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Geometric Proof of Rødseth's Formula for Frobenius Numbers

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Dedicated to the 75th birthday of A.A. Karatsuba

Abstract—Using a geometric interpretation of continued fractions, we give a new proof of Rødseth's formula for Frobenius numbers.

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1. INTRODUCTION

Let $a_1,\dots,a_n$ be jointly coprime positive integers (the greatest common divisor of all the numbers is 1). The *Frobenius number* $g(a_1,\dots,a_n)$ is the greatest integer m not representable as

$$x_1a_1+\ldots+x_na_n=m, \tag{1}$$

where $x_1,\dots,x_n$ are nonnegative integers. The *modified Frobenius number*

$$f(a_1,\dots,a_n)=g(a_1,\dots,a_n)+a_1+\ldots+a_n$$

is the greatest integer m that cannot be represented in the form (1) with positive integer coefficients $x_1,\dots,x_n$. The problem of finding $g(a_1,\dots,a_n)$ (or $f(a_1,\dots,a_n)$) is called the *Frobenius problem.*

For $n=2$, there is a well-known formula attributed to Sylvester (see [37]; background information can be found in [31]): $f(a,b)=ab$. If $n=3$, then $f(a,b,c)$ is expressible in terms of continued fractions (see the results of Selmer, Beyer, and Rødseth [32, 35]). There also exist other approaches to finding the Frobenius numbers with three arguments (see [31, Ch. 2] and later results in [20, 21]); however, from an analytic point of view, Rødseth's formula (see below) is the most convenient. It allows one to apply the technique used for analyzing the statistical properties of finite continued fractions (see, for example, [6]) to the study of Frobenius numbers. In particular, Rødseth's formula has helped to solve Arnold's problem on the weak asymptotics of Frobenius numbers (i.e., the asymptotics in the mean; see [13, problems 1999-8, 2003-5] and [1]): $f(a,b,c)\sim\frac{8}{\pi}\sqrt{abc}$ (see [7, 9]); to obtain, as a corollary, a proof of Davison's conjecture from [17]: the mean value of the normalized Frobenius numbers $\frac{f(a,b,c)}{\sqrt{abc}}$ is $8/\pi$; and to find the density for the distribution of normalized Frobenius numbers (see [8]):

$$p(t)=\begin{cases}0 & \text{if } t\in[0,\sqrt{3}],\\ \dfrac{12}{\pi}\left(\dfrac{t}{\sqrt{3}}-\sqrt{4-t^2}\right) & \text{if } t\in[\sqrt{3},2],\\ \dfrac{12}{\pi^2}\left(t\sqrt{3}\arccos\dfrac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}+\dfrac{3}{2}\sqrt{t^2-4}\log\dfrac{t^2-4}{t^2-3}\right) & \text{if } t\in[2,+\infty).\end{cases}$$

The existence of this density was established by Bourgain and Sinai in [2] (see also [36]).

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For $n \geq 4$, formulas for finding $f(a_1, \dots, a_n)$ are known only in some particular cases; probably, the most general result is that concerning chain sequences [4, 5]. It was proved that for fixed n the Frobenius number can be calculated in polynomial time (see [26]), whereas for arbitrary n the determination of $f(a_1, \dots, a_n)$ becomes an NP-hard problem (see [27]).

Recently, a number of results on the statistical properties of the Frobenius numbers with an arbitrary number of arguments have been obtained by the methods of the geometry of numbers (see [11, 12]) and the methods of ergodic theory (see [28, 29]); relevant experimental data can be found in [3] and [29]. Therefore, the geometric interpretation proposed in the present study for Rødseth's formula seems to be of interest. One may hope that this interpretation will also help understand the problem in higher dimensions, when the situation becomes much more complicated (see [19]). Note that in [14] Arnold suggested using the geometric interpretation of continued fractions for studying the Frobenius numbers; however, he did not obtain explicit formulas.

The proof of Rødseth's formula presented below is based on a few simple propositions (see the properties of L-shaped diagrams and Lemmas 1 and 2 below) that are well known. Along with short proofs of these propositions (which are not original and are presented only for the convenience of the reader), we give references to publications in which these propositions appeared independently. The references should illustrate the relationship between different problems and complement somewhat the historical remarks from [31].

2. DOUBLE-LOOP NETWORKS

When determining the Frobenius number $f(a, b, c)$, one can get rid of common divisors of the arguments by the Johnson formula from [25],

$$f(da, db, c) = df(a, b, c). \tag{2}$$

Therefore, below we will assume that $(a, b) = (a, c) = (b, c) = 1$.

Given a triple (a, b, c), we introduce a *double-loop network* which is an oriented graph on a vertices $0, 1, 2, \dots, a-1$ with edges of two types, $j \to j + b \pmod a$ and $j \to j + c \pmod a$, with weights (lengths) w_b and w_c, respectively. To solve the Frobenius problem, one should choose $w_b = b$ and $w_c = c$ (it takes time b to traverse an edge of the first type and time c to traverse an edge of the second type). To each route that starts at the zero vertex and passes x edges of length b and y edges of length c in time $t(x, y) = bx + cy$, we will assign a cell $K(x, y)$ (the coordinates of the cell are the coordinates of its lower left corner) and a number $n(x, y) = t(x, y) \pmod a$, the number of the final vertex of the route.

To describe the main parameters of a double-loop network (such as the diameter, the mean distance between vertices, the length of the shortest cycle, ...), one needs the description of the shortest paths between vertices. Due to the obvious symmetry, it suffices to restrict the analysis to paths that start at the zero vertex. A full description is given by an L-shaped diagram (it has the form of a rectangle with a cut-out upper right corner; see Figs. 2–4 below) which is constructed as follows.

The time instants $t(x, y)$ $(x, y \geq 0)$ are arranged in increasing order, $0 = t_0 < t_1 < \ldots < t_j < \ldots$, and, for each $t_j = t(x, y)$, provided that the number $n(x, y)$ is missing in the diagram, a cell $K(x, y)$ with number $n(x, y)$ is added to the diagram; *if several cells can be added simultaneously, then one adds the cell with the least ordinate.*

In other words, if a number $n = n(x, y)$ is written in a cell $K(x, y)$, then the shortest path $0 \to n$ passes through x edges of length b and y edges of length c. Since $(a, b) = (a, c) = 1$, each vertex of the graph is reachable and the diagram (henceforth denoted by $\mathcal{L}$) consists of exactly a cells.

Figure 1 shows a double-loop network constructed for the numbers $a = 7$, $b = 3$, and $c = 5$. The solid arrows are of length $b = 3$, and the dashed arrows are of length $c = 5$. Figure 2 shows the corresponding diagram $\mathcal{L}$ (bottom) and a diagram with the times of the shortest paths (top).

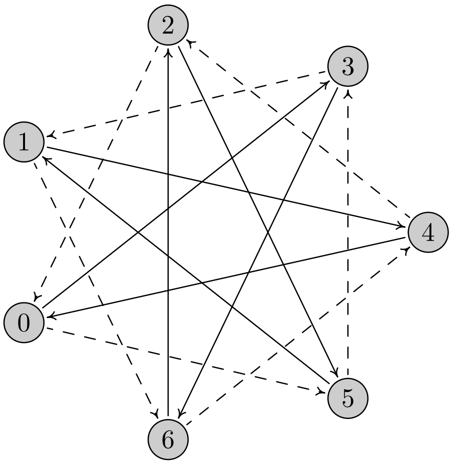

Fig. 1.

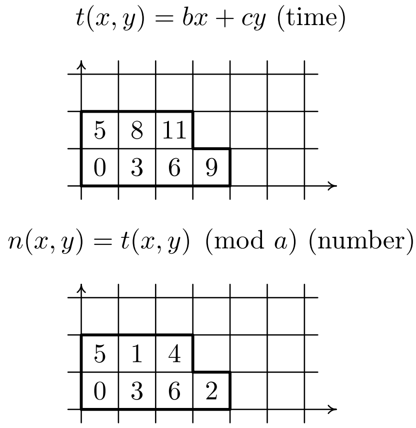

Fig. 2.

Double-loop networks appeared in [38] in connection with the problem of constructing multi-module memory structures and has been extensively studied (see surveys [15, 23, 24]). However, still earlier, L-shaped diagrams occurred in relation to the Frobenius problem in [16] and has been repeatedly used by different authors (see [10, 14, 19–21, 30, 32, 33, 35]).

Let us list the simplest properties of the arising diagrams:

1. If a cell $K(x, y)$ does not belong to the diagram $\mathcal{L}$, then all the other cells in the corner $\bigcup_{u,v\ge 0} K(x+u, y+v)$ do not belong to $\mathcal{L}$ either.
2. If $K(x, y) \notin \mathcal{L}$ but $K(x-1, y) \in \mathcal{L}$, then the number $n(x, y)$ is encountered in the first column of the diagram $\mathcal{L}$.
3. If $K(x, y) \notin \mathcal{L}$ but $K(x, y-1) \in \mathcal{L}$, then the number $n(x, y)$ is encountered in the first row of the diagram $\mathcal{L}$.
4. If $K(x, y) \notin \mathcal{L}$ but $K(x-1, y) \in \mathcal{L}$ and $K(x, y-1) \in \mathcal{L}$, then $n(x, y) = 0$.

These properties follow from the diagram compilation rule.

Property 1 is equivalent to the fact that if $K(x, y) \in \mathcal{L}$, then the cells $K(x-u, y-v)$ lie in $\mathcal{L}$ for all u and v with $0 \le u \le x$ and $0 \le v \le y$.

To check property 2, assume that the number $n(x, y)$ is encountered in a cell $K(x', y') \in \mathcal{L}$ that is not in the first column. Then $K(x-1, y), K(x'-1, y') \in \mathcal{L}$ and $n(x-1, y) = n(x'-1, y')$, which contradicts the compilation rule (the numbers cannot be repeated).

Property 3 is analogous to property 2.

Property 4 follows from properties 2 and 3: the number $n(x, y)$ must be encountered in the first row and the first column, i.e., $n(x, y) = n(0, 0) = 0$.

The properties of the constructed diagram are closely related to the properties of the lattice

$$\Lambda = \bigl\{(x, y) \in \mathbb{Z}^2 \colon\ bx + cy \equiv 0 \pmod a\bigr\}.$$

Lemma 1. *The diagram obtained by applying the above-described rule is L-shaped. The translations of $\mathcal{L}$ by the vectors of the lattice Λ tile the entire plane. The sides of the diagram are uniquely defined by any two vectors in the triple* (*see Fig.* 3) $\overrightarrow{OD}, \overrightarrow{AF}, \overrightarrow{BG}$ ($\overrightarrow{AF} = \overrightarrow{OD} + \overrightarrow{BG}$). *The coordinates of the vectors* $\overrightarrow{OD} = (x_0, y_0)$, $\overrightarrow{AF} = (x_1, y_1)$, *and* $\overrightarrow{BG} = (x_2, y_2)$ *are characterized by the following conditions.*

$\overrightarrow{OD}$: $x_0, y_0 > 0$, $(x_0, y_0) \in \Lambda$, $t(x_0, y_0) = \min_{x,y>0,\ (x,y)\in\Lambda} t(x, y)$; *if the minimum value of the form $t(x, y)$ is attained at several points, then the point with the least ordinate is chosen as* (x_0, y_0).

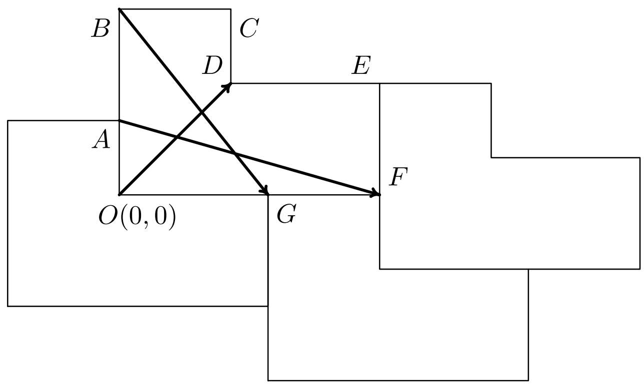

Fig. 3.

$\overrightarrow{AF}$: *x_1 is the least positive integer for which there exists a $y_1 \ge 0$ such that $(x_1, -y_1) \in \Lambda$ and $t(0, y_1) < t(x_1, 0)$.*

$\overrightarrow{BG}$: *y_2 is the least positive integer for which there exists an $x_2 \ge 0$ such that $(x_2, -y_2) \in \Lambda$ and $t(x_2, 0) \le t(0, y_2)$.*

Proof (see [10, 14, 22, 34, 38])**.** According to property 1, the diagram has the form of a "staircase" consisting of rectangular steps. By property 4, there may exist only one cell $K(x, y) \notin \mathcal{L}$ for which $K(x-1, y) \in \mathcal{L}$ and $K(x, y-1) \in \mathcal{L}$. Indeed, if there are two such cells $K(x_1, y_1)$ and $K(x_2, y_2)$, then $n(x_1, y_1) = n(x_2, y_2) = 0$ and, hence, $n(x_1 - 1, y_1) = n(x_2 - 1, y_2)$; but this contradicts the assumption that $K(x_1 - 1, y_1) \in \mathcal{L}$ and $K(x_2 - 1, y_2) \in \mathcal{L}$. Hence, the staircase consists of at most two steps, and the diagram is L-shaped.

Each number from 0 to $a - 1$ appears in the diagram exactly once. Therefore, translating $\mathcal{L}$ by the vectors of the lattice Λ, we obtain a tiling of the entire plane. Since $n(x_1, y_1) = n(x_2, y_2) \Leftrightarrow (x_1 - x_2, y_1 - y_2) \in \Lambda$, the characteristic properties of the vectors $\overrightarrow{AF}$ and $\overrightarrow{BG}$ follow from properties 2 and 3, respectively. The characteristic properties of the vector $\overrightarrow{OD}$ follow from the diagram compilation rule. $\square$

Remark 1. In some cases an L-shaped diagram may degenerate into a rectangle. Since the area of the diagram is a, the degenerate case corresponds to the equality $x_1 y_2 = a$. Hence, multiplying the congruences $bx_1 \equiv cy_1 \pmod{a}$ and $cy_2 \equiv bx_2 \pmod{a}$, we obtain $y_1 x_2 \equiv 0 \pmod{a}$. However, $0 \le y_1 < y_2$ and $0 \le x_2 < x_1$, i.e., $0 \le y_1 x_2 < x_1 y_2 = a$. Thus, $y_1 x_2 = 0$. If $y_1 = 0$, then $x_1 = a$, which is only possible under the condition $b(a-1) \le c$ $(t(a-1, 0) \le t(0, 1))$. If $x_2 = 0$, then $y_2 = a$, which occurs only if $c(a-1) \le b$ $(t(0, a-1) \le t(1, 0))$. These cases are of no interest from the viewpoint of finding the Frobenius number, because in the first case $g(a, b, c) = g(a, b) = ab - a - b$, while in the second $g(a, b, c) = g(a, c) = ac - a - c$.

In order that Lemma 1 remain valid in the degenerate cases as well, one should assume that the point D for a rectangular diagram is defined by the equality $\overrightarrow{OD} = \overrightarrow{AF} - \overrightarrow{BG}$. Then the characteristic property of the vector $\overrightarrow{OD}$ follows from the explicit form of the vectors $\overrightarrow{AF}$ and $\overrightarrow{BG}$.

Lemma 2. *Let $C = (x_C, y_C)$ and $E = (x_E, y_E)$. Then*

$$f(a, b, c) = \max\{t(x_C, y_C), t(x_E, y_E)\}.$$

Proof (see [14, 16, 34])**.** For integer numbers n, $0 \le n \le a-1$, define a function $t(n)$ as the time taken to reach the vertex n: $t(n(x, y)) = t(x, y)$ $(K(x, y) \in \mathcal{L})$. The diameter of the double-loop network can be expressed in terms of the coordinates of the points C and E:

$$D = \max_{0 \le n \le a-1} t(n) = \max_{K(x,y) \in \mathcal{L}} t(x, y) = \max\bigl\{t(x_C - 1, y_C - 1), t(x_E - 1, y_E - 1)\bigr\}.$$

The number $m \equiv n \pmod a$ $(0 \le n < a)$ is representable as $m = bx + cy + az$ $(x, y, z \ge 0)$ if and only if $m \ge t(n)$. Therefore,

$$g(a,b,c) = \max_{0 \le n \le a-1} t(n) - a = D - a = \max\{t(x_C, y_C), t(x_E, y_E)\} - a - b - c,$$

which is equivalent to the assertion of the lemma. □

Remark 2. Lemma 2 remains valid even under the weaker initial assumption that $(a, b, c) = 1$. For $(d, a) = 1$, the two oriented graphs on a vertices with edges of the form $j \to j + b \pmod a$ and $j \to j + c \pmod a$ (in the first graph) and $j \to j + db \pmod a$ and $j \to j + dc \pmod a$ (in the second graph) are isomorphic. If we compare the diameters of the corresponding double-loop networks and apply Lemma 2, then we obtain the Johnson formula (2): $f(a, db, dc) = df(a, b, c)$.

3. RØDSETH'S FORMULA

Let a, b, and c be positive integers, $(a, b) = (a, c) = (b, c) = 1$, and l be a solution to the congruence $bl \equiv c \pmod a$ such that $1 \le l \le a$. Rødseth's formula for $f(a, b, c)$ is based on the expansion of the number a/l in a reduced regular continued fraction

$$\frac{a}{l} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_m}}} \qquad (a_1, \ldots, a_m \ge 2).$$

Define sequences $\{s_j\}$ and $\{q_j\}$ $(0 \le j \le m + 1)$ by the conditions

$$s_{m+1} = 0, \quad s_m = 1, \qquad q_0 = 0, \quad q_1 = 1,$$
$$s_{j-1} = a_j s_j - s_{j+1}, \qquad q_{j+1} = a_j q_j - q_{j-1} \qquad (1 \le j \le m).$$

Then (see [32]) $s_0 = q_{m+1} = a$, $s_1 = l$, $q_m = l^{-1} \pmod a$, the sequence $\{s_j\}$ monotonically decreases, the sequence $\{q_j\}$ monotonically increases, and

$$0 = \frac{s_{m+1}}{q_{m+1}} < \frac{s_m}{q_m} < \ldots < \frac{s_1}{q_1} < \frac{s_0}{q_0} = \infty.$$

Hence, there is a unique number v such that $s_{v+1}/q_{v+1} \le c/b < s_v/q_v$. Rødseth proved that the coordinates of the points C and E, which define the sides of the diagram $\mathcal{L}$ (see Fig. 3), have the form $(s_v - s_{v+1}, q_{v+1})$ and $(s_v, q_{v+1} - q_v)$. Combined with Lemma 2, this allows one to find the Frobenius number $f(a, b, c)$.

Theorem 1 (Rødseth, 1978)**.** *The following formula is valid*:

$$f(a, b, c) = bs_v + cq_{v+1} - \min\{bs_{v+1}, cq_v\}.$$

The sequences $\{s_j\}$ and $\{q_j\}$ (see [7]) have the following geometric interpretation. Consider the convex hulls of nonzero points of the lattice Λ that lie in quadrants I and II. The boundaries of these hulls will be called sails and denoted by Π_+ and Π_-, respectively. The points $P_n = (q_n, s_n)$ $(0 \le n \le m + 1)$ are the points of Λ that lie on Π_-. For any n $(1 \le n \le m + 1)$, the vectors $e_n = (q_n, s_n)$ and $e_{n-1} = (q_{n-1}, s_{n-1})$ form a basis of Λ. The family of vertices of the sails Π_- and Π_+ is described by sequences that are similar to $\{s_j\}$ and $\{q_j\}$ but are constructed by expanding a/l in a classical continued fraction. The vectors connecting the origin with the vertices of Π_+ make up Π_-, and vice versa. In particular, the points $e_n - e_{n-1} = (q_n - q_{n-1}, s_n - s_{n-1})$ are vertices of the sail Π_+.

Figure 4 presents an example for $a = 17$, $b = 9$, and $c = 5$ $(l = 10)$.

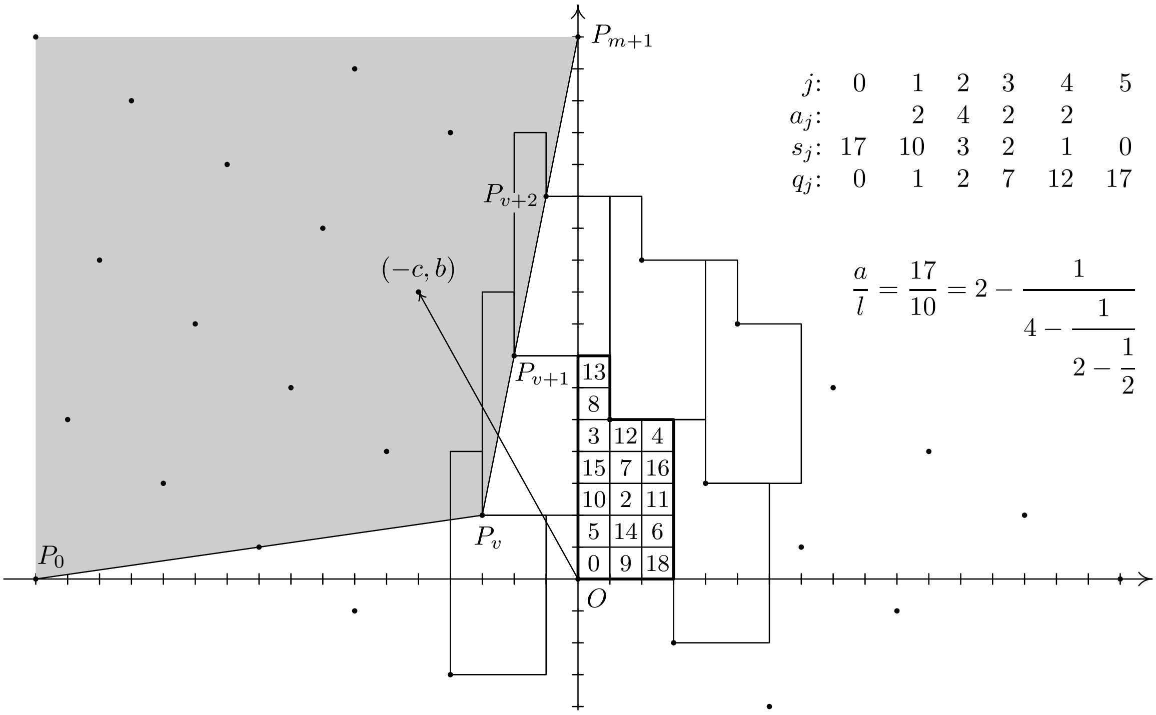

Fig. 4.

4. PROOF OF RØDSETH'S FORMULA

Lemma 3. *Suppose that the point of intersection of the vector $(-c,b)$ with the sail Π_- lies in the half-open interval $(P_v,P_{v+1}]$ $(0\le v\le m)$. Then the vectors $\overrightarrow{OD}$, $\overrightarrow{AF}$, and $\overrightarrow{BG}$, which determine the form of the L-shaped diagram, are given by $\overrightarrow{OD}=\overrightarrow{P_vP_{v+1}}$, $\overrightarrow{AF}=\overrightarrow{P_vO}$, and $\overrightarrow{BG}=\overrightarrow{P_{v+1}O}$; i.e.,*

$$\overrightarrow{OD}=(s_v-s_{v+1},q_{v+1}-q_v),\qquad \overrightarrow{AF}=(s_v,-q_v),\qquad \overrightarrow{BG}=(s_{v+1},-q_{v+1}).$$

Proof. Among the points of the lattice Λ that lie strictly inside the first quadrant, the point $D(x_0,y_0)$ is characterized by the fact that the linear form $t(x,y)=bx+cy$ attains its least possible value at the point D. (If the least value is attained at several points simultaneously, then, according to the rule, the point D is chosen so that its ordinate is as small as possible.) The point D must lie on the sail Π_+; hence, the equality $\overrightarrow{OD}=\overrightarrow{P_jP_{j+1}}$ holds for some j $(0\le j\le m)$. The minimality of the form $t(x,y)$ is equivalent to choosing, among all vectors of the form $\overrightarrow{P_jP_{j+1}}$, a vector with the longest projection to the vector (b,c). This is the vector $\overrightarrow{P_vP_{v+1}}$, because, when moving from bottom to top along the sail Π_+, we arrive at the point D along a vector that is below $(-c,b)$ (in Fig. 4, this is the vector $\overrightarrow{OP_v}$), and leave this point along a vector that is now not below $(-c,b)$ (in Fig. 4, this is the vector $\overrightarrow{OP_{v+2}}$).

The condition $t(0,y_1)<t(x_1,0)$ can be rewritten as $y_1/x_1<b/c$. By Lemma 1, to prove the equality $-\overrightarrow{AF}=\overrightarrow{OP_v}$, one should verify the following assertion: all points of the lattice Λ that are different from P_v and lie above the axis Ox and strictly below the ray $(-c,b)$ lie to the left of P_v. For the points below the ray OP_v, this assertion is obvious, whereas for the points inside the angle P_vOP_{v+1}, it follows from the fact that these points can be represented as $x\overrightarrow{OP_v}+y\overrightarrow{OP_{v+1}}$, where $y\ge 0$ and $x\ge 1$ (the latter inequality follows from the fact that the points lie strictly below the vector $(-c,b)$ and, hence, strictly below the vector $\overrightarrow{OP_{v+1}}$).

Similarly, to prove the equality $-\overrightarrow{BG}=\overrightarrow{OP_{v+1}}$, one should verify the following assertion: all points of the lattice Λ that are different from P_{v+1} and lie to the right of the axis Oy and not below the ray $(-c, b)$ lie above P_{v+1}. For the points to the right of the ray OP_{v+1}, this is obvious, whereas for the points inside the angle P_vOP_{v+1}, the assertion follows by representing these points as $x\overrightarrow{OP_v}+y\overrightarrow{OP_{v+1}}$, where $x \geq 0$ and $y \geq 1$ (the vector $(-c, b)$ is strictly above the vector $\overrightarrow{OP_v}$). □

Proof of Theorem 1. By Lemma 3, the points C and E have coordinates $(s_v - s_{v+1}, q_{v+1})$ and $(s_v, q_{v+1} - q_v)$, respectively. Substituting them into Lemma 2, we obtain the required formula for the Frobenius numbers. □

Remark 3. The formulas in Lemma 3 allow one to describe other Diophantine properties of the triple (a, b, c) (and relevant characteristics of the double-loop network). For example, for $(a, b, c) = 1$, one can find the quantity $N(a, b, c)$ equal to the number of positive integers m that are not representable as $m = ax+by+cz$ $(x, y, z \geq 0)$. The number $N(a, b, c)$ (which could naturally be called a *Sylvester number*, because the problem in [37] was devoted precisely to the determination of $N(a, b)$) is responsible for the mean distance between the vertices of the double-loop network (see [34] and [31, Theorem 5.3.1]). Rødseth proved that the *modified Sylvester number*

$$S(a, b, c) = N(a, b, c) + \frac{a+b+c-1}{2},$$

just as $f(a, b, c)$, satisfies the relation (see [33, Lemma 1])

$$S(da, db, c) = dS(a, b, c);$$

moreover, for $(a, b) = (a, c) = (b, c) = 1$ (see [33, Theorem 2]),

$$2S(a, b, c) = bs_v + cq_{v+1} - s_{v+1}q_v\big(b(s_v - s_{v+1}) + c(q_{v+1} - q_v)\big)/a.$$

In addition, the least value of the form $t(x, y) = bx + cy$ (which is responsible for the length of the shortest cycle) attained for nonnegative nontrivial solutions of the congruence $bx + cy \equiv 0 \pmod{a}$ $(x, y \geq 0)$ is $b(s_v - s_{v+1}) + c(q_{v+1} - q_v)$. Interchanging the arguments a, b, and c, one can find the elements of the *Johnson matrix*, which also allows one to find the Frobenius numbers [25, Theorem 4]. It is a more symmetric tool compared with Rødseth's formula and is convenient to apply in combination with the method of generating functions (see [18, 20, 21]).

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REFERENCES

1. V. I. Arnold, *Experimental Observation of Mathematical Facts* (MTsNMO, Moscow, 2006) [in Russian].
2. J. Bourgain and Ya. G. Sinai, "Limit Behaviour of Large Frobenius Numbers," Usp. Mat. Nauk **62** (4), 77–90 (2007) [Russ. Math. Surv. **62**, 713–725 (2007)].
3. I. S. Vorob'ev, "Experimental Study of the Frobenius Problem with Three Arguments," Dalnevost. Mat. Zh. **11** (1), 3–9 (2011).
4. I. D. Kan, "On the Frobenius Problem," Fundam. Prikl. Mat. **3** (3), 821–835 (1997).
5. I. D. Kan, "The Frobenius Problem for Classes of Polynomial Solvability," Mat. Zametki **70** (6), 845–853 (2001) [Math. Notes **70**, 771–778 (2001)].
6. A. V. Ustinov, *Applications of Kloosterman Sums in Arithmetic and Geometry* (Lambert Academic Publ., Saarbrücken, 2011) [in Russian].
7. A. V. Ustinov, "The Solution of Arnold's Problem on the Weak Asymptotics of Frobenius Numbers with Three Arguments," Mat. Sb. **200** (4), 131–160 (2009) [Sb. Math. **200**, 597–627 (2009)].

8. A. V. Ustinov, "On the Distribution of Frobenius Numbers with Three Arguments," Izv. Ross. Akad. Nauk, Ser. Mat. **74** (5), 145–170 (2010) [Izv. Math. **74**, 1023–1049 (2010)].
9. D. A. Frolenkov, "The Mean Value of Frobenius Numbers with Three Arguments," Izv. Ross. Akad. Nauk, Ser. Mat. (in press); arXiv: 1103.5427v1 [math.NT].
10. F. Aicardi, "On the Geometry of the Frobenius Problem," Funct. Anal. Other Math. **2**, 111–127 (2009).
11. I. Aliev and M. Henk, "Integer Knapsacks: Average Behavior of the Frobenius Numbers," Math. Oper. Res. **34** (3), 698–705 (2009).
12. I. Aliev, M. Henk, and A. Hinrichs, "Expected Frobenius Numbers," J. Comb. Theory A **118** (2), 525–531 (2011).
13. V. I. Arnold, *Arnold's Problems* (Springer, Berlin, 2004).
14. V. I. Arnold, "Geometry of Continued Fractions Associated with Frobenius Numbers," Funct. Anal. Other Math. **2**, 129–138 (2009).
15. J.-C. Bermond, F. Comellas, and D. F. Hsu, "Distributed Loop Computer Networks: A Survey," J. Parallel Distrib. Comput. **24**, 2–10 (1995).
16. A. Brauer and J. E. Shockley, "On a Problem of Frobenius," J. Reine Angew. Math. **211**, 215–220 (1962).
17. J. L. Davison, "On the Linear Diophantine Problem of Frobenius," J. Number Theory **48**, 353–363 (1994).
18. G. Denham, "Short Generating Functions for Some Semigroup Algebras," Electron. J. Comb. **10**, R36 (2003).
19. D. Einstein, D. Lichtblau, A. Strzebonski, and S. Wagon, "Frobenius Numbers by Lattice Point Enumeration," Integers **7** (1), A15 (2007).
20. L. G. Fel, "Frobenius Problem for Semigroups $\mathsf{S}(d_1, d_2, d_3)$," Funct. Anal. Other Math. **1**, 119–157 (2006).
21. L. G. Fel, "Analytic Representations in the Three-Dimensional Frobenius Problem," Funct. Anal. Other Math. **2**, 27–44 (2008).
22. G. R. Hofmeister, "Zu einem Problem von Frobenius," Norske Vid. Selsk. Skr., No. 5, 1–37 (1966).
23. F. K. Hwang, "A Survey on Double Loop Networks," in *Reliability of Computer and Communication Networks, New Brunswick, NJ, 1989* (Am. Math. Soc., Providence, RI, 1991), DIMACS Ser. Discrete Math. Theor. Comput. Sci. **5**, pp. 143–151.
24. F. K. Hwang, "A Complementary Survey on Double-Loop Networks," Theor. Comput. Sci. **263**, 211–229 (2001).
25. S. M. Johnson, "A Linear Diophantine Problem," Can. J. Math. **12**, 390–398 (1960).
26. R. Kannan, "Lattice Translates of a Polytope and the Frobenius Problem," Combinatorica **12** (2), 161–177 (1992).
27. R. M. Karp, "Reducibility among Combinatorial Problems," in *Complexity of Computer Computations*, Ed. by R. E. Miller and J. W. Thatcher (Plenum Press, New York, 1972), pp. 85–103.
28. J. Marklof, "The Asymptotic Distribution of Frobenius Numbers," Invent. Math. **181**, 179–207 (2010).
29. J. Marklof and A. Strömbergsson, "Diameters of Random Circulant Graphs," arXiv: 1103.3152v2 [math.CO].
30. M. Nijenhuis, "A Minimal-Math Algorithm for the 'Money Changing Problem'," Am. Math. Mon. **86**, 832–835 (1979); "Correction," Am. Math. Mon. **87**, 377 (1980).
31. J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem* (Oxford Univ. Press, Oxford, 2005).
32. Ö. J. Rödseth, "On a Linear Diophantine Problem of Frobenius," J. Reine Angew. Math. **301**, 171–178 (1978).
33. Ö. J. Rödseth, "Weighted Multi-connected Loop Networks," Discrete Math. **148**, 161–173 (1996).
34. E. S. Selmer, "On the Linear Diophantine Problem of Frobenius," J. Reine Angew. Math. **293–294**, 1–17 (1977).
35. E. S. Selmer and Ö. Beyer, "On the Linear Diophantine Problem of Frobenius in Three Variables," J. Reine Angew. Math. **301**, 161–170 (1978).
36. V. Shchur, Ya. Sinai, and A. Ustinov, "Limiting Distribution of Frobenius Numbers for $n = 3$," J. Number Theory **129** (11), 2778–2789 (2009).
37. J. J. Sylvester, "Question 7382," Educ. Times **37**, 26 (1884); "Mathematics from the Educational Times, with Additional Papers and Solutions," in *Mathematical Questions, with Their Solutions, from the "Educational Times"* (F. Hodgson, London, 1884), Vol. 41, p. 21.
38. C. K. Wong and D. Coppersmith, "A Combinatorial Problem Related to Multimodule Memory Organizations," J. Assoc. Comput. Mach. **21**, 392–402 (1974).

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