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On the distribution of Frobenius numbers with three arguments

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Abstract. We prove the existence of the limit density distribution for normalized Frobenius numbers with three arguments. The density is found explicitly.

Keywords: Frobenius numbers, Kloosterman sums, continued fractions.

Let A be a proposition. Then $[A]=1$ if A is true and $[A]=0$ otherwise.

For a positive integer q we denote by $\delta_q(a)$ the characteristic function of divisibility by q:

$$\delta_q(a)=[a\equiv 0 \pmod p]=\begin{cases} 1 & \text{if } \ a\equiv 0 \pmod q, \\ 0 & \text{if } \ a\not\equiv 0 \pmod q. \end{cases}$$

The symbol $*$ in sums of the form $\sum_{m=1}^{*n}\dots$ means that the summation is carried out over the reduced system of residues.

The sum of powers of the divisors of a positive integer q is given by the formula

$$\sigma_\alpha(q)=\sum_{d\mid q} d^\alpha.$$

1. Introduction

The *Frobenius number* $g(a_1,\dots,a_n)$ of jointly coprime positive integers $a_1,\dots,a_n$ is the largest positive integer m which is not representable in the form

$$x_1a_1+\dots+x_na_n=m, \tag{1}$$

where $x_1,\dots,x_n$ are non-negative integers. In the majority of problems related to Frobenius numbers, it is more convenient to consider the function

$$f(a_1,\dots,a_n)=g(a_1,\dots,a_n)+a_1+\dots+a_n,$$

which gives the largest positive integer m which is not representable in the form (1) with positive integer coefficients $x_1,\dots,x_n$.

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For $n=2$ there is a known formula attributed to Sylvester (see [1]; see also [2] for a discussion of the history of the problem), namely, $f(a,b)=ab$. When $n=3$, the problem of finding $f(a,b,c)$ reduces to the consideration of the case in which the arguments are pairwise coprime, and when

$$(a,b)=(a,c)=(b,c)=1, \qquad b\equiv lc \pmod a, \quad 1\leqslant l\leqslant a,$$

the value of $f(a,b,c)$ can be expressed using the terms of the continued fraction for the number l/a (see the results in [3] and [4] and also the formulae (5) and (6) below). When $n\geqslant 4$, there are no known formulae for $f(a_1,\dots,a_n)$. It has been proved that for a fixed n one can evaluate the Frobenius number in polynomial time (see [5]), while for an arbitrary n, finding $f(a_1,\dots,a_n)$ becomes an NP-hard problem (see [6]).

Problems concerning the behaviour of Frobenius numbers in the mean have been posed by Davison and Arnold (see [7], and [8], Problems 1999-8, 2003-5). A natural assumption (which was later confirmed in [10]; see also [11]) was used in [9] to prove the existence of the limit distribution as $N\to\infty$ for the quantity $f(a,b,c)N^{-3/2}$ with $1\leqslant a,b,c\leqslant N$. For an arbitrary $n\geqslant 4$, it was also proved in [9] that (under an additional technical assumption) for any $\varepsilon>0$ there is a $D=D(\varepsilon)$ such that

$$P_{N,\alpha}\bigl\{f(a_1,\dots,a_n)N^{-1-\frac{1}{n+1}}\leqslant D\bigr\}\geqslant 1-\varepsilon,$$

where $P_{N,\alpha}$ stands for the probability corresponding to the uniform distribution on the set

$$\bigl\{(a_1,\dots,a_n)\colon \alpha N<a_1,\ \dots,\ a_n\leqslant N,\ (a_1,\dots,a_n)=1\bigr\}, \qquad 0<\alpha<1.$$

A more precise result was obtained in [12] (see also [13]) by methods of the geometry of numbers, namely,

$$P_{N,0}\bigl\{f(a_1,\dots,a_n)N^{-1-\frac{1}{n+1}}>t\bigr\}\ll t^{-2}.$$

It was also proved there that the numbers $f(a_1,\dots,a_n)$ behave in the mean as $N^{1+\frac{1}{n+1}}$, although the ratio $f(a_1,\dots,a_n)/N^{1+\frac{1}{n+1}}$ can be unboundedly large.

The existence of a limit distribution function for the normalized Frobenius numbers $\frac{f(a_1,\dots,a_n)}{(a_1\cdots a_n)^{1/(n-1)}}$ was proved for any $n\geqslant 3$ in [14] using methods of ergodic theory.

Arnold's problem on the existence of weak asymptotic behaviour for the numbers $f(a,b,c)$ was solved in [15]. It was proved that, in the mean over the set

$$M_a(x_1,x_2)=\bigl\{(a,b,c)\colon 1\leqslant b\leqslant x_1a,\ 1\leqslant c\leqslant x_2a,\ (a,b,c)=1\bigr\}$$

for any $x_1,x_2>0$, the Frobenius numbers behave as $\frac{8}{\pi}\sqrt{abc}$:

$$\begin{aligned}\frac{1}{|M_a(x_1,x_2)|}&\sum_{(a,b,c)\in M_a(x_1,x_2)} f(a,b,c)\\ &=\frac{1}{|M_a(x_1,x_2)|}\sum_{(a,b,c)\in M_a(x_1,x_2)}\frac{8}{\pi}\sqrt{abc}+O_{x_1,x_2,\varepsilon}(a^{4/3+\varepsilon}). \qquad (2)\end{aligned}$$

As a consequence, Davison's conjectures [7] were proved in a stronger form, namely,

$$\frac{1}{|M_a(1,1)|}\sum_{(a,b,c)\in M_a(1,1)}\frac{f(a,b,c)}{\sqrt{abc}}=\frac{8}{\pi}+O_\varepsilon(a^{-1/12+\varepsilon}). \tag{3}$$

It turns out that the limit density for the distribution of normalized Frobenius numbers $\frac{f(a,b,c)}{\sqrt{abc}}$ with three arguments can be found explicitly. As in [15], this density is obtained by averaging the three arguments over two of them, and one obtains an explicit polynomial reduction in the remainder term.

Theorem. *Let a be a positive integer and let x_1, x_2, ε be positive real numbers. Then*

$$\frac{1}{|M_a(x_1,x_2)|}\sum_{(a,b,c)\in M_a(x_1,x_2)}\bigl[f(a,b,c)\leqslant\tau\sqrt{abc}\bigr] \\ =\int_0^\tau p(t)\,dt+O_\varepsilon(R(a;x_1,x_2;\tau)a^\varepsilon),$$

where

$$R(a;x_1,x_2;\tau)\ll_{x_1,x_2,\tau}a^{-1/6},$$

and the density $p(t)$ is given by the equations

$$p(t)=\begin{cases}0 & \text{if }\ t\in[0,\sqrt{3}],\\ \dfrac{12}{\pi}\left(\dfrac{t}{\sqrt{3}}-\sqrt{4-t^2}\right) & \text{if }\ t\in[\sqrt{3},2],\\ \dfrac{12}{\pi^2}\left(t\sqrt{3}\arccos\dfrac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}+\dfrac{3}{2}\sqrt{t^2-4}\log\dfrac{t^2-4}{t^2-3}\right) & \text{if }\ t\in[2,+\infty).\end{cases}$$

One can prove the equation

$$\int_0^\infty tp(t)\,dt=\frac{8}{\pi}$$

for the function $p(t)$, and this agrees with the formulae (2) and (3). It follows from this theorem that the Frobenius numbers $f(a,b,c)$ are of the order of $\sqrt{abc}$ for almost all number triples $(a,b,c)\in M_a(1,1)$. Using properties of the density, we can describe the cardinality of the exceptional set:

$$\frac{1}{|M_a(1,1)|}\sum_{(a,b,c)\in M_a(1,1)}\bigl[f(a,b,c)>\tau\sqrt{abc}\bigr]=\frac{9}{\pi^2\tau^2}+O\left(\frac{1}{\tau^4}\right)$$

for $a>\tau^{34+\varepsilon}$.

The proof of the theorem (as well as that of the result of [15]) uses Rödseth's formula for the Frobenius numbers. This formula contains the most natural interpretation in terms of lattices in the geometric theory of continued fractions (§ 2). The largest remainder terms arising in the proof are evaluated using van der Corput's approach and estimates for Kloosterman sums (§ 3). The problem of the distribution of the normalized Frobenius numbers is first reduced to that of the distribution of the values of the Rödseth function. This involves an averaging over lattices formed by solutions of linear congruences (§ 4). After auxiliary transformations using the symmetry of the Rödseth function (§ 5), the sum over lattices is replaced by a corresponding integral (§ 6). The triple integral expressing the distribution function is differentiated with respect to a parameter, and this leads to the representation of the desired density in the form of a double integral (§ 7). This integral is then evaluated explicitly (§ 8).

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2. The Rödseth function

Let l be a fixed integer, $1 \leqslant l \leqslant a$, $(l,a)=1$, and let $\bar{l}$ be a solution of the congruence $\bar{l}l \equiv 1 \pmod{a}$, where $1 \leqslant \bar{l} \leqslant a$. In accordance with [3], we consider the continued fraction expansion of a/l with minus signs,

$$\frac{a}{l} = a_1 - \cfrac{1}{a_2 - \ddots - \frac{1}{a_m}}, \qquad a_1,\dots,a_m \geqslant 2,$$

of length $m = m(a/l)$. We introduce the sequences $\{s_j\}$, $\{q_j\}$, $-1 \leqslant j \leqslant m$, by the conditions

$$\begin{gathered} s_m = 0, \qquad s_{m-1} = 1, \qquad q_{-1} = 0, \qquad q_0 = 1, \\ s_{j-1} = a_{j+1}s_j - s_{j+1}, \quad q_{j+1} = a_{j+1}q_j - q_{j-1}, \qquad 0 \leqslant j \leqslant m-1. \end{gathered} \tag{4}$$

One can readily prove the following properties of the numbers $\{s_j\}$ and $\{q_j\}$ (see [3] and [15]).

1°) $s_{-1} = q_m = a$, $s_0 = l$, $q_{m-1} = \bar{l}$.

2°) The sequence $\{s_j\}$ is monotone decreasing, the sequence $\{q_j\}$ is monotone increasing, and the following inequalities hold:

$$0 = \frac{s_m}{q_m} < \frac{s_{m-1}}{q_{m-1}} < \dots < \frac{s_0}{q_0} < \frac{s_{-1}}{q_{-1}} = \infty.$$

3°) For any n, $0 \leqslant n \leqslant m$, the vectors $e_n = (q_n, s_n)$ and $e_{n-1} = (q_{n-1}, s_{n-1})$ form a basis of the lattice

$$\Lambda_l = \bigl\{(x,y) \in \mathbb{Z}^2 \colon xl \equiv y \pmod{a}\bigr\},$$

and here $\left|\begin{smallmatrix} q_n & s_n \\ q_{n-1} & s_{n-1} \end{smallmatrix}\right| = \det \Lambda_l = a$.

4°) The points (q_n, s_n), $-1 \leqslant n \leqslant m$, are the vertices of the convex hull of the non-zero points of the lattice Λ_l that belong to the first coordinate quadrant.

5°) The quadruples $(q_n, s_{n-1}, q_{n-1}, s_n)$ with $1 \leqslant l < a$, $(l,a)=1$, and $0 \leqslant n \leqslant m(l/a)$ are in one-to-one correspondence with the solutions (u_1,u_2,v_1,v_2) of the equation $u_1u_2 - v_1v_2 = a$ for which $0 \leqslant v_1 < u_1 \leqslant a$, $(u_1,v_1)=1$, $0 \leqslant v_2 < u_2 \leqslant a$, and $(u_2,v_2)=1$.

6°) The inequalities $s_{n-1} - s_n \leqslant a/q_n$ and $q_n - q_{n-1} \leqslant a/s_{n-1}$ hold for $0 \leqslant n \leqslant m$.

We introduce the Rödseth function $\rho_{l,a}(t_1,t_2)$, which is defined for non-negative reals t_1, t_2 such that

$$\frac{s_n}{q_n} \leqslant \frac{t_2}{t_1} < \frac{s_{n-1}}{q_{n-1}},$$

by the equation

$$\rho_{l,a}(t_1,t_2) = t_1 s_{n-1} + t_2 q_n - \min\{t_1 s_n, t_2 q_{n-1}\}. \tag{5}$$

Then it was proved in [3] that for $(b,a)=1$ and $c \equiv bl \pmod a$, the Frobenius number can be evaluated using the formula

$$f(a,b,c) = \rho_{l,a}(b,c). \tag{6}$$

Remark 1. The function $\rho_{l,a}(t_1,t_2)$ is continuous and the equation (5) holds when

$$\frac{s_n}{q_n} \leqslant \frac{t_2}{t_1} \leqslant \frac{s_{n-1}}{q_{n-1}}.$$

3. An application of estimates for Kloosterman sums

Let q be a positive integer, a an integer, and f a non-negative function. Let $T[f]$ denote the number of solutions of the congruence $xy \equiv a \pmod q$ belonging to the domain $P_1 < x \leqslant P_2$, $0 < y \leqslant f(x)$:

$$T[f] = \sum_{P_1 < x \leqslant P_2} \sum_{0 < y \leqslant f(x)} \delta_q(xy - a).$$

It was proved in [16] that the evaluation of $T[f]$ reduces to finding the sum

$$S[f] = \frac{1}{q} \sum_{P_1 < x \leqslant P_2} \mu_{q,a}(x) f(x),$$

where $\mu_{q,a}(x)$ stands for the number of solutions of the congruence $xy \equiv a \pmod q$ when the variable y belongs to the interval $[1,q]$.

We now give a simplified version of the result in [17] which refines the corresponding theorem in [16]. This result uses bounds for the Kloosterman sums

$$K_q(l,m,n) = \sum_{x,y=1}^{q} \delta_q(xy-l) \exp\left\{2\pi i \frac{mx+ny}{q}\right\}$$

and van der Corput's approach to estimating trigonometric sums. The proof uses the inequality

$$|K_q(l,m,n)| \leqslant \sigma_0(q)\sigma_0((l,m,n,q))(lm,ln,mn,q)^{1/2} q^{1/2},$$

which generalizes Estermann's result [18],

$$|K_q(\pm 1,m,n)| \leqslant \sigma_0(q)(m,n,q)^{1/2} q^{1/2}.$$

Lemma 1. *Let P_1 and P_2 be real numbers such that $P = P_2 - P_1 \geqslant 2$, let $f(x)$ be a real non-negative function twice continuously differentiable on the entire interval $[P_1, P_2]$, and suppose that*

$$\frac{1}{A} \leqslant |f''(x)| \leqslant \frac{w}{A}$$

for some $A > 0$ and $w \geqslant 1$. Then the following asymptotic formula holds:

$$T[f] = S[f] - \frac{P}{2}\delta_q(a) + R[f],$$

where

$$R[f] \ll_{w,\varepsilon} (PA^{-1/3} + A^{1/2}D + q^{1/2})P^{\varepsilon}, \qquad D = (a, q).$$

Remark 2. It follows from the trivial formula

$$\sum_{y=1}^{Y} \delta_q(xy - a) = \frac{Y}{q}\mu_{q,a}(x) + O(1)$$

that, in the notation of Lemma 1,

$$T[f] = S[f] + O(P).$$

Lemma 2 ([15], Lemma 5). *Let P_1 and P_2 be real numbers such that $P = P_2 - P_1 > 0$ and suppose that a real non-negative function $f(x)$ is constant on the interval $[P_1, P_2]$. Then*

$$T[f] = S[f] + O\biggl(\biggl(q^{1/2} + \biggl(\frac{P}{q} + 1\biggr)D\biggr)q^{\varepsilon}\biggr),$$

where $D = (a, q)$.

In turn, the sum $S[f]$ is also regular, and finding it can be reduced to the evaluation of an integral.

Lemma 3 ([15], Lemma 6). *Let f be a function decreasing on the interval $[P_1, P_2]$ and let $f(P_1) - f(P_2) = Q$. Then, in the notation of Lemma 1,*

$$S[f] = \psi(a, q)\int_{P_1}^{P_2} f(x)\,dx + O(DQq^{-1+\varepsilon}),$$

where

$$\psi(a, q) = \frac{K_q(a, 0, 0)}{q^2} = \frac{1}{q}\sum_{k|(a,q)}\sum_{\delta|q/k}\frac{\mu(\delta)}{\delta} \ll q^{-1+\varepsilon}. \tag{7}$$

Lemma 4 ([15], Lemma 7). *Let $\tau > 0$, let a function $I(r)/r \in C([0, \tau])$ have finitely many intervals of monotonicity, let $|I(r)/r| \leqslant B$ for any $r \in [0, \tau]$, let $\psi(a, q)$ be as defined in (7), and let $1 \leqslant U\tau \leqslant a$. Then*

$$\sum_{q \leqslant \tau U} \psi(a, q) I\biggl(\frac{q}{U}\biggr) = \frac{\sigma_{-1}(a)}{\zeta(2)}\int_0^{\tau}\frac{I(r)}{r}\,dr + O(BU^{-1}a^{\varepsilon}).$$

4. Reduction to an integral involving the Rödseth function

Lemma 5 ([15], Remark 2). *Let Ω be a plane simply connected domain with rectifiable boundary, let V be the area of Ω, and let P be the perimeter of Ω. If Λ is a sublattice of $\mathbb{Z}^2$ of index d and $N(\Lambda)$ is the number of points of Λ belonging to Ω, then*

$$\left|N(\Lambda)-\frac{V}{d}\right|\leqslant 4(P+1).$$

Lemma 6. *Let $1\leqslant l<a$, $(l,a)=1$, let δ_1 and δ_2 be positive integers, let x_1 and x_2 be positive real numbers, and let $\{s_j\}$ and $\{q_j\}$ be the sequences given by the equations (4). We introduce an integer $r=r(l,a;x_1,x_2)$ by the inequalities*

$$\frac{s_r}{q_r}\leqslant\frac{x_2}{x_1}<\frac{s_{r-1}}{q_{r-1}}.$$

Then the sum

$$S_{l,a}(\delta_1,\delta_2;x_1,x_2;\tau)=\sum_{\substack{b\leqslant x_1a\\ \delta_1|b}}\sum_{\substack{c\leqslant x_2a\\ \delta_2|c}}\delta_a(bl-c)\big[\rho_{l,a}(b,c)\leqslant\tau\sqrt{abc}\,\big]$$

satisfies the asymptotic formula

$$S_{l,a}(\delta_1,\delta_2;x_1,x_2;\tau)$$
$$=a\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^{x_1}\int_0^{x_2}\big[\rho_{l,a}(t_1,t_2)\leqslant\tau\sqrt{at_1t_2}\,\big]\,dt_1\,dt_2+O(R_{l,a}(x_1,x_2)),$$

where

$$R_{l,a}(x_1,x_2)=x_1a\left(\frac{1}{s_0}+\cdots+\frac{1}{s_{r-1}}\right)+x_2a\left(\frac{1}{q_r}+\cdots+\frac{1}{q_{m-1}}\right).$$

Proof. Consider the lattice

$$\Lambda_l(\delta_1,\delta_2)=\big\{(x,y)\in\Lambda_l\colon\ \delta_1|x,\ \ \delta_2|y\big\}.$$

An arbitrary point (x,y) of Λ_l is of the form $(x,y)=ue_{-1}+ve_0$, where u and v are integers, $e_{-1}=(0,a)$, and $e_0=(1,l)$ (see the property 3°)). A point of this kind belongs to $\Lambda_l(\delta_1,\delta_2)$ if and only if the following congruences hold:

$$v\equiv 0\pmod{\delta_1},\qquad au+lv\equiv 0\pmod{\delta_2}.$$

Hence, $\Lambda_l(\delta_1,\delta_2)$ is a sublattice of index $\delta_1\delta_2/(a,\delta_1,\delta_2)$ in Λ_l.

Consider the sum

$$S_n=S_{l,a,n}(\delta_1,\delta_2;x_1,x_2)$$
$$=\sum_{\substack{b\leqslant x_1a\\ \delta_1|b}}\sum_{\substack{c\leqslant x_2a\\ \delta_2|c}}\left[(b,c)\in\Lambda_l(\delta_1,\delta_2),\ \frac{s_n}{q_n}\leqslant\frac{c}{b}<\frac{s_{n-1}}{q_{n-1}},\ \rho_{l,a}(b,c)\leqslant\tau\sqrt{abc}\right].$$

By the property 3°) of the sequences $\{s_j\}$ and $\{q_j\}$, all solutions of the congruence $bl \equiv c \pmod{a}$ for which $s_n/q_n \leqslant c/b < s_{n-1}/q_{n-1}$ are of the form

$$b(u,v) = uq_n + vq_{n-1}, \qquad c(u,v) = us_n + vs_{n-1}$$

with integers $u > 0$ and $v \geqslant 0$. Therefore,

$$S_n = \sum_{u>0}\sum_{v\geqslant 0}\bigl[b(u,v) \leqslant x_1 a,\ \delta_1|b(u,v),\ c(u,v) \leqslant x_2 a,\ \delta_2|c(u,v)\bigr]h_{l,a,n}(u,v),$$

where

$$h_{l,a,n}(u,v) = \bigl[\rho_{l,a}(uq_n + vq_{n-1}, us_n + vs_{n-1}) \leqslant \tau\sqrt{a(uq_n + vq_{n-1})(us_n + vs_{n-1})}\,\bigr].$$

For $n > r$, the only substance in the two conditions $b \leqslant x_1 a$ and $c \leqslant x_2 a$ is in the first. Thus,

$$S_n = \sum_{u>0}\sum_{v\geqslant 0}\bigl[uq_n + vq_{n-1} \leqslant x_1 a,\ \delta_1|uq_n + vq_{n-1},\ \delta_2|us_n + vs_{n-1}\bigr]h_{l,a,n}(u,v).$$

The variables u and v take their values in the triangular domain $u > 0$, $v \geqslant 0$, $uq_n + vq_{n-1} \leqslant x_1 a$, whose perimeter is $O(x_1 a/q_{n-1})$. Moreover, as noted above, $\Lambda_l(\delta_1, \delta_2)$ is a sublattice of index $\delta_1\delta_2/(a, \delta_1, \delta_2)$ in Λ_l. Therefore, applying Lemma 5, we see that (in the integrands, the previous variables of summation now take real values)

$$\begin{aligned} S_n &= \frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^a\int_0^a [uq_n + vq_{n-1} \leqslant x_1 a] h_{l,a,n}(u,v)\,du\,dv + O\biggl(\frac{x_1 a}{q_{n-1}}\biggr) \\ &= \frac{(a,\delta_1,\delta_2)}{a\delta_1\delta_2}\int_0^{x_1 a}\int_0^{x_2 a}\biggl[\frac{s_n}{q_n} \leqslant \frac{c}{b} < \frac{s_{n-1}}{q_{n-1}},\ \rho_{l,a}(b,c) \leqslant \tau\sqrt{abc}\biggr]db\,dc + O\biggl(\frac{x_1 a}{q_{n-1}}\biggr) \\ &= a\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^{x_1}\int_0^{x_2}\biggl[\frac{s_n}{q_n} \leqslant \frac{t_2}{t_1} < \frac{s_{n-1}}{q_{n-1}},\ \rho_{l,a}(t_1,t_2) \leqslant \tau\sqrt{at_1t_2}\biggr]dt_1\,dt_2 \\ &\qquad + O\biggl(\frac{x_1 a}{q_{n-1}}\biggr). \end{aligned} \tag{8}$$

Similarly, only the second of the two conditions $b \leqslant x_1 a$ and $c \leqslant x_2 a$ is kept for $n < r$, and this leads to the equation

$$\begin{aligned} S_n = a^2\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^{x_1}\int_0^{x_2}\biggl[\frac{s_n}{q_n} \leqslant \frac{t_2}{t_1} < \frac{s_{n-1}}{q_{n-1}},\ \rho_{l,a}(t_1,t_2) \leqslant \tau\sqrt{at_1t_2}\biggr]dt_1\,dt_2 \\ + O\biggl(\frac{x_1 a}{s_n}\biggr). \end{aligned} \tag{9}$$

If $n = r$, then the line $c/b = x_2/x_1$ on the plane Obc divides the angle $s_n/q_n \leqslant c/b < s_{n-1}/q_{n-1}$ into two parts. In the first ($s_n/q_n \leqslant c/b < x_2/x_1$), one must consider the condition $b \leqslant x_1 a$ and in the second ($x_2/x_1 \leqslant c/b < s_{n-1}/q_{n-1}$), the condition $c \leqslant x_2 a$. Therefore,

$$\begin{aligned} S_n = &\sum_{u>0}\sum_{v\geqslant 0}\bigl[b(u,v) \leqslant x_1 a,\ x_2 b(u,v) > x_1 c(u,v),\ \delta_1\,|\,b(u,v),\ \delta_2\,|\,c(u,v)\bigr]h_{l,a,r}(u,v) \\ &+ \sum_{u>0}\sum_{v\geqslant 0}\bigl[c(u,v) \leqslant x_2 a,\ x_2 b(u,v) \leqslant x_1 c(u,v),\ \delta_1\,|\,b(u,v),\ \delta_2\,|\,c(u,v)\bigr]h_{l,a,r}(u,v). \end{aligned}$$

The variables u and v take their values in the quadrangular domain $u > 0,\ v \geqslant 0$, $uq_r + vq_{r-1} \leqslant x_1 a,\ us_r + vs_{r-1} \leqslant x_2 a$, whose perimeter is estimated as (see the property 6°))

$$O\biggl(\frac{x_1 a}{q_r} + \frac{x_2 a}{s_{r-1}} + x_1(s_{r-1} - s_r) + x_2(q_r - q_{r-1})\biggr) = O\biggl(\frac{x_1 a}{q_r} + \frac{x_2 a}{s_{r-1}}\biggr).$$

Again applying Lemma 5, we arrive at the equation

$$S_r = a\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^{x_1}\int_0^{x_2}\biggl[\frac{s_r}{q_r} \leqslant \frac{t_2}{t_1} < \frac{s_{r-1}}{q_{r-1}},\ \rho_{l,a}(t_1,t_2) \leqslant \tau\sqrt{at_1t_2}\biggr]dt_1\,dt_2 + O\biggl(\frac{x_1 a}{q_r} + \frac{x_2 a}{s_{r-1}}\biggr). \qquad (10)$$

Adding the equations (8)–(10), we obtain the desired asymptotic formula for the sum,

$$S_{l,a}(\delta_1,\delta_2;x_1,x_2;\tau) = \sum_{n=0}^{m} S_n.$$

This completes the proof of the lemma.

Lemma 7. *Under the hypotheses of Lemma 6, for any* $\varepsilon > 0$ *the sum* $R_a(x_1,x_2) = \sum_{l=1}^{*a} R_{l,a}(x_1,x_2)$ *satisfies the bound*

$$R_a(x_1,x_2) \ll (x_1+x_2)a^{1/2+\varepsilon}.$$

Proof. Using the symmetry of the sum $R_a(x_1,x_2)$ (it is preserved under the changes $x_1 \leftrightarrow x_2,\ s_j \leftrightarrow q_{m-j-1}$) and the property 5°) of the sequences $\{s_j\}$ and $\{q_j\}$, we have the bound

$$R_a(x_1,x_2) \ll \sum_{u_1=1}^{a}\sum_{v_1=0}^{u_1-1}\sum_{u_2=1}^{a}\sum_{v_2=0}^{u_2-1}\biggl[u_1u_2 - v_1v_2 = a,\ \frac{v_2}{u_1} \leqslant \frac{x_2}{x_1}\biggr]\frac{x_1}{u_1}.$$

We introduce the new variables $x = u_2,\ y = u_1 - v_1,\ z = u_1,\ w = u_2 - v_2$. Then

$$\begin{aligned} R_a(x_1,x_2) &\ll \sum_{z=1}^{a}\frac{x_1}{z}\sum_{w\geqslant 0}\sum_{x\leqslant x_2z/x_1}\sum_{y\leqslant z}[zw + xy = a] \\ &\ll \sum_{z=1}^{a}\frac{x_1}{z}\sum_{x\leqslant x_2z/x_1}\sum_{y\leqslant z}[xy \leqslant a]\delta_z(xy - a). \end{aligned}$$

For any z one can partition the domain where the variables x and y take their values into parts ($x \leqslant \sqrt{a}$ and $x > \sqrt{a}$) in such a way that the size of each of them with respect to one of the coordinates does not exceed $\sqrt{a}$. Applying Remark 2 and Lemmas 2 and 3 to each of these parts, we obtain the desired bound:

$$\begin{aligned} R_a(x_1,x_2) &= \sum_{z\leqslant\sqrt{x_1a/x_2}}\frac{x_1}{z}\biggl(\frac{x_2z}{x_1} + \sqrt{a}\biggr)a^{\varepsilon} + \sum_{z>\sqrt{x_1a/x_2}}\frac{x_1}{z}\biggl(\frac{a}{z} + \sqrt{a}\biggr)a^{\varepsilon} \\ &\ll (x_1+x_2)a^{1/2+\varepsilon}. \end{aligned}$$

This completes the proof of the lemma.

Corollary 1. *Under the hypotheses of Lemma 6, the sum*

$$S_a(\delta_1,\delta_2;x_1,x_2;\tau)=\sum_{l=1}^{a}{}^{*}\,S_{l,a}(\delta_1,\delta_2;x_1,x_2;\tau)$$

satisfies the asymptotic formula

$$S_a(\delta_1,\delta_2;x_1,x_2;\tau)=a\frac{(a,\delta_1,\delta_2)}{\delta_1\delta_2}\int_0^{x_1}\int_0^{x_2}\rho_a(t_1,t_2;\tau)\,dt_1\,dt_2+O((x_1+x_2)a^{3/2+\varepsilon}),$$

where

$$\rho_a(t_1,t_2;\tau)=\sum_{l=1}^{a}{}^{*}[\rho_{l,a}(t_1,t_2)\leqslant\tau\sqrt{at_1t_2}\,].$$

Lemma 8. *Let x_1 and x_2 be positive real numbers. Then the sum*

$$F_a(x_1,x_2;\tau)=\sum_{(a,b,c)\in M_a(x_1,x_2)}[f(a,b,c)\leqslant\tau\sqrt{abc}\,] \tag{11}$$

satisfies the asymptotic formula

$$F_a(x_1,x_2;\tau)=$$
$$=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}\frac{a}{d_1d_2}\sum_{\delta_1|d_2a_1}\frac{\mu(\delta_1)}{\delta_1}\sum_{\delta_2|d_1a_1}\frac{\mu(\delta_2)}{\delta_2}(a,\delta_1,\delta_2)\int_0^{x_1d_2}\int_0^{x_2d_1}\rho_{a_1}(t_1,t_2;\tau)\,dt_1\,dt_2$$
$$+O((x_1+x_2)a^{3/2+\varepsilon}),$$

where $a_1=a/(d_1d_2)$.

Proof. To find the sum $F_a(x_1,x_2;\tau)$, we introduce the parameters $d_1=(a,b)$ and $d_2=(a,c)$ and write $b_1=b/d_1$, $c_1=c/d_2$, and $a_1=a/(d_1d_2)$. For any non-zero summand we have $(d_1,d_2)=1$. Therefore,

$$F_a(x_1,x_2;\tau)=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}\sum_{\substack{b\leqslant x_1a\\(b,a)=d_1}}\sum_{\substack{c\leqslant x_2a\\(c,a)=d_2}}[f(a,b,c)\leqslant\tau\sqrt{abc}\,]$$
$$=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}\sum_{\substack{b_1\leqslant x_1d_2a_1\\(b_1,d_2a_1)=1}}\sum_{\substack{c_1\leqslant x_2d_1a_1\\(c_1,d_1a_1)=1}}\big[f(d_1d_2a_1,d_1b_1,d_2c_1)\leqslant\tau d_1d_2\sqrt{a_1b_1c_1}\,\big].$$

Applying Johnson's identity (see [19])

$$f(a,b,c)=df\left(\frac{a}{d},\frac{b}{d},c\right),$$

we see that

$$F_a(x_1,x_2;\tau)=\sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}}\sum_{\substack{b_1\leqslant x_1d_2a_1\\(b_1,d_2a_1)=1}}\sum_{\substack{c_1\leqslant x_2d_1a_1\\(c_1,d_1a_1)=1}}[f(a_1,b_1,c_1)\leqslant\tau\sqrt{a_1b_1c_1}\,].$$

One can now express the Frobenius number in terms of the Rödseth function using the formula (6). Thus,

$$
\begin{aligned}
F_a(x_1,x_2;\tau) &= \sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}} \sideset{}{^*}\sum_{l=1}^{a_1} \sum_{\substack{b_1\leqslant x_1d_2a_1\\(b_1,d_2a_1)=1}} \sum_{\substack{c_1\leqslant x_2d_1a_1\\(c_1,d_1a_1)=1}} \delta_{a_1}(b_1l-c_1)\\
&\qquad\qquad\times\bigl[\rho_{l,a_1}(b_1,c_1)\leqslant\tau\sqrt{a_1b_1c_1}\,\bigr]\\
&= \sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}} \sum_{\delta_1|d_2a_1}\mu(\delta_1)\sum_{\delta_2|d_1a_1}\mu(\delta_2)\sideset{}{^*}\sum_{l=1}^{a_1} \sum_{\substack{b_1\leqslant x_1d_2a_1\\ \delta_1|b_1}} \sum_{\substack{c_1\leqslant x_2d_1a_1\\ \delta_2|c_1}} \delta_{a_1}(b_1l-c_1)\\
&\qquad\qquad\times\bigl[\rho_{l,a_1}(b_1,c_1)\leqslant\tau\sqrt{a_1b_1c_1}\,\bigr]\\
&= \sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}} \sum_{\delta_1|d_2a_1}\mu(\delta_1)\sum_{\delta_2|d_1a_1}\mu(\delta_2)S_{a_1}(\delta_1,\delta_2;x_1d_2,x_2d_1;\tau).
\end{aligned}
$$

Substituting the result of Corollary 1 into the last formula, we arrive at the statement of the lemma.

Remark 3. The same arguments applied to the sum

$$G_a(x_1,x_2;\tau)=\sum_{(a,b,c)\in M_a(x_1,x_2)} g(\tau)$$

with $g(\tau)\in[0,1]$ lead to the formula

$$
\begin{aligned}
G_a(x_1,x_2;\tau) &= \sum_{\substack{d_1d_2|a\\(d_1,d_2)=1}} \frac{a}{d_1d_2}\varphi\biggl(\frac{a}{d_1d_2}\biggr)\sum_{\delta_1|d_2a_1}\frac{\mu(\delta_1)}{\delta_1}\sum_{\delta_2|d_1a_1}\frac{\mu(\delta_2)}{\delta_2}(a,\delta_1,\delta_2)\\
&\qquad\times\int_0^{x_1d_2}\int_0^{x_2d_1} g(\tau)\,dt_1\,dt_2+O((x_1+x_2)a^{3/2+\varepsilon}).
\end{aligned}
$$

Remark 4. Since the function $\rho_{l,a}(t_1,t_2)$ is homogeneous, it follows that

$$\rho_{l,a}(t_1,t_2)=t_1\rho_{l,a}\biggl(\frac{t_2}{t_1}\biggr),$$

where

$$\rho_{l,a}(\xi)=s_{n-1}+\xi q_n-\min\{s_n,\xi q_{n-1}\}$$

for $s_n/q_n\leqslant\xi<s_{n-1}/q_{n-1}$. Therefore, knowing the density ρ_a,

$$\rho_a(\xi;\tau)=\sideset{}{^*}\sum_{l=1}^{a}\bigl[\rho_{l,a}(\xi)\leqslant\tau\sqrt{a\xi}\,\bigr], \tag{12}$$

one can readily find the desired function

$$\rho_a(t_1,t_2;\tau)=\rho_a(t_2/t_1;\tau). \tag{13}$$

5. Auxiliary transformations

By the property 5°) of the sequences $\{s_j\}$ and $\{q_j\}$,

$$\rho_a(\xi;\tau) = \sum_{u_1=1}^{a} \sideset{}{^*}\sum_{v_1=0}^{u_1-1} \sum_{u_2=1}^{a} \sideset{}{^*}\sum_{v_2=0}^{u_2-1} \biggl[u_1u_2 - v_1v_2 = a,\ \frac{v_2}{u_1} \leqslant \xi < \frac{u_2}{v_1},\ u_2 + \xi u_1 - \min\{v_2, \xi v_1\} \leqslant \tau\sqrt{a\xi}\biggr].$$

Considering the cases $v_2 > \xi v_1$ and $v_2 \leqslant \xi v_1$ separately, we can write the desired density in the form

$$\rho_a(\xi;\tau) = \lambda^*(a;\xi;\tau) + \eta^*(a;\xi;\tau),$$

where

$$\lambda^*(a;\xi;\tau) = \sum_{u_1=1}^{a} \sideset{}{^*}\sum_{v_1=0}^{u_1-1} \sum_{u_2=1}^{a} \sideset{}{^*}\sum_{v_2=0}^{u_2-1} \biggl[u_1u_2 - v_1v_2 = a,\ \frac{v_2}{u_1} \leqslant \xi < \frac{v_2}{v_1},\ u_2 + \xi(u_1 - v_1) \leqslant \tau\sqrt{a\xi}\biggr],$$

$$\eta^*(a;\xi;\tau) = \sum_{u_1=1}^{a} \sideset{}{^*}\sum_{v_1=0}^{u_1-1} \sum_{u_2=1}^{a} \sideset{}{^*}\sum_{v_2=0}^{u_2-1} \biggl[u_1u_2 - v_1v_2 = a,\ \frac{v_2}{v_1} \leqslant \xi < \frac{u_2}{v_1},\ u_2 - v_2 + \xi u_1 \leqslant \tau\sqrt{a\xi}\biggr].$$

In view of Remark 1, the change of variables $u_1 \leftrightarrow u_2$ and $v_1 \leftrightarrow v_2$ leads to the equation $\eta^*(a;\xi;\tau) = \lambda^*(a;1/\xi;\tau)$. Therefore,

$$\rho_a(\xi;\tau) = \lambda^*(a;\xi;\tau) + \lambda^*(a;1/\xi;\tau). \tag{14}$$

To evaluate the function $\lambda^*(a;\xi;\tau)$, we rewrite the equation $u_1u_2 - v_1v_2 = a$ in the form

$$u_1(u_2 - v_2) + v_2(u_1 - v_1) = a$$

and introduce the variables $x = u_1,\ y = u_2 - v_2,\ z = u_1 - v_1,\ w = v_2$. Similarly, the sum $\lambda^*(a;\xi;\tau)$ can be written in the form

$$\lambda^*(a;\xi;\tau) = \sum_{x=1}^{a} \sideset{}{^*}\sum_{z=1}^{x} \sum_{y=1}^{a} \sideset{}{^*}\sum_{w=0}^{a-1} \biggl[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z},\ y + w + \xi z \leqslant \tau\sqrt{a\xi}\biggr].$$

Getting rid of the coprimeness conditions, we obtain

$$\lambda^*(a;\xi;\tau) = \sum_{d_1d_2|a} \mu(d_1)\mu(d_2)\lambda\biggl(\frac{a}{d_1d_2}; \frac{d_1\xi}{d_2}; \tau\biggr), \tag{15}$$

where

$$\lambda(a;\xi;\tau) = \sum_{x\geqslant1} \sum_{z=1}^{x} \sum_{y\geqslant1} \sum_{w\geqslant0} \biggl[xy + wz = a,\ \frac{w}{x} \leqslant \xi < \frac{w}{x-z},\ y + w + \xi z \leqslant \tau\sqrt{a\xi}\biggr]. \tag{16}$$

6. An integral representation for the distribution function

Lemma 9. *The sum $\lambda(a;\xi;\tau)$ defined by the equation (16) satisfies the asymptotic formula*

$$\lambda(a;\xi;\tau)=\frac{a\sigma_{-1}(a)}{\zeta(2)}\Phi(\tau)+O_\varepsilon(R_0(a,\xi,\tau)),$$

where

$$\Phi(\tau)=\int_0^\tau\frac{I(r,\tau)}{r}dr,$$

$$I(r,\tau)=\int_r^\infty\int_r^\infty\Big[1\leqslant\alpha\beta\leqslant 1+r^2,\ \alpha+\beta-\tau\leqslant\frac{\alpha\beta-1}{r}\Big]d\alpha\,d\beta, \tag{17}$$

$$R_0(a,\xi,\tau)=\big(a^{5/6}\tau^{5/3}\xi^{1/2}+a^{3/4}\tau^{3/2}(\xi^{3/4}+\xi^{1/2})+a^{1/2}\tau(\xi^{1/2}+\xi^{-1/2})\big)a^\varepsilon.$$

Proof. We note that for known values of w, x, and y such that $xy\equiv a \pmod w$, the value of z can be found uniquely from the formula

$$z=\frac{a-xy}{w}.$$

Therefore, in view of the condition $z\leqslant x$, we can represent the sum $\lambda(a;\xi;\tau)$ in the form

$$\begin{aligned}\lambda(a;\xi;\tau)&=\sum_{w\leqslant\tau\sqrt{a\xi}}\ \sum_{x\geqslant w/\xi}\ \sum_{y\geqslant 1}\delta_w(xy-a)\big[\max\{y_0(x),y_1(x)\}\leqslant y<y_2(x)\big]\\&=\sum_{w\leqslant\tau\sqrt{a\xi}}\ \sum_{y\geqslant 1}\ \sum_{x\geqslant w/\xi}\delta_w(xy-a)\big[\max\{x_0(y),x_1(y)\}\leqslant x<x_2(y)\big]\\&=\sum_{w\leqslant\tau\sqrt{a\xi}}\ \sum_{(x,y)\in\Omega}\delta_w(xy-a),\end{aligned}$$

where

$$y_0(x)=\frac{a+\frac{w^2}{\xi}-w\tau\sqrt{\frac{a}{\xi}}}{x-\frac{w}{\xi}},\qquad y_1(x)=\frac{a}{x}-w,\qquad y_2(x)=\frac{a}{x}-w+\frac{w^2}{\xi x},$$

$$x_0(y)=\frac{a+\frac{w^2}{\xi}-w\tau\sqrt{\frac{a}{\xi}}}{y}+\frac{w}{\xi},\qquad x_1(y)=\frac{a}{w+y},\qquad x_2(y)=\frac{1}{w+y}\Big(a+\frac{w^2}{\xi}\Big),$$

and the domain $\Omega=\Omega(a,\xi,\tau)$ is given by the inequalities

$$x\geqslant\frac{w}{\xi},\qquad \max\{y_0(x),y_1(x)\}\leqslant y<y_2(x),$$

or by the equivalent conditions

$$x\geqslant\frac{w}{\xi},\qquad \max\{x_0(y),x_1(y)\}\leqslant x<x_2(y).$$

For the sum $\lambda(a;\xi;\tau)$ not to be identically zero, the domain $y_0(x)\leqslant y<y_2(x)$ must be non-empty. The equation $y_1(x)=y_2(x)$ reduces to a quadratic equation

with discriminant $-3r^2+2r\tau+\tau^2-4$, where $r=w/\sqrt{a\xi}$. The discriminant can take non-negative values only for $\tau\geqslant\sqrt{3}$. Thus, the domain is empty for $\tau<\sqrt{3}$, and the function $\lambda(a;\xi;\tau)$ (as well as $I(r,\tau)$) is identically equal to zero. Therefore, in what follows, we everywhere assume that the inequality $\tau\geqslant\sqrt{3}$ (corresponding to the Rödseth bound $f(a,b,c)\geqslant\sqrt{3abc}$ [20]) is satisfied.

We introduce the parameters

$$U_0=\sqrt{a+\frac{w^2}{\xi}-w\tau\sqrt{\frac{a}{\xi}}}\leqslant\sqrt{a},\qquad U_1=\sqrt{a},\qquad U_2=\sqrt{a+\frac{w^2}{\xi}}$$

and decompose the domain Ω into parts by the horizontal lines

$$y=U_0,\qquad y=U_1-w,\qquad y=U_2-w$$

and the vertical lines

$$x=U_0+\frac{w}{\xi},\qquad x=U_1,\qquad x=U_2.$$

For $x\leqslant U_0+w/\xi$ we apply Lemma 1 to the function $y_0(x)$ by partitioning the domain where the variable x takes its values into half-open intervals of the form $(X+w/\xi,2X+w/\xi]$. On each of them we have

$$y_0''(x)\asymp\frac{a+\frac{w^2}{\xi}-w\tau\sqrt{\frac{a}{\xi}}}{(x-\frac{w}{\xi})^3}\asymp\frac{U_0^2}{X^3}=\frac{1}{A_0(X)},\qquad A_0(X)\leqslant U_0\leqslant U_2.$$

For $x\geqslant U_0+w/\xi$ $(y\leqslant U_0)$ we apply Lemma 1 to the function $x_0(y)$ by partitioning the domain where the variable y takes its values into half-open intervals of the form $(Y,2Y]$. On each of these intervals we have

$$x_0''(y)\asymp\frac{a+\frac{w^2}{\xi}-w\tau\sqrt{\frac{a}{\xi}}}{y^3}\asymp\frac{U_0^2}{Y^3}=\frac{1}{A_0(Y)},\qquad A_0(Y)\leqslant U_0\leqslant U_2.$$

For the function $y_1(x)$ we apply Lemma 1 for $x\leqslant U_1$, and for $x\in(X,2X]$ we have

$$y_1''(x)\asymp\frac{a}{x^3}\asymp\frac{U_1^2}{X^3}=\frac{1}{A_1(X)},\qquad A_1(X)\leqslant U_1\leqslant U_2.$$

For the function $x_1(y)$ we apply Lemma 1 for $y\leqslant U_1-w$ $(x\geqslant U_1)$. When $y\in(Y-w,2Y-w]$ we have

$$x_1''(y)\asymp\frac{a}{(w+y)^3}\asymp\frac{U_1^2}{Y^3}=\frac{1}{A_1(Y)},\qquad A_1(Y)\leqslant U_1\leqslant U_2.$$

For the function $y_2(x)$ we apply Lemma 1 for $x\leqslant U_2$, and for $x\in(X,2X]$ we have

$$y_2''(x)\asymp\frac{a+\frac{w^2}{\xi}}{x^3}\asymp\frac{U_2^2}{X^3}=\frac{1}{A_2(X)},\qquad A_2(X)\leqslant U_2.$$

For the function $x_2(y)$ we apply Lemma 1 for $y \leqslant U_2 - w \quad (x \geqslant U_2)$, and for $y \in (Y-w, 2Y-w]$ we have

$$x_1''(y) \asymp \frac{a+\frac{w^2}{\xi}}{(w+y)^3} \asymp \frac{U_2^2}{Y^3} = \frac{1}{A_2(Y)}, \qquad A_2(Y) \leqslant U_2.$$

We apply Lemma 2 to the rectangular parts of the partition of Ω and Lemma 3 to the sums $S[x_i(y)]$, $S[y_i(x)]$, $i=0,1,2$. Using the relations

$$S[f-g] = S[f]-S[g], \qquad x_2(y)-x_1(y) \leqslant \frac{w}{\xi}, \quad y_2(x)-y_1(x) \leqslant \frac{w^2}{\xi x} \leqslant w,$$

we arrive at the leading term

$$\sum_{w \leqslant \tau\sqrt{a\xi}} \psi(a,w) \iint_{\Omega(a,\xi,\tau)} dx\, dy$$

and the remainder

$$\begin{aligned} R_0(a,\xi,\tau) \ll \sum_{w \leqslant \tau\sqrt{a\xi}} \Bigg(\left(a+\frac{w^2}{\xi}\right)^{1/3} + \left(a+\frac{w^2}{\xi}\right)^{1/4} D_w + w^{1/2} \\ + \delta_w(a)\left(a+\frac{w^2}{\xi}\right)^{1/2} + \frac{D_w}{w}\left(w\left(1+\frac{1}{\xi}\right) + \left(a+\frac{w^2}{\xi}\right)^{1/2} + w^{1/2}\right)\Bigg) a^{\varepsilon} \\ \ll \left(a^{5/6}\tau^{5/3}\xi^{1/2} + a^{3/4}\tau^{3/2}(\xi^{1/2}+\xi^{3/4}) + a^{1/2}\tau(\xi^{1/2}+\xi^{-1/2})\right)a^{\varepsilon}. \end{aligned}$$

In the leading term we make the change of the variables $x = \alpha\sqrt{a/\xi}$, $y = \beta\sqrt{a\xi}$ and write $r = w/\sqrt{a\xi}$. We then obtain

$$\lambda(a;\xi;\tau) = \sum_{w \leqslant \tau\sqrt{a\xi}} \psi(a,w) I\left(\frac{w}{\sqrt{a\xi}}, \tau\right) + O_{\varepsilon}(R_0(a,\xi,\tau)),$$

where

$$I(r,\tau) = a\int_r^{\infty} d\alpha \int_0^{\infty} \left[\max\{\beta_0(\alpha),\beta_1(\alpha)\} \leqslant \beta < \beta_2(\alpha)\right] d\beta,$$

$$\beta_0(\alpha) = \frac{1+r^2-r\tau}{\alpha - r}, \qquad \beta_1(\alpha) = \frac{1}{\alpha} - r, \qquad \beta_2(\alpha) = \frac{1+r^2}{\alpha} - r.$$

The change $\beta \to \beta - r$ reduces the integral $I(r,\tau)$ to the form (17). Applying Lemma 4 and using the bound $I(r,\tau) \ll r \leqslant \tau$, we arrive at the desired formula for the sum $\lambda(a;\xi;\tau)$. This completes the proof of the lemma.

Corollary 2. *In the notation of Lemma 9, the function $\rho_a(\xi;\tau)$ given by (12) satisfies the asymptotic formula*

$$\rho_a(\xi;\tau) = \frac{2\varphi(a)}{\zeta(2)}\Phi(\tau) + O_{\varepsilon}(R(a,\xi,\tau)),$$

where

$$R(a,\xi,\tau) = \left(a^{5/6}\tau^{5/3}(\xi^{1/2}+\xi^{-1/2}) + a^{3/4}\tau^{3/2}(\xi^{3/4}+\xi^{-3/4}) + a^{1/2}\tau(\xi^{1/2}+\xi^{-1/2})\right)a^{\varepsilon}.$$

Proof. Using the equation

$$\sum_{d_1d_2|a}\mu(d_1)\mu(d_2)\frac{a}{d_1d_2}\sigma_{-1}\left(\frac{a}{d_1d_2}\right)=\varphi(a),$$

we see by the formula (15) that

$$\lambda^*(a;\xi;\tau)=\frac{\varphi(a)}{\zeta(2)}\Phi(\tau)+O_\varepsilon(R_0(a,\xi,\tau)).$$

Substituting this equation into (14), we arrive at the statement of the corollary.

Corollary 3. *The sum $F_a^*(x_1,x_2;\tau)$ defined by* (11) *satisfies the equation*

$$F_a^*(x_1,x_2;\tau)=\frac{2}{\zeta(2)}|M_a(x_1,x_2)|\Phi(\tau)+O_\varepsilon(R(a;x_1,x_2;\tau)),$$

where

$$R(a;x_1,x_2;\tau)=\big(a^{11/6}\tau^{5/3}x_1^{1/2}x_2^{1/2}(x_1+x_2)+a^{7/4}\tau^{3/2}x_1^{1/4}x_2^{1/4}(x_1^{3/2}+x_2^{3/2})\\+a^{3/2}(x_1+x_2)(x_1^{1/2}x_2^{1/2}\tau+1)\big)a^\varepsilon.$$

Proof. The desired formula is obtained by substituting the result of Corollary 2 into the equation (13), then applying Lemma 8 to the result, and finally using Remark 3.

7. An integral representation for the density

To find the density $p(\tau)$, we differentiate the triple integral defining $\Phi(\tau)$ with respect to the parameter τ.

Lemma 10. *The density $p(\tau)=\frac{2}{\zeta(2)}\Phi'(\tau)$ can be represented in the form*

$$p(\tau)=\frac{2}{\zeta(2)}\int_0^\infty\int_0^\infty\Big[\alpha+\beta>\tau,\ (\alpha+\beta-\tau)^2\leqslant\alpha\beta-1,\\ \frac{\alpha\beta-1}{\alpha+\beta-\tau}\leqslant\min\{\alpha,\beta\}\Big]\frac{d\alpha\,d\beta}{\alpha+\beta-\tau}.$$

Proof. Note first that for $r>\tau$, the inner double integral in the equation

$$\Phi(\tau)=\int_0^\tau\frac{dr}{r}\int_r^\infty\int_r^\infty\Big[1\leqslant\alpha\beta\leqslant 1+r^2,\ \alpha+\beta-\tau\leqslant\frac{\alpha\beta-1}{r}\Big]d\alpha\,d\beta$$

vanishes. We introduce a small parameter $\Delta>0$ and consider the difference

$$\Phi(\tau+\Delta)-\Phi(\tau)=\int_0^\infty\frac{dr}{r}\int_r^\infty\int_r^\infty\Big[1\leqslant\alpha\beta\leqslant 1+r^2,\\ \alpha+\beta-\tau-\Delta\leqslant\frac{\alpha\beta-1}{r}<\alpha+\beta-\tau\Big]d\alpha\,d\beta\\ =\Phi_1(\Delta,\tau)+\Phi_2(\Delta,\tau),$$

where

$$\Phi_1(\Delta,\tau)=\int_0^\infty\frac{dr}{r}\int_r^\infty\int_r^\infty\biggl[1\leqslant\alpha\beta\leqslant 1+r^2,$$
$$\alpha+\beta-\tau-\Delta\leqslant 0<\alpha+\beta-\tau,\ \frac{\alpha\beta-1}{r}<\alpha+\beta-\tau\biggr]\,d\alpha\,d\beta,$$

$$\Phi_2(\Delta,\tau)=\int_0^\infty\frac{dr}{r}\int_r^\infty\int_r^\infty\biggl[1\leqslant\alpha\beta\leqslant 1+r^2,\ \alpha+\beta-\tau-\Delta>0,$$
$$\frac{\alpha\beta-1}{\alpha+\beta-\tau}<r\leqslant\frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\biggr]\,d\alpha\,d\beta.$$

In the first case, one may assume that $\tau\geqslant 2$ because otherwise the integral $\Phi_1(\Delta,\tau)$ vanishes for $\Delta<2-\tau$. It follows from the conditions

$$\alpha+\beta=\tau+O(\Delta),\qquad \alpha\beta=1+O(\Delta\tau)$$

that

$$\alpha=\frac{\tau\pm\sqrt{\tau^2-4}}{2}+O(\Delta),\qquad \beta=\frac{\tau\pm\sqrt{\tau^2-4}}{2}+O(\Delta).$$

Therefore, the area of the domain where α and β take their values does not exceed $O(\Delta^2)$. On the other hand, for a fixed $\alpha\geqslant 1$ (if $\alpha<1$, then it follows from the inequality $\alpha\beta\geqslant 1$ that $\beta>1$, and one can reason in a similar way), the variable β takes its values in an interval whose length does not exceed

$$\frac{1+r^2}{\alpha}-\frac{1}{\alpha}=\frac{r^2}{\alpha}\leqslant r^2.$$

Therefore, the area of the domain where the parameters α and β take their values can also be estimated as $O(r^2\Delta)$. Hence,

$$\Phi_1(\Delta,\tau)\ll\int_0^{\Delta^{1/3}}\frac{r^2\Delta}{r}\,dr+\int_{\Delta^{1/3}}^{\tau}\frac{\Delta^2}{r}\,dr\ll\Delta^{5/3}.$$

Therefore, $\Phi_1(\Delta,\tau)/\Delta\to 0$ as $\Delta\to 0$ and

$$p(\tau)=\lim_{\Delta\to 0}\frac{\Phi(\tau+\Delta)-\Phi(\tau)}{\Delta}=\lim_{\Delta\to 0}\frac{\Phi_2(\Delta,\tau)}{\Delta}.$$

Let the integration with respect to the variable r be the innermost in the integral $\Phi_2(\Delta,\tau)$,

$$\Phi_2(\Delta,\tau)=\iint_{\Omega(\Delta)}d\alpha\,d\beta\int_0^\infty\biggl[\sqrt{\alpha\beta-1}\leqslant r\leqslant\min\{\alpha,\beta\},$$
$$\frac{\alpha\beta-1}{\alpha+\beta-\tau}<r\leqslant\frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\biggr]\frac{dr}{r}.$$

Here and below,

$$\Omega(\Delta)=\bigl\{(\alpha,\beta)\colon \alpha,\beta>0,\ \alpha\beta\geqslant 1,\ \alpha+\beta>\tau+\Delta\bigr\}.$$

The case in which the closed interval $\left[\frac{\alpha\beta-1}{\alpha+\beta-\tau}, \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\right]$ contains the point $\sqrt{\alpha\beta-1}$ makes a small contribution to $\Phi_2(\Delta,\tau)$. Indeed, for a fixed value $t = \alpha+\beta-\tau > \Delta$, the variables α and β satisfy the inequalities $t-\Delta \leqslant \sqrt{\alpha\beta-1} \leqslant t$. Therefore, the variables take their values in intervals whose total length is $O(\sqrt{\Delta})$, and we have the bounds

$$\iint_{\Omega(\Delta)} d\alpha\, d\beta \int_0^\infty \left[\frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \sqrt{\alpha\beta-1} \leqslant r \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\right]\frac{dr}{r}$$
$$\ll \iint_{\Omega(\Delta)} \left[\frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \sqrt{\alpha\beta-1} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\right]\frac{d\alpha\, d\beta}{\alpha+\beta-\tau}$$
$$\ll \Delta \int_\Delta^\tau \frac{\sqrt{\Delta}}{t}dt \ll \Delta^{3/2-\varepsilon}.$$

The case in which the closed interval $\left[\frac{\alpha\beta-1}{\alpha+\beta-\tau}, \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\right]$ contains the point $\min\{\alpha,\beta\}$ also makes a small contribution to $\Phi_2(\Delta,\tau)$. To obtain the corresponding bounds, we note that for $\tau < 2$ and $\Delta < 2-\tau$ it follows from the inequalities $\alpha\beta \geqslant 1$ and $\min\{\alpha,\beta\} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}$ that this contribution vanishes. When $\tau \geqslant 2$, we may assume by symmetry that $\alpha \leqslant \beta$. Then for a fixed β we obtain

$$\int_0^\infty \Bigl[\alpha+\beta > \tau-\Delta,\ \sqrt{\alpha\beta-1} \leqslant \min\{\alpha,\beta\},$$
$$\frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\Bigr]d\alpha$$
$$\times \int_0^\infty \left[\frac{\alpha\beta-1}{\alpha+\beta-\tau} < r \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\right]\frac{dr}{r}$$
$$\ll \int_0^\infty \Bigl[\alpha+\beta > \tau-\Delta,\ \sqrt{\alpha\beta-1} \leqslant \min\{\alpha,\beta\},$$
$$\frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\Bigr]\frac{\Delta}{\alpha+\beta-\tau}\,d\alpha$$
$$\ll -\Delta\log\Delta.$$

Moreover, for $\alpha \leqslant \beta$ it follows from the inequalities $\frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}$ that β changes within the limits

$$\beta_1(\Delta) \leqslant \beta \leqslant \beta_1, \qquad \beta_2 \leqslant \beta \leqslant \beta_2(\Delta),$$

where

$$\beta_{1,2} = \frac{\tau \mp \sqrt{\tau^2-4}}{2}, \qquad \beta_{1,2}(\Delta) = \frac{\tau+\Delta \mp \sqrt{(\tau+\Delta)^2-4}}{2},$$
$$\beta_1 - \beta_1(\Delta) \ll \sqrt{\Delta}, \qquad \beta_2(\Delta) - \beta_2 \ll \sqrt{\Delta}.$$

Therefore, in this case the contribution can also be estimated as $O(\Delta^{3/2-\varepsilon})$. Hence,

$$\begin{aligned}\Phi_2(\Delta,\tau) = \iint_{\Omega(\Delta)} \left[\sqrt{\alpha\beta-1} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\}\right] d\alpha\, d\beta \\ \times \int_0^\infty \left[\frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant r < \frac{\alpha\beta-1}{\alpha+\beta-\tau-\Delta}\right]\frac{dr}{r} + O(\Delta^{3/2-\varepsilon}) \\ = \iint_{\Omega(\Delta)} \left[\sqrt{\alpha\beta-1} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\}\right] \\ \times \left(\frac{\Delta}{\alpha+\beta-\tau} + O\left(\left(\frac{\Delta}{\alpha+\beta-\tau}\right)^2\right)\right) d\alpha\, d\beta + O(\Delta^{3/2-\varepsilon}).\end{aligned}$$

Here for a fixed $t = \alpha+\beta-\tau > \Delta$, the variables α and β take their values in intervals whose length is $O(t)$. Therefore,

$$\begin{aligned}\iint_{\Omega(\Delta)} \left[\sqrt{\alpha\beta-1} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\}\right]\left(\frac{\Delta}{\alpha+\beta-\tau}\right)^2 d\alpha\, d\beta \\ \ll \int_\Delta^\tau \frac{\Delta^2}{t}\, dt \ll \Delta^{2-\varepsilon}.\end{aligned}$$

Thus,

$$\frac{\Phi_2(\Delta,\tau)}{\Delta} = \iint_{\Omega(\Delta)} \left[\sqrt{\alpha\beta-1} \leqslant \frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\}\right] \frac{d\alpha\, d\beta}{\alpha+\beta-\tau} + O(\Delta^{1/2-\varepsilon}).$$

Passage to the limit as $\Delta \to 0$ completes the proof of the lemma.

8. Evaluation of the density

To find the explicit form of the density $p(\tau)$, we draw the curves involved in the integral representation

$$\begin{aligned}p(\tau) = \frac{2}{\zeta(2)} \int_0^\infty\int_0^\infty \Big[\alpha+\beta>\tau,\ (\alpha+\beta-\tau)^2 \leqslant \alpha\beta-1, \\ \frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\}\Big] \frac{d\alpha\, d\beta}{\alpha+\beta-\tau}.\end{aligned}$$

For $\sqrt{3} < \tau < 2$, the line $\alpha+\beta=\tau$ and the ellipse $(\alpha+\beta-\tau)^2 = \alpha\beta-1$ are represented in Fig. 1 by continuous arcs, and the hyperbola $\alpha\beta=1$ by a dotted arc. For $\tau > 2$, we add to these curves the horizontal lines $\beta = \beta_1$, β_2, τ and the vertical lines $\alpha = \alpha_1$, α_2, τ in Fig. 2, where

$$\alpha_1 = \beta_1 = \frac{\tau - \sqrt{\tau^2-4}}{2}, \qquad \alpha_2 = \beta_2 = \frac{\tau+\sqrt{\tau^2-4}}{2}.$$

Lemma 11. *The density $p(\tau)$ satisfies the equations*

$$p(\tau) = \begin{cases} 0 & \text{if } \tau \in [0, \sqrt{3}], \\ \dfrac{12}{\pi}\left(\dfrac{\tau}{\sqrt{3}} - \sqrt{4-\tau^2}\right) & \text{if } \tau \in [\sqrt{3}, 2], \\ \dfrac{12}{\pi^2}\left(\tau\sqrt{3}\arccos\dfrac{\tau + 3\sqrt{\tau^2-4}}{4\sqrt{\tau^2-3}} + \dfrac{3}{2}\sqrt{\tau^2-4}\,\log\dfrac{\tau^2-4}{\tau^2-3}\right) & \text{if } \tau \in [2, +\infty). \end{cases}$$

Proof. As noted above, the domain of integration is empty and $p(\tau) = 0$ for $\tau < \sqrt{3}$.

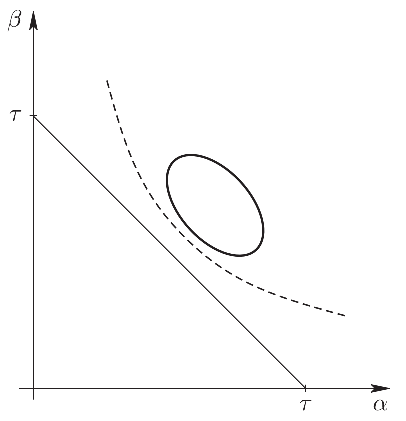

Figure 1

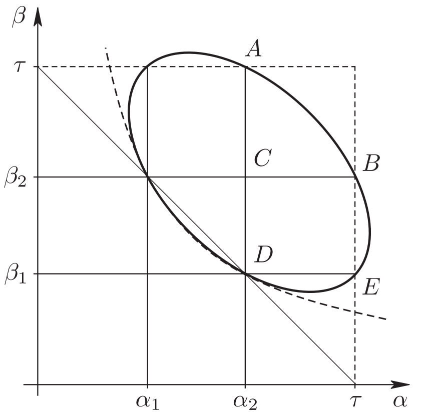

Figure 2

Consider the case when $\sqrt{3} < \tau < 2$. If $\alpha + \beta > \tau$, then $\frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha, \beta\}$. Therefore, the domain where the variables take their values is given by the condition $(\alpha+\beta-\tau)^2 \leqslant \alpha\beta - 1$ (the inequality $\alpha + \beta > \tau$ follows from the inequality $\alpha\beta \geqslant 1$). Thus, the integration is carried out over the interior of the ellipse $(\alpha + \beta - \tau)^2 = \alpha\beta - 1$ (see Fig. 1). We introduce the variable $x = \alpha + \beta - \tau$. Then

$$\begin{aligned} \frac{\zeta(2)}{2} p(\tau) &= \int_0^\infty \int_0^\infty \left[(\alpha + \beta - \tau)^2 \leqslant \alpha\beta - 1\right] \frac{d\alpha\, d\beta}{\alpha + \beta - \tau} \\ &= \int_0^\infty \frac{dx}{x} \int_0^\infty \left[x^2 + 1 \leqslant \alpha(x + \tau - \alpha)\right] d\alpha \\ &= \int_0^\infty \left[(x+\tau)^2 - 4(x^2+1) \geqslant 0\right] \frac{\sqrt{(x+\tau)^2 - 4(x^2+1)}}{x} dx \\ &= \int_{x_1}^{x_2} \frac{\sqrt{(x+\tau)^2 - 4(x^2+1)}}{x} dx, \end{aligned}$$

where

$$x_{1,2} = \frac{\tau \mp \sqrt{4\tau^2 - 12}}{3}.$$

The change of variable $y = x - \frac{\tau}{3}$ leads to the integral ($b = \frac{\tau}{3}$, $a^2 = \frac{4\tau^2-12}{9}$)

$$\int \frac{\sqrt{a^2-y^2}}{y+b}\,dy = \sqrt{a^2-y^2} + b\arctan\frac{y}{\sqrt{a^2-y^2}} - \sqrt{b^2-a^2}\arctan\frac{a^2+by}{\sqrt{b^2-a^2}\sqrt{a^2-y^2}},$$

for which

$$\int_{-a}^{a} \frac{\sqrt{a^2-y^2}}{y+b}\,dy = \pi(b - \sqrt{b^2-a^2}).$$

Hence,

$$p(\tau) = \frac{2\pi}{\zeta(2)}\left(\frac{\tau}{\sqrt{3}} - \sqrt{4-\tau^2}\right) = \frac{12}{\pi}\left(\frac{\tau}{\sqrt{3}} - \sqrt{4-\tau^2}\right).$$

Now consider the case when $\tau \geqslant 2$. As above, the inequality $(\alpha+\beta-\tau)^2 \leqslant \alpha\beta-1$ defines the interior of an ellipse. The condition $\frac{\alpha\beta-1}{\alpha+\beta-\tau} \leqslant \min\{\alpha,\beta\}$ is equivalent to the condition $\alpha \in [0,\alpha_1]\cup[\alpha_2,\infty)$, $\beta \in [0,\beta_1]\cup[\beta_2,\infty)$. Therefore, the density $p(\tau)$ can be represented in the form

$$p(\tau) = \frac{2}{\zeta(2)}(J_1 + 2J_2),$$

where J_1 stands for the integral over the domain bounded by the segments AC and BC and an arc of the ellipse AB, and J_2 for the integral over the segment of the ellipse cut out by the line segment DE (see Fig. 2).

The integral J_1 can also be found by the change of variable $x = \alpha+\beta-\tau$:

$$J_1 = \int_{\sqrt{\tau^2-4}}^{\alpha_2} \frac{x-\sqrt{\tau^2-4}}{x}\,dx + \int_{\alpha_2}^{\frac{\tau+2\sqrt{\tau^2-3}}{3}} \frac{\sqrt{(x+\tau)^2-4(x^2+1)}}{x}\,dx.$$

Applying the formula

$$\int \frac{\sqrt{(x+\tau)^2-4(x^2+1)}}{x}\,dx = \sqrt{(x+\tau)^2-4(x^2+1)} - \frac{\tau}{\sqrt{3}}\arcsin\frac{\tau-3x}{2\sqrt{\tau^2-3}} - \sqrt{\tau^2-4}\,\log\frac{2(\tau^2-4)+2x\tau+2\sqrt{\tau^2-4}\sqrt{(x+\tau)^2-4(x^2+1)}}{x}$$

in this case, we obtain

$$J_1 = \frac{\tau}{\sqrt{3}}\arccos\frac{\tau+3\sqrt{\tau^2-4}}{4\sqrt{\tau^2-3}} + \frac{1}{2}\sqrt{\tau^2-4}\,\log\frac{\tau^2-4}{\tau^2-3}.$$

We write the integral J_2 in the form

$$J_2 = \int_{\frac{2}{3}(\tau-\sqrt{\tau^2-3})}^{\beta_1} d\beta \int_{\alpha_1(\beta)}^{\alpha_2(\beta)} \frac{d\alpha}{\alpha+\beta-\tau} = \int_{\frac{2}{3}(\tau-\sqrt{\tau^2-3})}^{\beta_1} \big(\log(\alpha_2(\beta)+\beta-\tau) - \log(\alpha_1(\beta)+\beta-\tau)\big)\,d\beta,$$

where

$$\alpha_{1,2}(\beta) = \frac{2\tau - \beta \mp \sqrt{-3\beta^2 + 4\beta\tau - 4}}{2}.$$

An antiderivative of the integrand can be given explicitly:

$$\int \bigl(\log(\alpha_2(\beta) + \beta - \tau) - \log(\alpha_1(\beta) + \beta - \tau)\bigr)\, d\beta = F_0(\beta),$$

where

$$\begin{aligned} F_0(\beta) = (\beta_1 - \beta)\log(\beta_1 - \beta) + (\beta_2 - \beta)\log(\beta_2 - \beta) + \frac{\tau}{\sqrt{3}} \arcsin \frac{3\beta - 2\tau}{2\sqrt{\tau^2 - 3}} \\ + 2\beta \log \frac{\beta + \sqrt{-3\beta^2 + 4\beta\tau - 4}}{2} \\ - \beta_1 \log\bigl(2\beta_1\tau - 4 + \beta(2\tau - 3\beta_1) + \beta_1\sqrt{-3\beta^2 + 4\beta\tau - 4}\bigr) \\ - \beta_2 \log\bigl(2\beta_2\tau - 4 + \beta(2\tau - 3\beta_2) + \beta_2\sqrt{-3\beta^2 + 4\beta\tau - 4}\bigr). \end{aligned}$$

Applying the Newton–Leibniz formula, we obtain the value of the integral J_2:

$$F_0(\beta_1) = \frac{\sqrt{\tau^2 - 4}}{2} \log(\tau^2 - 4) - \tau \log 2 - \frac{\tau}{\sqrt{3}} \arcsin \frac{\tau + 3\sqrt{\tau^2 - 4}}{4\sqrt{\tau^2 - 3}} - \beta_2 \log(\tau^2 - 3),$$

$$F_0\biggl(\frac{2}{3}\Bigl(\tau - \sqrt{\tau^2 - 3}\Bigr)\biggr) = -\frac{\pi\tau}{2\sqrt{3}} - \tau \log 2 - \frac{\tau}{2} \log(\tau^2 - 3),$$

$$J_2 = J_1 = \frac{\tau}{\sqrt{3}} \arccos \frac{\tau + 3\sqrt{\tau^2 - 4}}{4\sqrt{\tau^2 - 3}} + \frac{1}{2}\sqrt{\tau^2 - 4} \log \frac{\tau^2 - 4}{\tau^2 - 3}.$$

Thus,

$$p(\tau) = \frac{6J_1}{\zeta(2)} = \frac{12}{\pi^2}\biggl(\tau\sqrt{3} \arccos \frac{\tau + 3\sqrt{\tau^2 - 4}}{4\sqrt{\tau^2 - 3}} + \frac{3}{2}\sqrt{\tau^2 - 4} \log \frac{\tau^2 - 4}{\tau^2 - 3}\biggr).$$

This proves the lemma.

Proof of the theorem. Substituting the density $p(t)$ evaluated in Lemma 11 into Corollary 3, we arrive at the statement of the theorem with the remainder term $O_\varepsilon(R(a; x_1, x_2; \tau)a^\varepsilon)$, where

$$\begin{aligned} R(a; x_1, x_2; \tau) = a^{-1/6}\tau^{5/3} x_1^{-1/2} x_2^{-1/2}(x_1 + x_2) \\ + a^{-1/4}\tau^{3/2} x_1^{-3/4} x_2^{-3/4}(x_1^{3/2} + x_2^{3/2}) \\ + a^{-1/2}(x_1 + x_2)(x_1^{-1/2} x_2^{-1/2}\tau + x_1^{-1} x_2^{-1}) \ll_{x_1, x_2, \tau} a^{-1/6}. \end{aligned}$$

This proves the theorem.

9. Properties of the density

The graph of the function $p(t)$ is shown in Fig. 3. We now prove the main properties of the density.

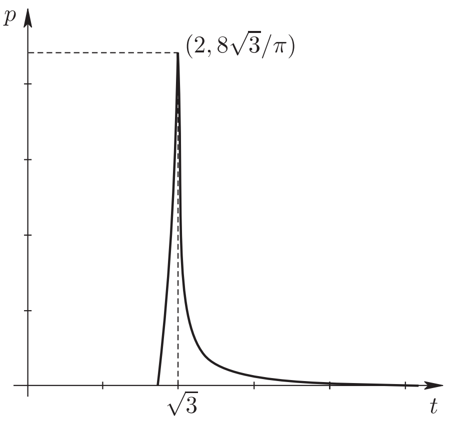

Figure 3

Lemma 12. *The following relations hold for the function* $p(t)$:

1) $p(t)$ *is increasing on the closed interval* $[\sqrt{3}, 2]$ *and decreasing on the half-open interval* $[2, +\infty)$, $\lim_{t\to 2-0} p'(t) = +\infty$, *and* $\lim_{t\to 2+0} p'(t) = -\infty$;

2) $p(t) = \frac{18}{\pi^2 t^3} + O\big(\frac{1}{t^5}\big)$, $t \to \infty$;

3) $\int_0^\infty p(t)\, dt = 1$;

4) $\int_0^\infty tp(t)\, dt = \frac{8}{\pi}$.

Proof. The properties 1) and 2) follow immediately from the formula for $p(t)$ which was proved in Lemma 11.

To prove the property 3), we note that, by Lemma 9,

$$\int_0^\infty p(t)\, dt = \frac{2}{\zeta(2)} \lim_{\tau\to\infty} \Phi(\tau) = \frac{2}{\zeta(2)} \int_0^\infty \frac{dr}{r} \int_r^\infty \int_r^\infty [1 \leqslant \alpha\beta \leqslant 1 + r^2]\, d\alpha\, d\beta.$$

Hence,

$$\begin{aligned}\int_0^\infty p(t)\, dt &= \frac{2}{\zeta(2)} \int_0^1 \left((1 + r^2) \log\left(1 + \frac{1}{r^2}\right) - \log\frac{1}{r^2} - r^2 \right) \frac{dr}{r} \\ &\quad + \frac{2}{\zeta(2)} \int_1^\infty \left((1 + r^2) \log\left(1 + \frac{1}{r^2}\right) - 1 \right) \frac{dr}{r} \\ &= \frac{2}{\zeta(2)} \left(\left(-\frac{1}{2} + \frac{\zeta(2)}{4} + \log 2 \right) + \left(\frac{1}{2} + \frac{\zeta(2)}{4} - \log 2 \right) \right) = 1.\end{aligned}$$

An antiderivative for the function $tp(t)$ can be found explicitly:

$$F_1(t) = \int \pi t\left(\frac{t}{\sqrt{3}} - \sqrt{4-t^2}\right)dt = \frac{\pi}{3}\left(\frac{t^3}{\sqrt{3}} + (4-t^2)^{3/2}\right),$$

$$F_2(t) = \int t\left(t\sqrt{3}\arccos\frac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}} + \frac{3}{2}\sqrt{t^2-4}\log\frac{t^2-4}{t^2-3}\right)dt$$

$$= \frac{t^3}{\sqrt{3}}\arccos\frac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}} + 4\arctan\sqrt{t^2-4} + \frac{(t^2-4)^{3/2}}{2}\log\frac{t^2-4}{t^2-3}.$$

Here

$$F_1(\sqrt{3}) = \frac{4\pi}{3}, \qquad F_1(2) = \frac{8\pi}{3\sqrt{3}}, \qquad F_2(2) = \frac{8\pi}{3\sqrt{3}}, \qquad F_2(\infty) = 2\pi.$$

Hence,

$$\int_0^\infty tp(t)\,dt = \frac{2}{\zeta(2)}\left(2\pi - \frac{4\pi}{3}\right) = \frac{8}{\pi}.$$

This proves the lemma.

Corollary 4. *It follows from the properties of the function $p(t)$ that for almost all triples of numbers $(a,b,c) \in M_a(x_1,x_2)$, the Frobenius numbers $f(a,b,c)$ are of the order of $\sqrt{abc}$. In the simplest case, when $x_1 = x_2 = 1$ and $a > \tau^{34+\varepsilon}$, we have*

$$\frac{1}{|M_a(x_1,x_2)|}\sum_{(a,b,c)\in M_a(x_1,x_2)}[f(a,b,c) > \tau\sqrt{abc}\,] = \frac{9}{\pi^2\tau^2} + O\left(\frac{1}{\tau^4}\right).$$

10. Concluding remarks

The case of the function $n(a,b,c)$ which (for $(a,b,c)=1$) is equal to the number of positive integers m that are not representable in the form

$$m = ax + by + cz, \qquad x,y,z \geqslant 0,$$

is of special interest. This function, together with the Frobenius function, describes the work of two-contour networks (where $g(a,b,c)$ corresponds to the maximal diameter of the network and $n(a,b,c)$ to the average diameter; see [20] and [21]). The modified quantity

$$N(a,b,c) = n(a,b,c) + \frac{a+b+c-1}{2},$$

like $f(a,b,c)$, satisfies for $d \mid (a,b)$ the equation (see [20], Lemma 1)

$$N(a,b,c) = dN\left(\frac{a}{d},\frac{b}{d},c\right).$$

Rödseth proved the formula (see [20], Theorem 2)

$$2N(a,b,c) = bs_{n-1} + cq_n - \frac{s_nq_{n-1}}{a}\big(b(s_{n-1}-s_n) + c(q_n-q_{n-1})\big), \qquad (18)$$

which holds (like (6)) for $(a,b)=(a,c)=(b,c)=1$ and $s_n/q_n \leqslant c/b \leqslant s_{n-1}/q_{n-1}$. The approach suggested in the present paper enables us to describe the distribution of the values $N(a,b,c)$ and other similar functions. It follows from the proof of Lemma 9 that the probability density

$$\frac{1}{\zeta(2)\max\{r,\frac{\alpha\beta-1}{r}\}}\biggl[\max\biggl\{r,\frac{\alpha\beta-1}{r}\biggr\}\leqslant\min\{\alpha,\beta\},\ \alpha\beta\geqslant 1\biggr]$$

describes the joint distribution of the triples (α,β,r), where

$$\alpha=\frac{u_1}{\sqrt{a/\xi}}=\frac{q_n}{\sqrt{a/\xi}},\qquad \beta=\frac{u_2}{\sqrt{a\xi}}=\frac{s_{n-1}}{\sqrt{a\xi}},\qquad r=\frac{v_2}{\sqrt{a\xi}}=\frac{s_n}{\sqrt{a\xi}}.$$

The same density describes the distribution of L-shape diagrams arising in the study of two-contour networks (see [20]). For $r>\frac{\alpha\beta-1}{r}$ this density (after adding to the symmetric part) becomes the density

$$p(\alpha,\beta,r)=\frac{2}{\zeta(2)r}\bigl[r\leqslant\min\{\alpha,\beta\},\ 1\leqslant\alpha\beta\leqslant 1+r^2\bigr],$$

which occurs implicitly in the proof of Lemma 10. Using this density, we can write down the distribution function for the quantities depending on the quadruples $(q_n,s_{n-1},q_{n-1},s_n)=(u_1,u_2,v_1,v_2)$ and on an additional parameter ξ which satisfy the relation

$$f(u_1,u_2,v_1,v_2;\xi)=f(u_2,u_1,v_2,v_1;1/\xi).$$

For example, for normalized Frobenius numbers we have

$$f(u_1,u_2,v_1,v_2;\xi)=\frac{1}{\sqrt{a\xi}}\bigl(u_2+\xi u_1-\min\{v_2,\xi v_1\}\bigr).$$

By (18),

$$f(u_1,u_2,v_1,v_2;\xi)=\frac{1}{\sqrt{a\xi}}\bigl(u_2+\xi u_1-v_1v_2(u_2-v_2+\xi(u_1-v_1))a^{-1}\bigr)$$

corresponds to the numbers $2N(a,b,c)/\sqrt{abc}$, and this leads to the distribution function

$$\begin{aligned}\widetilde{\Phi}(\tau)&=\frac{2}{\zeta(2)}\int_0^\infty\int_0^\infty\int_0^\infty\biggl[\alpha+\beta-(\alpha\beta-1)\biggl(\alpha+\beta-r-\frac{\alpha\beta-1}{r}\biggr)\leqslant\frac{\tau}{2}\biggr]\\&\qquad\times p(\alpha,\beta,r)\,d\alpha\,d\beta\,dr\\&=\frac{2}{\zeta(2)}\int_0^\infty\frac{dr}{r}\int_r^\infty\int_r^\infty\biggl[1\leqslant\alpha\beta\leqslant 1+r^2,\\&\qquad\alpha+\beta-\frac{\alpha\beta-1}{r}+\frac{(\alpha-r)(\beta-r)(\alpha\beta-1)}{r}\leqslant\frac{\tau}{2}\biggr]d\alpha\,d\beta.\end{aligned}$$

We can write down the distribution functions for the ratio $N(a,b,c)/f(a,b,c)$ in a similar form (thus implicitly answering Arnold's question on the expectation of this ratio; see [8], Problem 1999–9) and for the Frobenius pseudo-numbers with three arguments (see [22]).

Bibliography

[1] J. J. Sylvester, "Problem 7382", *Educational Times* **37** (1884), p. 26; "Mathematical questions with their solution", *Educational Times* (with additional papers and solutions) **41** (1884), p. 21.

[2] J. L. Ramírez Alfonsin, *The Diophantine Frobenius problem*, Oxford Lecture Ser. Math. Appl., vol. 30, Oxford Univ. Press, Oxford 2005.

[3] Ö. J. Rödseth, "On a linear Diophantine problem of Frobenius", *J. Reine Angew. Math.* **301** (1978), 171–178.

[4] E. S. Selmer and Ö. Beyer, "On the linear Diophantine problem of Frobenius in three variables", *J. Reine Angew. Math.* **301** (1978), 161–170.

[5] R. Kannan, "Lattice translates of a polytope and the Frobenius problem", *Combinatorica* **12**:2 (1992), 161–177.

[6] R. M. Karp, "Reducibility among combinatorial problems", *Complexity of computer computations* (Yorktown Heights, NY 1972), Plenum, New York 1972, pp. 85–103; Russian transl., *Kiberneticheskii Sbornik* (*N.S.*), 1975, 16–38.

[7] J. L. Davison, "On the linear Diophantine problem of Frobenius", *J. Number Theory* **48**:3 (1994), 353–363.

[8] V. I. Arnold, *Arnold's problems*, Fazis, Moscow 2000; English transl., Springer-Verlag, Berlin 2004.

[9] J. Bourgain and Ya. G. Sinai, "Limit behaviour of large Frobenius numbers", *Uspekhi Mat. Nauk* **62**:4 (2007), 77–90; English transl., *Russian Math. Surveys* **62**:4 (2007), 713–725.

[10] A. V. Ustinov, "On the statistical properties of elements of continued fractions", *Dokl. Akad. Nauk* **424**:4 (2009), 459–461; English transl., *Dokl. Math.* **79**:1 (2009), 87–89.

[11] V. Shchur, Ya. Sinai, and A. Ustinov, "Limiting distribution of Frobenius numbers for $n=3$", *J. Number Theory* **129**:11 (2009), 2778–2789.

[12] I. Aliev and M. Henk, "Integer knapsacks: average behavior of the Frobenius numbers", *Math. Oper. Res.* **34**:3 (2009), 698–705.

[13] I. Aliev, M. Henk, and A. Hinrichs, "Expected Frobenius numbers", *J. Combin. Theory Ser. A* (to appear), arXiv: abs/0910.2620.

[14] J. Marklof, "The asymptotic distribution of Frobenius numbers", *Invent. Math.* **181**:1 (2010), 179–207.

[15] A. V. Ustinov, "The solution of Arnold's problem on the weak asymptotics of Frobenius numbers with three arguments", *Mat. Sb.* **200**:4 (2009), 131–160; English transl., *Sb. Math.* **200**:4 (2009), 597–627.

[16] V. A. Bykovskii, "Asymptotic properties of integral points (a_1, a_2), satisfying the congruence $a_1 a_2 \equiv l \pmod q$", *Analytic number theory and function theory.* 4, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), vol. 112, Nauka, Leningrad 1981, pp. 5–25; English transl., *J. Soviet Math.* **25**:2 (1984), 975–988.

[17] A. V. Ustinov, "On the number of solutions of the congruence $xy \equiv l (\operatorname{mod} q)$ under the graph of a twice continuously differentiable function", *Algebra i Analiz* **20**:5 (2008), 186–216; English transl., *St. Petersburg Math. J.* **20**:5 (2009), 597–627.

[18] T. Estermann, "On Kloosterman's sum", *Mathematika* **8** (1961), 83–86.

[19] S. M. Johnson, "A linear diophantine problem", *Canad. J. Math.* **12** (1960), 390–398.

[20] Ö. J. Rödseth, "Weighted multi-connected loop networks", *Discrete Math.* **148**:1–3 (1996), 161–173.

[21] C. K. Wong and D. Coppersmith, "A combinatorial problem related to multimodule memory organizations", *J. Assoc. Comput. Mach.* **21**:3 (1974), 392–402.

[22] J. C. Rosales and P. A. García-Sáanchez, "Numerical semigroups with embedding dimension three", *Arch. Math.* (*Basel*) **83**:6 (2004), 488–496.

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