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Spin chains and Arnold's problem on the Gauss-Kuz'min statistics for quadratic irrationals

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Abstract. New results related to number theoretic model of spin chains are proved. We solve Arnold's problem on the Gauss-Kuz'min statistics for quadratic irrationals.

Bibliography: 24 titles.

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§ 1. Introduction

In [1], a number-theoretic model for spin chains was presented; this model uses Farey series (for the subsequent results, see [2]–[4]). In this model, to a finite chain of spins each of which can be directed upwards ($\uparrow$) or downwards ($\downarrow$), a product of the matrices

$$A=\begin{pmatrix}1&0\\1&1\end{pmatrix},\qquad B=\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

is assigned, according to the rule $\uparrow\ = A$ and $\downarrow\ = B$. For example,

$$\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\ =\ \uparrow^3\downarrow^2\uparrow^4\ = A^3B^2A^4.$$

By the *energy* of a given configuration we mean the quantity

$$E(\uparrow^{a_1}\downarrow^{a_2}\uparrow^{a_3}\dots)=\log(\operatorname{Tr}(A^{a_1}B^{a_2}A^{a_3}\dots)).$$

Let G be the free multiplicative monoid generated by the matrices A and B. From a physical viewpoint, the asymptotic behaviour of the number of configurations with a given energy,

$$\Phi(N)=\bigl|\{C\in G:\operatorname{Tr}C=N\}\bigr|,\qquad N\geqslant 3,$$

and the number of configurations in which the energy does not exceed a given quantity,

$$\Psi(N)=\bigl|\{C\in G:3\leqslant\operatorname{Tr}C\leqslant N\}\bigr|=\sum_{3\leqslant n\leqslant N}\Phi(n),$$

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are of interest. The conjecture that

$$\Phi(N) \sim \frac{N}{2} \log N \tag{1.1}$$

was presented in [1] and, at the same time, the asymptotic formula

$$\Psi(N) = \frac{N^2 \log N}{\zeta(2)} + O(N^2 \log\log N) \tag{1.2}$$

was proved in [3].

The conjecture (1.1) was disproved in [4]. It turns out that the arithmetic function $\Phi(N)(N\log N)^{-1}$ has a smooth limit distribution. In [5], the two-term asymptotic formula

$$\Psi(N) = N^2(c_1 \log N + c_0) + O_\varepsilon(N^{7/4+\varepsilon}) \tag{1.3}$$

was obtained for the quantity $\Psi(N)$, where

$$c_1 = \frac{1}{\zeta(2)}, \qquad c_0 = \frac{1}{\zeta(2)}\left(\gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\right).$$

Problems concerning the asymptotic behaviour of $\Phi(N)$ and $\Psi(N)$ are closely related to the distribution of quadratic irrationals and the closed geodesics corresponding to these irrationals on the modular surface (see [6] and [5]). For a reduced quadratic irrational ω (which has a purely periodic representation in the form of a continued fraction) we let $\rho(\omega)$ denote the length which is defined as the length of the corresponding closed geodesic. As was proved in [6],

$$\sum_{\rho(\omega)<x} 1 \sim \frac{e^x \log 2}{2\zeta(2)}. \tag{1.4}$$

The relationship between reduced quadratic irrationals and finite products of the matrices A and B (see [3]) was used in [5] to obtain an asymptotic formula with an explicit estimate for the remainder term,

$$\sum_{\rho(\omega)<x} 1 = \frac{e^x \log 2}{2\zeta(2)} + O_\varepsilon(e^{(7/8+\varepsilon)x}). \tag{1.5}$$

In this paper, we prove the asymptotic formula

$$\Psi(N) = N^2(c_1 \log N + c_0) + O(N^{3/2} \log^4 N), \tag{1.6}$$

which refines equation (1.3), and a formula refining (1.5), namely,

$$\sum_{\rho(\omega)<x} 1 = \frac{e^x \log 2}{2\zeta(2)} + O(x^4 e^{3x/4}).$$

Equation (1.6) is a special case of a more general result concerning the Gauss-Kuz'min statistics for spin chains (see Theorems 1 and 2). Another consequence of this result gives a solution to Arnold's problem (see [7], Problem 1993–11) on the statistical properties of the partial quotients of quadratic irrationals. Let $x, y \in [0,1]$ be real numbers and

$$r(x,y;N) = \sum_{\substack{\omega\in\mathscr{R}\\ \varepsilon_0(\omega)\leqslant N}} \biggl[\omega \leqslant x,\ -\frac{1}{\omega^*} \leqslant y\biggr].$$

Here $\mathscr{R}$ is the set of reduced quadratic irrationals, $\varepsilon_0(\omega)$ is the fundamental solution of Pell's equation

$$X^2 - \Delta Y^2 = 4,$$

$\Delta = B^2 - 4AC$, where $AX^2 + BX + C$ is the minimal polynomial of ω, and ω^* stands for the number conjugate to ω; moreover, $[A]$ stands for 1 if the statement A is true and for 0 otherwise. Then (see Theorem 3)

$$r(x,y;N) = \frac{\log(1+xy)}{2\zeta(2)} N^2 + O(N^{3/2}\log^4 N),$$

that is, Gauss-Kuz'min statistics for the quadratic irrationals are described by the same distribution function $\log_2(1+xy)$ and the same corresponding density

$$\frac{1}{\log 2}\cdot\frac{1}{(1+xy)^2}$$

as occur in the Gauss-Kuz'min statistics for the rationals and for almost all reals.

The proofs of the theorems use the approach suggested in [5].

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§ 2. Application of bounds for the Kloosterman sums

The main tool for solving problems which can be reduced to the distribution of solutions of the congruence $xy \equiv \pm 1 \pmod q$ is the following lemma.

Lemma 1. *Let q be a positive integer and $0 \leqslant P_1, P_2 \leqslant q$. Then*

$$\sum_{0<x\leqslant P_1}\sum_{0<y\leqslant P_2} \delta_q(xy\pm 1) = \frac{\varphi(q)}{q^2}P_1P_2 + O(\psi_1(q)), \tag{2.1}$$

where $\psi_1(q) = \sigma_0(q)\log^2(q+1)q^{1/2}$.

For a proof see, for example, [8].

In the next lemma, an asymptotic formula for the number of solutions of the congruence $xy \equiv \pm 1 \pmod q$ under the graph of the simplest linear function can be proved in a similar way.

Lemma 2. *Let q be a positive integer, let $0 \leqslant P_1, P_2 \leqslant q$, let a be an integer, and let $f(x) = a \pm x$ be a linear function for which $0 \leqslant f(P_1), f(P_2) \leqslant q$. Then the sum*

$$S_f(P_1, P_2) = \sum_{P_1 < x \leqslant P_2} \sum_{0 < y \leqslant f(x)} \delta_q(xy \pm 1)$$

admits the following asymptotic formula (for any choice of sign in the symbol $\pm$):

$$S_f(P_1, P_2) = \frac{\varphi(q)}{q^2} \int_{P_1}^{P_2} f(x)\,dx + O(\psi_2(q)),$$

where $\psi_2(q) = \sigma_0(q)\log(q+1)(\sigma_0(q) + \log(q+1))q^{1/2}$.

Proof. Assume that $f(x) = a + x$. The case of $f(x) = a - x$ can be proved similarly. Expand the function

$$F(x, y) = [P_1 < x \leqslant P_2,\ 0 < y \leqslant f(x)]$$

in a finite Fourier series,

$$F(x, y) = \sum_{-q/2 < m, n \leqslant q/2} \widehat{F}(m, n) e^{2\pi i(mx+ny)/q},$$

with the Fourier coefficients

$$\widehat{F}(m, n) = \frac{1}{q^2} \sum_{x,y=1}^{q} F(x, y) e^{-2\pi i(mx+ny)/q}.$$

Then the given sum can be represented in the form

$$S_f(P_1, P_2) = \sum_{x,y=1}^{q} F(x, y)\delta_q(xy \pm 1) = \sum_{-q/2 < m, n \leqslant q/2} \widehat{F}(m, n) K_q(m, \mp n),$$

where

$$K_q(m, n) = \sum_{x,y=1}^{q} \delta_q(xy - 1) e^{2\pi i(mx+ny)/q}$$

are Kloosterman sums. Distinguishing the term with $m = n = 0$, we obtain the equation

$$S_f(P_1, P_2) = \frac{\varphi(q)}{q^2} \int_{P_1}^{P_2} f(x)\,dx + O(1) + R, \tag{2.2}$$

where

$$R = {\sum_{-q/2 < m, n \leqslant q/2}}' \widehat{F}(m, n) K_q(m, \mp n).$$

Here and below, a dash $'$ on the summation sign means that the term for which all variables of summation vanish is omitted.

Using the bound

$$|K_q(m, n)| \leqslant \sigma_0(q)(q, m, n)^{1/2} q^{1/2}$$

for the Kloosterman sums (see [9]), we obtain the following inequalities for the remainder R:

$$|R| \leqslant \sigma_0(q) q^{1/2} \sideset{}{'}\sum_{-q/2<m,n\leqslant q/2} \widehat{F}(m,n)(q,m,n)^{1/2} \leqslant \sigma_0(q) q^{1/2}(R_1+R_2+R_3+R_4), \tag{2.3}$$

where

$$R_1 = \sideset{}{'}\sum_{-q/2<m\leqslant q/2} |\widehat{F}(m,0)|(m,q), \qquad R_2 = \sideset{}{'}\sum_{-q/2<n\leqslant q/2} |\widehat{F}(0,n)|(n,q),$$

$$R_3 = \sideset{}{'}\sum_{-q/2<m\leqslant q/2} |\widehat{F}(m,-m)|(m,q),$$

$$R_4 = \sideset{}{'}\sum_{-q/2<m\leqslant q/2} \sideset{}{'}\sum_{\substack{-q/2<n\leqslant q/2\\ m+n\neq 0,q}} |\widehat{F}(m,n)|(n,m,q)^{1/2}.$$

We will estimate the Fourier coefficients of the function F. If $n=0$, then

$$\widehat{F}(m,0) = \frac{1}{q^2} \sum_{P_1<x\leqslant P_2} (a+x) e^{-2\pi i m x/q},$$

$$\biggl|\sum_{P_1<x\leqslant P_2} e^{-2\pi i m x/q}\biggr| = \biggl|\frac{e^{-2\pi i m(P_2-P_1)/q}-1}{e^{-2\pi i m/q}-1}\biggr| \leqslant \frac{1}{|\sin(\pi m/q)|} \leqslant \frac{q}{|m|},$$

$$\biggl|\sum_{0<x\leqslant P} x e^{-2\pi i m x/q}\biggr| = \biggl|\frac{P e^{-2\pi i m(P+1)/q}}{e^{-2\pi i m/q}-1} - \frac{e^{-2\pi i m/q}(e^{-2\pi i m P/q}-1)}{(e^{-2\pi i m/q}-1)^2}\biggr|$$

$$\leqslant \frac{Pq}{|m|} + \frac{q^2}{|m|^2} \ll \frac{q^2}{|m|},$$

$$R_1 \ll \sum_{m=1}^{q} \frac{(m,q)}{m} \leqslant \sum_{d\mid q} d \sum_{\substack{m=1\\ d\mid m}}^{q} \frac{1}{m} \ll \sigma_0(q)\log(q+1).$$

If $n\neq 0$, then

$$\widehat{F}(m,n) = \frac{1}{q^2}\cdot\frac{e^{-2\pi i n/q}}{e^{-2\pi i n/q}-1}\biggl(e^{-2\pi i n a/q}\sum_{P_1<x\leqslant P_2} e^{-2\pi i(m+n)x/q} - \sum_{P_1<x\leqslant P_2} e^{-2\pi i m x/q}\biggr). \tag{2.4}$$

Therefore, for $m=0$ we have

$$\widehat{F}(0,n) \ll \frac{1}{q^2}\cdot\frac{q}{|n|}\biggl(\frac{q}{|n|}+q\biggr) \ll \frac{1}{|n|}$$

and the remainder R_2 can be estimated in just the same way as the remainder R_1,

$$R_2 \ll \sum_{n=1}^{q} \frac{(n,q)}{n} \ll \sigma_0(q)\log(q+1).$$

If $m+n=0$ and $m\neq 0$, then, by (2.4),

$$|\widehat{F}(m,-m)| \ll \frac{1}{q^2}\cdot\frac{q}{|m|}\biggl(q+\frac{q}{|m|}\biggr) \ll \frac{1}{|m|}.$$

Hence, the same bound is obtained for the remainder R_3,

$$R_3 \ll \sum_{m=1}^{q}\frac{(m,q)}{m} \ll \sigma_0(q)\log(q+1).$$

In the remaining cases ($m\neq 0$, $n\neq 0$, $m+n\neq 0,q$), by (2.4) we have

$$\begin{aligned}|\widehat{F}(m,n)| &\leqslant \frac{1}{q^2}\cdot\frac{1}{|\sin(\pi n/q)|}\biggl(\frac{1}{|\sin(\pi(m+n)/q)|}+\frac{1}{|\sin(\pi m/q)|}\biggr)\\ &\ll \frac{1}{|n|}\biggl(\frac{1}{|m+n|}+\frac{1}{|q-m-n|}+\frac{1}{|q+m+n|}+\frac{1}{|m|}\biggr).\end{aligned}$$

In particular, if m and n have different signs, then

$$|\widehat{F}(m,n)| \ll \frac{1}{|n|\cdot|m-n|},$$

and, if the signs are the same, then

$$|\widehat{F}(m,n)| \ll \frac{1}{|n|}\biggl(\frac{1}{q-|m|-|n|}+\frac{1}{|m|}\biggr).$$

Therefore, $R_4 \ll R_{4,1}+R_{4,2}$, where

$$R_{4,1} = \sum_{\substack{m,n\leqslant q/2\\ m\neq n}}\frac{(m,n,q)^{1/2}}{n\cdot|m-n|}, \qquad R_{4,2} = \sum_{\substack{m,n\leqslant q/2\\ m+n\neq q}}\frac{(m,n,q)^{1/2}}{n}\biggl(\frac{1}{q-m-n}+\frac{1}{m}\biggr).$$

Introducing the variables $d=(m,n,q)$, $m_1=md^{-1}$, and $n_1=nd^{-1}$, we obtain the following bound for the first sum:

$$\begin{aligned}R_{4,1} &\ll \sum_{d\mid q} d^{1/2}\sum_{\substack{m,n\leqslant q/2\\ m\neq n,\, d\mid(m,n)}}\frac{1}{n\cdot|m-n|} \ll \sum_{d\mid q}\frac{1}{d^{3/2}}\sum_{\substack{m_1,n_1\leqslant q\\ m_1\neq n_1}}\frac{1}{n_1\cdot|m_1-n_1|}\\ &\ll \sum_{\substack{m_1,n_1\leqslant q\\ m_1\neq n_1}}\frac{1}{n_1\cdot|m_1-n_1|} \ll \log^2(q+1).\end{aligned}$$

The second sum can be estimated similarly to the first,

$$R_{4,2} \ll \sum_{d|q} d^{1/2} \sum_{\substack{m,n\leqslant q/2\\ m+n\neq q,\, d|(m,n)}} \frac{1}{n}\biggl(\frac{1}{q-m-n}+\frac{1}{m}\biggr)$$
$$\ll \sum_{d|q} \frac{1}{d^{3/2}} \sum_{\substack{m_1,n_1\leqslant q/(2d)\\ m_1+n_1\neq q/d}} \frac{1}{n_1}\biggl(\frac{1}{q/d-m_1-n_1}+\frac{1}{m_1}\biggr)$$
$$\ll \sum_{\substack{m_1,n_1\leqslant q/(2d)\\ m_1+n_1\neq q/d}} \frac{1}{n_1}\biggl(\frac{1}{q/d-m_1-n_1}+\frac{1}{m_1}\biggr) \ll \log^2(q+1).$$

Thus, $R_4 \ll \log^2(q+1)$.

Substituting the above bounds for the remainders R_1, R_2, R_3, and R_4 into (2.3), we arrive at the relation

$$R \ll \sigma_0(q)\log(q+1)(\sigma_0(q)+\log(q+1))q^{1/2} = \psi_2(q),$$

and, taking account of equation (2.2), this leads to the statement of the lemma.

§ 3. Spin chains and continued fractions

We let $\mathscr{M}$ denote the set of all integer matrices

$$S = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} p(S) & p'(S) \\ q(S) & q'(S) \end{pmatrix}$$

with determinant ± 1 for which

$$1 \leqslant q \leqslant q', \qquad 0 \leqslant p \leqslant q, \qquad 1 \leqslant p' \leqslant q'.$$

This set is partitioned into two disjoint sets $\mathscr{M}_+$ and $\mathscr{M}_-$, which consist of matrices with determinants $+1$ and -1, respectively. The elements of the set $\mathscr{M}$ form a multiplicative semigroup and are in a one-to-one correspondence with the (non-empty) families of positive integers constructed using the rule (see [10])

$$(a_1, a_2, \dots, a_n) \mapsto \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}.$$

The inverse map is constructed using the equations

$$\frac{p}{q} = [0; a_1, \dots, a_{n-1}], \qquad \frac{p'}{q'} = [0; a_1, \dots, a_n],$$
$$\frac{p}{p'} = [0; a_n, \dots, a_2], \qquad \frac{q}{q'} = [0; a_n, \dots, a_1].$$

As in [5] and [3], to evaluate the function $\Psi(N)$, we consider the products of the matrices $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of even and odd length separately. To be definite, we assume that the products begin with the matrix B. Following the notation in [5], we introduce the sets W_{ev} and W_{odd} as follows:

$$W_{\mathrm{ev}}(N) = \bigl\{(a_1,\dots,a_{2m}) \in \mathbb{N}^{2m} : m \geqslant 1,\ \mathrm{Tr}(B^{a_1}A^{a_2}\dots B^{a_{2m-1}}A^{a_{2m}}) \leqslant N\bigr\},$$
$$W_{\mathrm{odd}}(N) = \bigl\{(a_1,\dots,a_{2m+1}) \in \mathbb{N}^{2m+1} : m \geqslant 1,\ \mathrm{Tr}(B^{a_1}A^{a_2}\dots A^{a_{2m}}B^{a_{2m+1}}) \leqslant N\bigr\}.$$

We set the family of positive integers $(a_1,\dots,a_n)$ in correspondence with the continued fraction $[0;a_1,\dots,a_n]$ and the sequence of approximants $p_k/q_k = [0;a_1,\dots,a_k]$, $0 \leqslant k \leqslant n$. The properties of the continued fractions imply that

$$JB^{a_1}A^{a_2}\dots B^{a_{2m-1}}A^{a_{2m}}J = J\begin{pmatrix} q_{2m} & q_{2m-1} \\ p_{2m} & p_{2m-1}\end{pmatrix}J = \begin{pmatrix} p_{2m-1} & p_{2m} \\ q_{2m-1} & q_{2m}\end{pmatrix} \in \mathscr{M}_+,$$
$$JB^{a_1}A^{a_2}\dots A^{a_{2m}}B^{a_{2m+1}} = J\begin{pmatrix} q_{2m} & q_{2m+1} \\ p_{2m} & p_{2m+1}\end{pmatrix} = \begin{pmatrix} p_{2m} & p_{2m+1} \\ q_{2m} & q_{2m+1}\end{pmatrix} \in \mathscr{M}_-,$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$. Therefore, the quantities

$$\Psi_{\mathrm{ev}}(N) = |W_{\mathrm{ev}}(N)|, \qquad \Psi_{\mathrm{odd}}(N) = |W_{\mathrm{odd}}(N)|$$

can also be defined by the equalities

$$\Psi_{\mathrm{ev}}(N) = \bigl|\{S \in \mathscr{M}_+ : \mathrm{Tr}(S) = p(S) + q'(S) \leqslant N\}\bigr|,$$
$$\Psi_{\mathrm{odd}}(N) = \bigl|\{S \in \mathscr{M}_- : p(S) > 0,\ p'(S) + q(S) \leqslant N\}\bigr|.$$

By hypothesis all the products under consideration begin with the matrix B, and so

$$\Psi(N) = 2\bigl(\Psi_{\mathrm{ev}}(N) + \Psi_{\mathrm{odd}}(N)\bigr).$$

To describe the behaviour of the partial quotients in the continued fractions for real numbers, it is convenient to use the measure (see [11])

$$d\lambda = \frac{1}{\log 2}\cdot\frac{du\,dv}{(1+uv)^2}.$$

Let a real number $\alpha \in [0,1]$ be given by an infinite continued fraction $\alpha = [0;a_1,a_2,\dots,a_n,\dots]$ and let $p_n(\alpha)/q_n(\alpha) = [0;a_1,\dots,a_n]$ and $r_n(\alpha) = [0;a_{n+1},a_{n+2},\dots]$. Then $\alpha = [0;a_1,\dots,a_n + r_n(\alpha)]$, $q_{n-1}(\alpha)/q_n(\alpha) = [0;a_n,\dots,a_1]$, and the behaviour of the elements of the continued fraction near the index n is described in the mean by the function (the Gauss-Kuz'min statistics treated in a generalized sense)

$$F_n(x,y) = \int_0^1 \left[r_n(\alpha) \leqslant x,\ \frac{q_{n-1}(\alpha)}{q_n(\alpha)} \leqslant y\right] d\alpha.$$

Here

$$F_n(x,y) \to \log_2(1+xy) = \frac{1}{\log 2}\int_0^x\int_0^y \frac{du\,dv}{(1+uv)^2}, \qquad n \to \infty.$$

In particular, for $y = 1$ we deal with the Gauss measure

$$d\mu = \frac{1}{\log 2}\cdot\frac{du}{1+u}$$

and the corresponding distribution function $\log_2(1+x)$.

The spin chains model under consideration is closely connected with continued fractions. Therefore, to describe the properties of generic configurations, it is natural to introduce characteristics similar to the Gauss-Kuz'min statistics. The object we introduce characterizes local properties of spin configurations.

For real $x, y \in [0,1]$ we write

$$W_{\mathrm{ev}}(x,y;N) = \left\{ \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathscr{M}_+ : p' \leqslant xq',\, q \leqslant yq',\, p+q' \leqslant N \right\},$$

$$W_{\mathrm{odd}}(x,y;N) = \left\{ \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathscr{M}_- : p > 0,\, p' \leqslant xq',\, q \leqslant yq',\, p'+q \leqslant N \right\},$$

$$\Psi_{\mathrm{ev}}(x,y;N) = |W_{\mathrm{ev}}(x,y;N)|, \qquad \Psi_{\mathrm{odd}}(x,y;N) = |W_{\mathrm{odd}}(x,y;N)|.$$

In particular, $\Psi_{\mathrm{ev}}(1,1;N) = \Psi_{\mathrm{ev}}(N)$ and $\Psi_{\mathrm{odd}}(1,1;N) = \Psi_{\mathrm{odd}}(N)$.

Arnold conjectured (see [7], Problem 1993–11) that the partial quotients of rationals and quadratic irrationals behave in the mean in just the same way as those for almost all reals. This statement was proved by Lochs (see [12]) for the rationals, in the simplest case when the averaging is carried out over the fractions a/b, $1 \leqslant a \leqslant b \leqslant R$ ($R \to \infty$). For the case when the averaging is carried out over the points in a sector $1 \leqslant a \leqslant b$, $a^2+b^2 \leqslant R^2$ ($R \to \infty$), as was suggested in the original setting of the problem, the conjecture was proved by Avdeeva and Bykovskii (see [13] and [14], and also [15] and [8]). The known Gauss-Kuz'min statistics for finite continued fractions enabled us to solve the Sinaĭ problem on the statistical properties of trajectories of particles in two-dimensional crystal lattices (see [16]), to obtain new results on the behaviour in the mean of various versions of the Euclidean algorithm (see [17], [18]), and to find the distribution density of the normalized Frobenius numbers with three arguments (see [19]).

It turns out that the quantities $\Psi_{\mathrm{ev}}(x,y;N)$ an $\Psi_{\mathrm{odd}}(x,y;N)$, viewed as functions of x and y, exhibit fundamentally different behaviour. The even chains satisfy the Gauss-Kuz'min law (as do the rational numbers in the Arnold problem), whereas the odd ones do not.

The relationship between the behaviour of the function $\Psi_{\mathrm{ev}}(N)$ and the distribution of quadratic irrationals, which was noted in [5] and [3], helps to prove Arnold's conjecture for the quadratic irrationals and to refine the asymptotic formula (1.5).

§ 4. Spin chains and the Gauss-Kuz'min statistics

Theorem 1. *Let $0 \leqslant x, y \leqslant 1$ and $N \geqslant 2$. Then the following asymptotic formula holds*:

$$\Psi_{\mathrm{ev}}(x,y;N) = \frac{\log(1+xy)}{2\zeta(2)} N^2 + O(N^{3/2}\log^4 N) \tag{4.1}$$

with an absolute constant in the remainder term.

Proof. We transform the given quantity,

$$\begin{aligned} \Psi_{\mathrm{ev}}(x,y;N) &= \sum_{\left(\begin{smallmatrix} t & u \\ v & q \end{smallmatrix}\right) \in \mathscr{M}_+} [u \leqslant xq,\, v \leqslant yq,\, t+q \leqslant N] \\ &= \sum_{u \leqslant xN} \sum_{q \geqslant u/x} \sum_{t \leqslant yu+1/q} \delta_u(tq-1)[t+q \leqslant N]. \end{aligned} \tag{4.2}$$

There is at most one value of the variable t lying in the interval $yu < t \leqslant yu + 1/q$. Therefore,

$$\sum_{u\leqslant xN}\sum_{q\geqslant u/x}\sum_{yu<t\leqslant yu+1/q}\delta_u(tq-1)[t+q\leqslant N] \\ \ll \sum_{u\leqslant N}\sum_{q\leqslant N}\delta_u(t_y(u)q-1) \ll \sum_{u\leqslant N}\frac{N}{u} \ll N^{1+\varepsilon}$$

for $t_y(u) = \lceil yu \rceil$. Thus,

$$\Psi_{\mathrm{ev}}(x,y;N) = \sum_{u\leqslant xN}\sum_{t\leqslant yu}\sum_{q\geqslant u/x}\delta_u(tq-1)[t+q\leqslant N] + O(N). \tag{4.3}$$

It follows from the equation

$$\sum_{y=1}^{Y}\delta_q(xy-1) \ll \frac{Y}{q} + 1$$

that the quantity $\Psi_{\mathrm{ev}}(x,y;N)$ admits the bound

$$\Psi_{\mathrm{ev}}(x,y;N) \ll \sum_{u\leqslant N}\sum_{t\leqslant yu}\sum_{q\leqslant N}\delta_u(tq-1) \ll \sum_{u\leqslant N} yu\frac{N}{u} \ll yN^2.$$

Since $\Psi_{\mathrm{ev}}(x,y;N)$ is symmetric with respect to x and y, the bound

$$\Psi_{\mathrm{ev}}(x,y;N) \ll xN^2$$

also holds. Therefore,

$$\Psi_{\mathrm{ev}}(x,y;N) \ll N^{3/2}, \qquad \log(1+xy)N^2 \ll N^{3/2}$$

for $\min\{x,y\} \leqslant N^{-1/2}$, and formula (4.1) holds. Thus, it is sufficient to prove that (4.1) holds under the assumption that $x \geqslant N^{-1/2}$. In formula (4.3), Lemmas 1 and 2 can be applied to the inner double sum. Taking the formula (see [20], Ch. II, Problem 19)

$$\sum_{x=1}^{q}\sum_{y=1}^{Y}\delta_q(xy-1) = \sum_{y=1}^{Y}{}^{*}\, 1 = \frac{\varphi(q)}{q}Y + O(\sigma_0(q))$$

into account (here and below, an asterisk * means that the variable of summation ranges over the reduced system of residues), we obtain the equation

$$\Psi_{\mathrm{ev}}(x,y;N) = \sum_{u\leqslant xN}\biggl(\frac{\varphi(u)}{u^2}\int_0^{yu}dt\int_{u/x}^{\infty}[t+q\leqslant N]\,dq \\ + O\biggl(\frac{N}{u}\sigma_0(u) + \psi_2(u)\biggr)\biggr) + O(N).$$

It follows from the standard bounds

$$\sum_{u\leqslant M}\sigma_0(u) \ll M\log(M+1), \qquad \sum_{u\leqslant M}\sigma_0^2(u) \ll M\log^3(M+1)$$

that

$$\Psi_{\mathrm{ev}}(x,y;N)=\sum_{u\leqslant N/x}\frac{\varphi(u)}{u^2}\int_0^{yu}dt\int_{u/x}^{\infty}[t+q\leqslant N]\,dq+O(N^{3/2}\log^4 N).$$

Applying the identity ($u=\delta u_1$)

$$\sum_{u\leqslant M}\frac{\varphi(u)}{u^2}f(u)=\sum_{\delta\leqslant M}\frac{\mu(\delta)}{\delta^2}\sum_{u_1\leqslant M/\delta}\frac{f(\delta u_1)}{u_1} \tag{4.4}$$

and introducing the variables $\alpha=t/u=t/(\delta u_1)$ and $\beta=q/u=q/(\delta u_1)$, we arrive at the asymptotic formula

$$\Psi_{\mathrm{ev}}(x,y;N)=\sum_{\delta\leqslant N}\mu(\delta)S\biggl(\frac{N}{\delta}\biggr)+O(N^{3/2}\log^4 N), \tag{4.5}$$

where

$$S(N)=\sum_{u\leqslant xN}u\int_0^{y}d\alpha\int_{1/x}^{\infty}[\alpha+\beta\leqslant Nu^{-1}]\,d\beta.$$

We represent the sum $S(N)$ in the form

$$S(N)=S_1(N)+S_2(N),$$

where

$$S_1(N)=\sum_{u\leqslant xN/(xy+1)}u\int_0^{y}d\alpha\int_{1/x}^{Nu^{-1}-\alpha}d\beta,$$

$$S_2(N)=\sum_{xN/(xy+1)<u\leqslant xN}u\int_0^{Nu^{-1}-1/x}d\alpha\int_{1/x}^{Nu^{-1}-\alpha}d\beta.$$

After evaluating the integrals

$$\int_0^{y}d\alpha\int_{1/x}^{Nu^{-1}-\alpha}d\beta=\biggl(\frac{N}{u}-\frac{1}{x}\biggr)y-\frac{y^2}{2},$$

$$\int_0^{Nu^{-1}-1/x}d\alpha\int_{1/x}^{Nu^{-1}-\alpha}d\beta=\frac{1}{2}\biggl(\frac{N}{u}-\frac{1}{x}\biggr)^2$$

we arrive at the asymptotic formulae

$$S_1(N)=\frac{xy(3xy+2)}{4(xy+1)^2}N^2+O(N),$$

$$S_2(N)=\frac{\log(xy+1)}{2}N^2-\frac{xy(3xy+2)}{4(xy+1)^2}N^2+O\biggl(\frac{N}{x}\biggr),$$

$$S(N)=\frac{\log(xy+1)}{2}N^2+O\biggl(\frac{N}{x}\biggr).$$

By assumption, $x\geqslant N^{-1/2}$. Hence, $Nx^{-1}\ll N^{3/2}$. Therefore, substituting the asymptotic formula for the sum $S(N)$ into (4.5), we obtain the desired equation for $\Psi_{\mathrm{ev}}(x,y;N)$.

Theorem 2. *Let $0 \leqslant x, y \leqslant 1$ and $N \geqslant 2$. Then*

$$\begin{aligned}\Psi_{\mathrm{odd}}(x,y;N) =& \frac{N^2}{2\zeta(2)}\biggl(\log N + \log\frac{xy}{x+y} + \gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\biggr)\\ &+ O(N^{3/2}\log^4 N) + O\biggl(\frac{x+y}{xy}N\log N\biggr).\end{aligned}$$

Proof. Repeating the arguments in the proof of Theorem 1, we arrive at the equations

$$\begin{aligned}\Psi_{\mathrm{odd}}(x,y;N) &= \sum_{\left(\begin{smallmatrix} t & u \\ v & q\end{smallmatrix}\right)\in\mathscr{M}_-} [t>0,\ u\leqslant xq,\ v\leqslant yq,\ u+v\leqslant N]\\ &= \sum_{t\leqslant N}\sum_{u,v\geqslant t}\delta_t(uv-1)\biggl[v\geqslant\frac{t}{x}+\frac{1}{u},\ u\geqslant\frac{t}{y}+\frac{1}{v},\ u+v\leqslant N\biggr]\\ &= \sum_{t\leqslant xyN/(x+y)}\ \sum_{u\geqslant t/y}\ \sum_{v\geqslant t/x}\delta_t(uv-1)[u+v\leqslant N] + O(N\log N).\end{aligned}$$

The boundary of the domain in which u and v vary intersects at most $O(N/t)$ squares of the form $[at,(a+1)t]\times[bt,(b+1)t]$. Therefore, applying Lemmas 1 and 2, we obtain the equations

$$\begin{aligned}\Psi_{\mathrm{odd}}(x,y;N) &= \sum_{t\leqslant xyN/(x+y)}\biggl(\frac{\varphi(t)}{t^2}\int_{t/y}^{\infty}du\int_{t/x}^{\infty}[u+v\leqslant N]\,dv + O\biggl(\frac{N}{t}\psi_2(t)\biggr)\biggr)\\ &\qquad + O(N\log N)\\ &= \sum_{t\leqslant xyN/(x+y)}\frac{\varphi(t)}{t^2}\int_{t/y}^{\infty}du\int_{t/x}^{\infty}[u+v\leqslant N]\,dv + O(N^{3/2}\log^4 N).\end{aligned}$$

Using (4.4) again and changing to the variables $\alpha = u/t = u/(\delta t_1)$ and $\beta = v/t = v/(\delta t_1)$, we see that

$$\Psi_{\mathrm{odd}}(x,y;N) = \sum_{\delta\leqslant xyN/(x+y)}\mu(\delta)T\biggl(\frac{N}{\delta}\biggr) + O(N^{3/2}\log^4 N), \tag{4.6}$$

where

$$\begin{aligned}T(N) &= \sum_{t\leqslant xyN/(x+y)} t\int_{1/y}^{\infty}d\alpha\int_{1/x}^{\infty}\biggl[\alpha+\beta\leqslant\frac{N}{t}\biggr]\,d\beta = \sum_{t\leqslant xyN/(x+y)}\frac{t}{2}\biggl(\frac{N}{t}-\frac{1}{x}-\frac{1}{y}\biggr)^2\\ &= \frac{N^2}{2}\biggl(\log N + \log\frac{xy}{x+y} + \gamma - \frac{3}{2}\biggr) + O\biggl(\frac{x+y}{xy}N\biggr).\end{aligned}$$

Substituting the asymptotic formula for $T(N)$ into (4.6) and applying the formula

$$\sum_{\delta\leqslant M}\frac{\mu(\delta)}{\delta^2}\log\delta = \frac{\zeta'(2)}{\zeta^2(2)} + O\biggl(\frac{\log(M+1)}{M}\biggr),$$

we obtain the statement of the theorem.

Corollary 1. *For $N \geqslant 2$*

$$\Psi(N) = \frac{N^2}{\zeta(2)}\left(\log N + \gamma - \log 2 - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + O(N^{3/2}\log^4 N).$$

Remark. Comparing the results of Theorems 1 and 2, we can conclude that the even and odd spin chains have a fundamentally different structure in the mean. It seems that these cases should be separated and studied independently. In contrast to the leading term in Theorem 1, the leading term in Theorem 2 does not depend on x and y, because it is obtained by summing matrices of the form $\left(\begin{smallmatrix} t & u \\ v & q \end{smallmatrix}\right)$ in which $v = o(q)$ and $u = o(q)$.

§ 5. Gauss-Kuz'min statistics for quadratic irrationals

Let $AX^2+BX+C \in \mathbb{Z}[X]$ $(A > 0, (A,B,C) = 1)$ be the minimal polynomial of a quadratic irrational ω and let $\varDelta = B^2 - 4AC$. We denote the number conjugate to ω by ω^*. In the field $\mathbb{Q}(\sqrt{\Delta})$, the number ω has *trace* $\operatorname{tr}(\omega) = \omega + \omega^* = -B/A$ and *norm* $\mathscr{N}(\omega) = \omega\omega^* = C/A$. A quadratic irrational ω is said to be *reduced* if it can be decomposed into a purely periodic continued fraction,

$$\omega = [0; \overline{a_1, a_2, \dots, a_n}], \tag{5.1}$$

of period $n = \operatorname{per}(\omega)$. Here, by the Galois theorem (see [21]),

$$-\frac{1}{\omega^*} = [0; \overline{a_n, \dots, a_1}].$$

We denote the set of all reduced quadratic irrationals by $\mathscr{R}$. The *length* of a number $\omega \in \mathscr{R}$ is the quantity $\rho(\omega) = 2\log\varepsilon_0$, where $\varepsilon_0 = \frac{1}{2}(x_0 + \sqrt{\Delta}y_0)$ is the fundamental solution of Pell's equation

$$X^2 - \Delta Y^2 = 4.$$

The term 'length' is used because, on the modular surface $\mathbb{H}/\operatorname{PSL}_2(\mathbb{Z})$, where $\mathbb{H} = \{(x,y) : y > 0\}$ stands for the upper half-plane, there is a closed geodesic corresponding to a pair of quadratic irrationals ω and ω^* (the projection of the geodesic joining ω and ω^*) whose length in the classical metric $ds^2 = (dx^2+dy^2)y^{-2}$ is precisely equal to $\rho(\omega)$ (see [6] and [22]).

For a reduced quadratic irrational $\omega = [0; \overline{a_1, \dots, a_n}]$ we write

$$\operatorname{per}_e(\omega) = \begin{cases} n & \text{if } n = \operatorname{per}(\omega) \text{ is even,} \\ 2n & \text{if } n = \operatorname{per}(\omega) \text{ is odd.} \end{cases}$$

In this case, the fundamental unit can be found using Smith's formula (see [23] and [24], § 2.4),

$$\varepsilon_0^{-1}(\omega) = \omega T(\omega) T^2(\omega) \dots T^{\operatorname{per}_e(\omega)-1}(\omega),$$

where $T(\alpha)$ stands for the Gauss map $T(\alpha) = \{1/\alpha\}$.

For the manipulations below, we need the following properties of the reduced quadratic irrationals and their corresponding fundamental units (see [3], Propositions 2.1 and 4.1).

1°. For every positive integer k

$$0 < \operatorname{tr}(\varepsilon_0^k(\omega)) - \varepsilon_0^k(\omega) < \frac{1}{2}.$$

2°. If $\omega = [0; \overline{a_1, \ldots, a_{2m}}]$, $l = \operatorname{per}_e(\omega)$, $2m = kl$, then

$$\operatorname{Tr}(B^{a_1} A^{a_2} \ldots A^{a_{2m}}) = \operatorname{tr}(\varepsilon_0^k(\omega)).$$

Let

$$r(N) = \sum_{\substack{\omega \in \mathscr{R} \\ \varepsilon_0(\omega) \leqslant N}} 1 = \pi_0(2 \log N),$$

where $\pi_0(x)$ stands for the number of reduced quadratic irrationals whose length does not exceed x. The next statement can be extracted from the proofs of Propositions 4.3 and 4.5 in [3] (see also [5]).

Lemma 3. *For every integer* $N \geqslant 2$

$$r(N) = \Psi_{\mathrm{ev}}(N) + O(N \log N).$$

Proof. We first assume that $N \geqslant 2$ is a real number. By property 2°, the map

$$(a_1, \ldots, a_{2m}) \mapsto (k, \omega),$$

which assigns the quadratic irrational $\omega = [0; \overline{a_1, \ldots, a_n}]$ and the number $k = 2m / \operatorname{per}_e(\omega)$ to a family of positive integers $(a_1, \ldots, a_{2m})$, is a bijection between the set $W_{\mathrm{ev}}(N)$ and the set of pairs (k, ω), where $k \in \mathbb{N}$, $\omega \in \mathscr{R}$ and $\operatorname{tr}(\varepsilon_0^k(\omega)) \leqslant N$. Hence,

$$\Psi_{\mathrm{ev}}(N) = \sum_{k=1}^{\infty} \sum_{\substack{\omega \in \mathscr{R} \\ \operatorname{tr}(\varepsilon_0^k(\omega)) \leqslant N}} 1 = \sum_{k \leqslant 2 \log N} \widetilde{r}_k(N), \tag{5.2}$$

where

$$\widetilde{r}_k(N) = \sum_{\substack{\omega \in \mathscr{R} \\ \operatorname{tr}(\varepsilon_0^k(\omega)) \leqslant N}} 1.$$

We can impose the condition $k \leqslant 2 \log N$ because the inequality

$$\varepsilon_0(\omega) \geqslant \varepsilon_0\left(\frac{1+\sqrt{5}}{2}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^2 > e^{1/2}$$

holds for every $\omega \in \mathscr{R}$.

It follows from (5.2) that $\widetilde{r}_1(N) \leqslant \Psi_{\mathrm{ev}}(N) \ll N^2$. By property 1°,

$$r\left(\left(N - \frac{1}{2}\right)^{1/k}\right) \leqslant \widetilde{r}_k(N) \leqslant r(N^{1/k}). \tag{5.3}$$

Therefore, $r(N) \ll N^2$, $\widetilde{r}_k(N) \ll N^{2/k}$, and

$$\sum_{2 \leqslant k \leqslant 2 \log N} \widetilde{r}_k(N) \ll \sum_{2 \leqslant k \leqslant 2 \log N} N^{2/k} \ll N \log N, \tag{5.4}$$

$$\Psi_{\mathrm{ev}}(N) = \widetilde{r}_1(N) + O(N \log N).$$

It follows from (5.3) that

$$r\left(N-\frac{1}{2}\right) \leqslant \widetilde{r}_1(N) \leqslant r(N).$$

Thus,

$$\Psi_{\mathrm{ev}}(N)+O(N\log N) \leqslant r(N) \leqslant \Psi_{\mathrm{ev}}\left(N+\frac{1}{2}\right)+O(N\log N).$$

However, the equation $\Psi_{\mathrm{ev}}(N+1/2)=\Psi_{\mathrm{ev}}(N)$ holds for integer N, and therefore the statement of the lemma follows from the estimates for $r(N)$ obtained above.

Corollary 2. *For $N \geqslant 2$*

$$r(N)=\frac{\log 2}{2\zeta(2)}N^2+O(N^{3/2}\log^4 N).$$

Corollary 3. *Let $x \geqslant 1$. Then*

$$\sum_{\substack{\omega\in\mathscr{R}\\ \rho(\omega)\leqslant x}} 1=\frac{e^x\log 2}{2\zeta(2)}+O(x^4e^{3x/4}).$$

To evaluate the Gauss-Kuz'min statistics for the reduced quadratic irrationals, we introduce the quantity $r(x,y;N)$ for the reals $x,y\in[0,1]$ and for $N\geqslant 2$ (we assume that the sequence of partial quotients is extended to the negative indices by periodicity) as follows:

$$\begin{aligned} r(x,y;N) &= \sum_{\substack{\omega\in\mathscr{R}\\ \varepsilon_0(\omega)\leqslant N}} \frac{1}{\mathrm{per}_e(\omega)} \sum_{j=1}^{\mathrm{per}_e(\omega)} [[0;a_{j+1},a_{j+2},\dots]\leqslant x,\ [0;a_j,a_{j-1},\dots]\leqslant y] \\ &= \sum_{\substack{\omega\in\mathscr{R}\\ \varepsilon_0(\omega)\leqslant N}} \frac{1}{\mathrm{per}_e(\omega)} \sum_{j=1}^{\mathrm{per}_e(\omega)} \left[\omega_j\leqslant x,\ -\frac{1}{\omega_j^*}\leqslant y\right], \end{aligned}$$

where $\omega_j=T^j(\omega)=[0;a_{j+1},a_{j+2},\dots]$. In particular, $r(1,1;N)=r(N)$. For all equivalent numbers $\omega_j=T^j(\omega)$ we count the Gauss-Kuz'min statistics only once, and the sum $r(x,y;N)$ can also be represented in the form

$$r(x,y;N)=\sum_{\substack{\omega\in\mathscr{R}\\ \varepsilon_0(\omega)\leqslant N}} \left[\omega\leqslant x,\ -\frac{1}{\omega^*}\leqslant y\right],$$

that is, the sum $r(x,y;N)$ describes the behaviour in the mean of the closed geodesics on the modular surface.

Theorem 3. *Let $0\leqslant x,y\leqslant 1$ and $N\geqslant 2$. Then*

$$r(x,y;N)=\frac{\log(1+xy)}{2\zeta(2)}N^2+O(N^{3/2}\log^4 N).$$

Proof. Consider the sums

$$\widetilde{r}_k(x,y;N)=\sum_{\substack{\omega\in\mathscr{R}\\ \operatorname{tr}(\varepsilon_0^k(\omega))\leqslant N}}\left[\omega\leqslant x,\ -\frac{1}{\omega^*}\leqslant y\right]$$

and

$$\widetilde{r}(x,y;N)=\sum_{\substack{\omega\in\mathscr{R}\\ \operatorname{tr}(\varepsilon_0(\omega))\leqslant N}}\left[\omega\leqslant x,\ -\frac{1}{\omega^*}\leqslant y\right]=\widetilde{r}_1(x,y;N).$$

By definition, $\widetilde{r}_k(x,y;N)\leqslant\widetilde{r}_k(N)$. Thus, it follows from (5.4) that

$$\sum_{2\leqslant k\leqslant 2\log N}\widetilde{r}_k(x,y;N)=O(N\log N)$$

and

$$\widetilde{r}(x,y;N)=\sigma(x,y;N)+O(N\log N),$$

where

$$\sigma(x,y;N)=\sum_{1\leqslant k\leqslant 2\log N}\ \sum_{\substack{\omega\in\mathscr{R}\\ \operatorname{tr}(\varepsilon_0^k(\omega))\leqslant N}}\frac{1}{\omega^*}\left[\omega\leqslant x,\ -\frac{1}{\omega^*}\leqslant y\right].$$

By property 2° the sum which has arisen can be represented in the form

$$\sigma(x,y;N)=\sum_{\substack{(a_1,\dots,a_{2m})\\ \operatorname{Tr}(B^{a_1}\dots A^{a_{2m}})\leqslant N}}\bigl[[0;\overline{a_1,\dots,a_{2m}}]\leqslant x,\ [0;\overline{a_{2m},\dots,a_1}]\leqslant y\bigr].$$

A pair of positive integers (q,q') $(q\leqslant q')$ can be completed to a matrix $\left(\begin{smallmatrix}p&p'\\q&q'\end{smallmatrix}\right)\in\mathscr{M}$ in two ways at most. Therefore, the number of matrices

$$JB^{a_1}A^{a_2}\dots B^{a_{2m-1}}A^{a_{2m}}J=\begin{pmatrix}p&p'\\q&q'\end{pmatrix}$$

for which $q'\leqslant\sqrt{N}$ can be estimated by $O(N)$. For the families $(a_1,\dots,a_{2m})$ to which a matrix with $q'>\sqrt{N}$ corresponds we have

$$0<[0;\overline{a_1,\dots,a_{2m}}]-[0;a_1,\dots,a_{2m}]=[0;\overline{a_1,\dots,a_{2m}}]-\frac{p'}{q'}\leqslant\frac{1}{(q')^2}<\frac{1}{N},$$
$$0<[0;\overline{a_{2m},\dots,a_1}]-[0;a_{2m},\dots,a_1]=[0;\overline{a_{2m},\dots,a_1}]-\frac{q}{q'}\leqslant\frac{1}{(q')^2}<\frac{1}{N}.$$

Hence, on the one hand,

$$\sigma(x,y;N)\leqslant\sum_{\left(\begin{smallmatrix}p&p'\\q&q'\end{smallmatrix}\right)\in\mathscr{M}_+}\left[q'>\sqrt{N},\ \frac{p'}{q'}\leqslant x,\ \frac{q}{q'}\leqslant y,\ p+q'\leqslant N\right]+O(N)$$
$$=\sum_{\left(\begin{smallmatrix}p&p'\\q&q'\end{smallmatrix}\right)\in\mathscr{M}_+}\left[\frac{p'}{q'}\leqslant x,\ \frac{q}{q'}\leqslant y,\ p+q'\leqslant N\right]+O(N)=\Psi_{\mathrm{ev}}(x,y;N)+O(N), \tag{5.5}$$

On the other hand,

$$\sigma(x,y;N) \geqslant \sum_{\left(\begin{smallmatrix} p & p' \\ q & q' \end{smallmatrix}\right) \in \mathscr{M}_+} \left[q' > \sqrt{N},\ \frac{p'}{q'} \leqslant x - \frac{1}{N},\ \frac{q}{q'} \leqslant y - \frac{1}{N},\ p + q' \leqslant N\right] + O(N)$$

$$= \Psi_{\mathrm{ev}}\left(x - \frac{1}{N}, y - \frac{1}{N}; N\right) + O(N),$$

that is,

$$\Psi_{\mathrm{ev}}\left(x - \frac{1}{N}, y - \frac{1}{N}; N\right) + O(N) \ll \sigma(x,y;N) \ll \Psi_{\mathrm{ev}}(x,y;N) + O(N). \qquad (5.6)$$

If $\min\{x,y\} \leqslant N^{-1/2}$, then Theorem 3 follows from Theorem 1 and from the bound (5.5). If $\min\{x,y\} > N^{-1/2}$, then, by Theorem 1,

$$\Psi_{\mathrm{ev}}\left(x - \frac{1}{N}, y - \frac{1}{N}; N\right) = \Psi_{\mathrm{ev}}(x,y;N) + O(N^{3/2}\log^4 N). \qquad (5.7)$$

Combining the above relations (5.6) and (5.7), we obtain the asymptotic formula $\sigma(x,y;N) = \Psi_{\mathrm{ev}}(x,y;N) + O(N^{3/2}\log^4 N)$. Thus, by Theorem 1,

$$\widetilde{r}(x,y;N) = \Psi_{\mathrm{ev}}(x,y;N) + O(N^{3/2}\log^4 N) = \frac{\log(1+xy)}{2\zeta(2)}N^2 + O(N^{3/2}\log^4 N).$$

To complete the proof of the theorem, it remains to note that, by property 1°, the desired function $r(x,y;N)$ is connected with $\widetilde{r}(x,y;N)$ by the inequalities

$$\widetilde{r}\left(x,y;N - \frac{1}{2}\right) \leqslant r(x,y;N) \leqslant \widetilde{r}(x,y;N).$$

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