int_{U_2}^{a} dx \int_0^a \big[y_3(x) \leqslant y < y_4(x,t)\big]\, dy + O(R_2(a;\xi)). \end{aligned}$$

The double integrals are straightforward to calculate:

$$\begin{aligned} \int_{U_2}^{a} dx \int_0^a \big[y_3(x) \leqslant y < y_4(x)\big]\, dy &= aI_4\left(\frac{z}{U_2}\right), \\ \int_{U_2}^{a} dx \int_0^a \big[y_3(x) \leqslant y < y_4(x,t)\big]\, dy &= aI_4\left(\frac{z}{U_2}, \frac{U_1}{t}\right), \end{aligned}$$

where $I_4(r) = I_4(r,1)$ and

$$I_4(r,\tau) = (1+r^2)\log(1+r^2) - r^2\log r + r\tau\log\tau - r(r+\tau)\log(r+\tau). \tag{29}$$

Hence by Lemma 7,

$$\begin{aligned} W_2(a;\xi) &= \frac{a^{3/2}\xi^{1/2}\sigma_{-1}(a)}{\zeta(2)} \left(\int_0^1 \frac{I_4(r)}{r}\, dr + \int_1^\infty d\tau \int_0^{1/\tau} \frac{I_4(r,\tau)}{r}\, dr \right) + O(R_2(a;\xi)) \\ &= \frac{a^{3/2}\xi^{1/2}\sigma_{-1}(a)}{\zeta(2)} \left(\frac{1}{3} + \frac{\pi}{3} - \frac{4}{3}\log 2 \right) + O(R_2(a;\xi)). \end{aligned} \tag{30}$$

Adding (28) and (30) together we obtain the required result.

Lemma 10. *The sum*

$$Z(a;\xi) = \sum_{x \geqslant 1} \sum_{z=1}^{x} \sum_{y \geqslant 1} \sum_{w \geqslant 0} \left[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z} \right] z$$

has the asymptotic representation

$$Z(a;\xi) = \frac{2(\pi-2)}{3\zeta(2)} \sigma_{-1}(a) a^{3/2} \xi^{-1/2} + O(\xi^{-1} R(a,\xi)),$$

where the remainder term $R(a,\xi)$ is defined in (19).

Proof. We split $Z(a;\xi)$ into four sums:

$$Z(a;\xi) = Z_1(a;\xi) + Z_2(a;\xi) + Z_3(a;\xi) + Z_4(a;\xi), \tag{31}$$

where

$$Z_1(a;\xi) = \sum_{t\leqslant U_2} \sum_{w\leqslant \xi t} \sum_{x,y\geqslant 1} \sum_{t\leqslant z\leqslant x} \left[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z}\right],$$

$$Z_2(a;\xi) = \sum_{t\leqslant U_2} \sum_{\xi t< w\leqslant U_1} \sum_{x,y\geqslant 1} \sum_{t\leqslant z\leqslant x} \left[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z}\right],$$

$$Z_3(a;\xi) = \sum_{t> U_2} \sum_{w\leqslant a/t} \sum_{x,y\geqslant 1} \sum_{t\leqslant z\leqslant x} \left[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z}\right],$$

$$Z_4(a;\xi) = \sum_{t\leqslant U_2} \sum_{t<z<U_2} \sum_{x\geqslant z} \sum_{y\geqslant 1} \sum_{w>U_1} \left[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z}\right],$$

The condition $z \geqslant t$ holds automatically for the nonzero terms of the sum $Z_1(a;\xi)$, therefore

$$Z_1(a;\xi) = \sum_{t\leqslant U_2} \sum_{w\leqslant \xi t} \sum_{x,y\geqslant 1} \delta_w(xy-a)\big[y_1(x) \leqslant y < y_2(x)\big].$$

We can single out the leading terms as in $Y_1(a;\xi)$; namely,

$$\begin{aligned} Z_1(a;\xi) &= \sum_{t\leqslant U_2} \sum_{w\leqslant \xi t} \psi(a,w)\left(\int_t^a dx \int_0^a \big[y_1(x) \leqslant y < y_2(x)\big]\,dy + O(R_5(a,t,w;\xi))\right) \\ &= a\sum_{t\leqslant U_2} \sum_{w\leqslant \xi t} \psi(a,w)\left(I_4\left(\frac{w}{U_1},\frac{t}{U_2}\right) + O(R_5(a,t,w;\xi))\right), \end{aligned}$$

where $I_4(r,\tau)$ is defined by equality (29) and

$$\begin{aligned} R_5(a,t,w;\xi) = \left(a^{1/3} + a^{1/4}D_w + w^{1/2} + \frac{D_w a^{3/4}}{(a-wt)^{1/2}}\right)a^{\varepsilon} \\ + \delta_w(a)\min\left\{U,\frac{a}{t}\right\} + D_w(1+\xi^{-1}). \end{aligned}$$

To estimate the remainder we observe that

$$\sum_{wt<a} \frac{a^{3/4}}{(a-wt)^{1/2}} \ll a^{3/4+\varepsilon}\sum_{n=1}^{a}\frac{1}{\sqrt{n}} \ll a^{5/4+\varepsilon}.$$

The other terms in $R_5(a,t,w;\xi)$ are estimated as in Lemma 8. Hence

$$\sum_{t\leqslant U_2} \sum_{w\leqslant \xi t} R_5(a,t,w;\xi) \ll (a^{4/3} + a^{5/4}(\xi^{1/4} + a\xi^{-1}))a^{\varepsilon} \ll R_1(a;\xi)\xi^{-1}.$$

By Lemma 7,

$$\begin{aligned} Z_1(a;\xi) &= \frac{a^{3/2}\xi^{-1/2}\sigma_{-1}(a)}{\zeta(2)}\int_0^1 d\tau \int_0^{\tau} \frac{I_4(r,\tau)}{r}\,dr + O(R_1(a;\xi)\xi^{-1}) \\ &= \frac{a^{3/2}\xi^{-1/2}\sigma_{-1}(a)}{\zeta(2)}\left(\frac{3}{2} - \frac{\pi}{3} + \frac{\zeta(2)}{4} - \log 2\right) + O(R_1(a;\xi)\xi^{-1}). \end{aligned}$$

We calculate $Z_3(a;\xi)$ in a similar way:

$$\begin{aligned}
Z_3(a;\xi) &= \sum_{t>U_2}\sum_{w\leqslant a/t}\psi(a,w)\biggl(\int_t^a dx\int_0^a \bigl[y_1(x)\leqslant y<y_2(x)\bigr]\,dy+O(R_5(a,t,w;\xi))\biggr)\\
&= a\sum_{t>U_2}\sum_{w\leqslant a/t}\psi(a,w)I_4\biggl(\frac{w}{U_1},\frac{t}{U_2}\biggr)+O(R_1(a;\xi)\xi^{-1})\\
&= \frac{a^{3/2}\xi^{-1/2}\sigma_{-1}(a)}{\zeta(2)}\int_1^\infty d\tau\int_0^{1/\tau}\frac{I_4(r,\tau)}{r}\,dr+O(R_1(a;\xi)\xi^{-1})\\
&= \frac{a^{3/2}\xi^{-1/2}\sigma_{-1}(a)}{\zeta(2)}\biggl(-\frac{1}{6}+\frac{\pi}{3}-\frac{\zeta(2)}{4}-\frac{1}{3}\log 2\biggr)+O(R_1(a;\xi)\xi^{-1}).
\end{aligned}$$

Analogous transformations lead to the following representation for the sum $Z_2(a;\xi)$:

$$\begin{aligned}
Z_2(a;\xi) &= \sum_{t\leqslant U_2}\sum_{\xi t<w\leqslant U_1}\psi(a,w)\biggl(\int_{w/\xi}^a dx\int_0^a \bigl[y_1(x)\leqslant y<y_2(x)\bigr]\,dy+O(R_5(a,t,w;\xi))\biggr)\\
&= a\sum_{t\leqslant U_2}\sum_{\xi t<w\leqslant U_1}\psi(a,w)I_5\biggl(\frac{w}{U_1},\frac{t}{U_2}\biggr)+O(R_1(a;\xi)\xi^{-1}),
\end{aligned}$$

where

$$I_5(r,\tau)=(1+r^2)\log(1+r^2)-r(r+\tau)\log(r+\tau)-r(\tau-r)\log r+r(\tau-r).$$

By Lemma 7,

$$\begin{aligned}
Z_2(a;\xi) &= \frac{a^{3/2}\xi^{-1/2}\sigma_{-1}(a)}{\zeta(2)}\int_0^1 d\tau\int_\tau^1\frac{I_5(r,\tau)}{r}\,dr+O(R_1(a;\xi)\xi^{-1})\\
&= \frac{a^{3/2}\xi^{-1/2}\sigma_{-1}(a)}{\zeta(2)}\biggl(-\frac{5}{4}+\frac{\pi}{3}+\frac{2}{3}\log 2\biggr)+O(R_1(a;\xi)\xi^{-1}).
\end{aligned}$$

The sum $Z_4(a;\xi)$ is calculated similarly to $W_2(a;\xi)$ (using Lemmas 4 and 6):

$$\begin{aligned}
Z_4(a;\xi) = \sum_{t\leqslant U_2}\sum_{t<z<U_2}\biggl(&\psi(a,z)\int_{U_2}^a dx\int_0^a\bigl[y_3(x)\leqslant y<y_4(x,t)\bigr]\,dy\\
&+O(R_4(a,t,z;\xi))\biggr).
\end{aligned}$$

Here we have

$$\sum_{t\leqslant U_2}\sum_{z<U_2}R_4(a,t,z;\xi)\ll R(a;\xi)\xi^{-1}.$$

Finding the leading term reduces to integrating the function $I_3(r)$ defined by equality (27):

$$\begin{aligned}
Z_4(a;\xi) &= \frac{a^{3/2}\xi^{-1/2}\sigma_{-1}(a)}{\zeta(2)}\int_0^1 d\tau\int_\tau^1\frac{I_4(r,\tau)}{r}\,dr+O(R(a;\xi)\xi^{-1})\\
&= \frac{a^{3/2}\xi^{-1/2}\sigma_{-1}(a)}{\zeta(2)}\biggl(-\frac{17}{12}+\frac{\pi}{3}+\frac{2}{3}\log 2\biggr)+O(R(a;\xi)\xi^{-1}).
\end{aligned}$$

Substituting the values of the sums $Z_1(a;\xi)$, $Z_2(a;\xi)$, $Z_3(a;\xi)$ and $Z_4(a;\xi)$ which we have obtained into (31) we arrive at the conclusion of the lemma.

§ 9. The proof of the main result

Corollary 3. *Let $x_1, x_2 \in [0,1]$. Then the density $\rho_a(\xi)$ has the asymptotic expression*

$$\rho_a(\xi) = \frac{8}{\pi}\,\varphi(a)a^{1/2}\xi^{1/2} + O\big((a^{4/3}(1+\xi) + a^{5/4}(\xi^{5/4}+\xi^{-1/4}))a^{\varepsilon}\big).$$

Proof. Substituting the expressions from Lemmas 8–10 into equality (13) we obtain

$$\lambda(a;\xi) = \frac{4}{\pi}\,\sigma_{-1}(a)a^{3/2}\xi^{1/2} + O\big((a^{4/3}(1+\xi) + a^{5/4}(\xi^{5/4}+\xi^{-1/4}))a^{\varepsilon}\big).$$

Hence formula (12) yields

$$\lambda^*(a;\xi) = \frac{4}{\pi}\,\varphi(a)a^{1/2}\xi^{1/2} + O\big((a^{4/3}(1+\xi) + a^{5/4}(\xi^{5/4}+\xi^{-1/4}))a^{\varepsilon}\big).$$

Substituting this expression into (11) we arrive at the required result.

Proof of Theorem 1. From formula (10), on the basis of Corollary 3 we obtain

$$\rho_a(t_1,t_2) = \frac{8}{\pi}\,\varphi(a)\sqrt{t_1t_2a} + O\big((a^{4/3}(t_1+t_2) + a^{5/4}(t_1^{5/4}t_2^{-1/4}+t_1^{-1/4}t_2^{5/4}))a^{\varepsilon}\big).$$

Substituting this into the asymptotic formula from Lemma 3 and taking Remark 3 into account we arrive at the result of the theorem with the remainder term as required.

To prove Theorem 2 we can partition the square $[1,N]^2$ containing the pairs (b,c) into smaller squares with side length $N^{11/12}$ and can use Theorem 1 in each of these squares.

Remark 6. The ideas underlying the approach used in the proof of Theorem 1 are close to Porter's approach in [28] (see also [18]). In it the following asymptotic formula for the mean value of the length of continued fractions for rational numbers with equal denominators was obtained:

$$\frac{1}{\varphi(d)}\sum_{\substack{1\leqslant c\leqslant d\\ (c,d)=1}} s\biggl(\frac{c}{d}\biggr) = \frac{2\log 2}{\zeta(2)}\log d + C_P - 1 + O_{\varepsilon}(d^{-1/6+\varepsilon}),$$

where the constant

$$C_P = \frac{2\log 2}{\zeta(2)}\biggl(\frac{3\log 2}{2} + 2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - 1\biggr) - \frac{1}{2}$$

is now known as the *Porter constant*. The key point in both cases is root estimates for Kloosterman sums and van der Corput's method, which explains why there is the same decrease of the exponent in the remainder terms. A more precise result was obtained in [17] for averaging over the numerators and denominators:

$$E(R)=\frac{2}{R(R+1)}\sum_{d\leqslant R}\sum_{c\leqslant d}s\biggl(\frac{c}{d}\biggr)=\frac{2\log 2}{\zeta(2)}\log R+C'_P+O(R^{-1}\log^4 R)$$

with absolute constant C'_P. Hence in the case when $f(a,b,c)$ is averaged over all the three arguments we can conjecture the following result.

Conjecture 4. The estimate

$$\frac{1}{x_1x_2x_3N^{9/2}}\sum_{a\leqslant x_1N}\sum_{b\leqslant x_2N}\sum_{\substack{c\leqslant x_3N\\(a,b,c)=1}}\biggl(f(a,b,c)-\frac{8}{\pi}\sqrt{abc}\biggr)=O_{\varepsilon,x_1,x_2,x_3}(N^{-1/2+\varepsilon})$$

holds.

Remark 7. The method used in the proof of Theorem 1 also enables us to describe the distribution density of $f(a,b,c)/\sqrt{abc}$. It turns out that

$$\frac{1}{|M_a(x_1,x_2)|}\sum_{(a,b,c)\in M_a(x_1,x_2)}\bigl[f(a,b,c)\leqslant\tau\sqrt{abc}\,\bigr]=\int_0^\tau p(t)\,dt+O_\varepsilon(R(a;x_1,x_2;\tau)a^\varepsilon),$$

where

$$R(a,x_1,x_2,\tau)\ll_{x_1,x_2,\tau}a^{-1/6}$$

and the density $p(t)$ is defined by

$$p(t)=\begin{cases}0 & \text{for } t\in[0,\sqrt{3}\,];\\ \dfrac{12}{\pi}\biggl(\dfrac{t}{\sqrt{3}}-\sqrt{4-t^2}\biggr) & \text{for } t\in[\sqrt{3}\,,2];\\ \dfrac{12}{\pi^2}\biggl(t\sqrt{3}\arccos\dfrac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}+\dfrac{3}{2}\sqrt{t^2-4}\,\log\dfrac{t^2-4}{t^2-3}\biggr) & \text{for } t\in[2,+\infty).\end{cases}$$

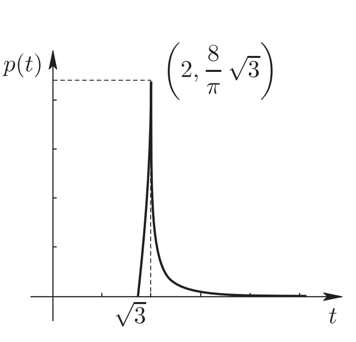

Figure 1. The graph of the density $p(t)$

This density has the following properties:

1) the function $p(t)$ increases on $[\sqrt{3}\,,2]$ and decreases on the semi-infinite interval $[2,+\infty)$, $\lim_{t\to 2-0} p'(t)=+\infty$ and $\lim_{t\to 2+0} p'(t)=-\infty$;
2) $p(t)=\dfrac{18}{\pi^2t^3}+O\biggl(\dfrac{1}{t^5}\biggr)$ as $t\to\infty$;
3) $\displaystyle\int_0^\infty p(t)\,dt=1$;
4) $\displaystyle\int_0^\infty tp(t)\,dt=\frac{8}{\pi}\,$.

The author is going to present the proof in a forthcoming paper.

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A. V. Ustinov
Khabarovsk Section of the Institute of Applied Mathematics, Far-Eastern Branch of RAS
E-mail: ustinov@iam.khv.ru

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