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SHORT COMMUNICATIONS

On the Three-Dimensional Vahlen Theorem

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1. INTRODUCTION

Let $\gamma^{(1)},\dots,\gamma^{(s)}$ be the basis nodes of a full-rank lattice

$$\Gamma=\{m_1\gamma^{(1)}+\dots+m_s\gamma^{(s)}:m_1,\dots,m_s\in\mathbb{Z}\}\subset\mathbb{R}^s.$$

The lattice is always assumed to be "in general position," namely, the coordinate hyperplanes contain no nodes of the lattice except for the origin. (For the general case, see [1].) Denote by G_s the finite group acting on $GL_s(\mathbb{R})$ and generated by the following elementary transformations: 1) change of signs of the elements of a row or a column; 2) transposition of rows or columns. Two matrices are said to be *equivalent* if one of them can be obtained from another by a transformation in G_s. Suppose that $\{\gamma^{(1)},\gamma^{(2)}\}$ is a basis of the two-dimensional lattice Γ for which there are no nonzero nodes $\gamma=(\gamma_1,\gamma_2)\in\Gamma$ such that

$$|\gamma_1|<\max\{|\gamma_1^{(1)}|,|\gamma_1^{(2)}|\},\qquad |\gamma_2|<\max\{|\gamma_2^{(1)}|,|\gamma_2^{(2)}|\}.$$

Voronoi proved (see [2]) that this is possible if and only if

$$\Phi\begin{pmatrix}\gamma_1^{(1)} & \gamma_1^{(2)}\\ \gamma_2^{(1)} & \gamma_2^{(2)}\end{pmatrix}=\begin{pmatrix}a_1 & -b_1\\ a_2 & b_2\end{pmatrix},\qquad \text{where}\quad 0\le b_1\le a_1,\quad 0\le a_2\le b_2,.$$

for some transformation $\Phi\in G_2$. A basis of a lattice is said to be a *Voronoi basis* if the matrix corresponding to this basis is equivalent to a matrix of the form $\left(\begin{smallmatrix}a_1 & -b_1\\ a_2 & b_2\end{smallmatrix}\right)$. As was noted in [3], Vahlen's theorem concerning the approximation of numbers by convergents (see [4], [5]) has the following interpretation in terms of lattices: for every Voronoi basis $\{\gamma^{(1)},\gamma^{(2)}\}$,

$$\min\{|\gamma_1^{(1)}\gamma_2^{(1)}|,|\gamma_1^{(2)}\gamma_2^{(2)}|\}\le\frac{1}{2}\det\Gamma.$$

In the same paper, a refinement of Vahlen's theorem was suggested,

$$|\gamma_1^{(1)}\gamma_2^{(1)}|+|\gamma_1^{(2)}\gamma_2^{(2)}|\le\det\Gamma, \tag{1}$$

which follows from the inequality

$$a_1b_2+a_2b_1-a_1a_2-b_1b_2=(a_1-b_1)(b_2-a_2)\ge0.$$

Avdeeva and Bykovskii [3] proved an analog of the inequality (1) for Minkowski bases (three-dimensional analogs of the Voronoi bases). Below we present a short simpler proof of the Avdeeva–Bykovskii theorem. Moreover, their result is extended to arbitrary minimal three-vector systems (for the definitions, see below).

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2. MINIMAL SYSTEMS AND MINKOWSKI BASES

For a nonempty point set $T \subset \mathbb{R}^3$, write

$$|T|_i = \max\{|x_i| : x = (x_1, x_2, x_3) \in T\}, \qquad i = 1, 2, 3,$$
$$\Pi(T) = [-|T|_1, |T|_1] \times [-|T|_2, |T|_2] \times [-|T|_3, |T|_3].$$

Let Γ be a full-rank lattice in $\mathbb{R}^3$. By a *system of nodes of order* t of the lattice Γ we mean an arbitrary finite family of nonzero nodes of Γ of the form $(\gamma^{(1)}, \dots, \gamma^{(t)})$, where $\gamma^{(i)} \neq \pm\gamma^{(j)}$, $1 \le i < j \le t$. A system S of the lattice Γ is said to be *minimal* if there are no nonzero nodes $\gamma \in \Gamma$ placed strictly inside $\Pi(S)$.

Minkowski proved [6], [7, Article 109] that every minimal system of three vectors $(\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})$ is either degenerate (and then $\gamma^{(1)} \pm \gamma^{(2)} \pm \gamma^{(3)} = 0$ for some combination of signs) or forms a basis of the lattice (*Minkowski basis*). Using elementary transformations, one can reduce the matrix of every Minkowski basis to one of the following canonical forms (see [6], [7], and also [8]–[10]):

$$\begin{pmatrix} a_1 & b_1 & -c_1 \\ -a_2 & b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{pmatrix}, \qquad \begin{pmatrix} a_1 & b_1 & c_1 \\ -a_2 & b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{pmatrix} \tag{2}$$

with nonnegative a_i, b_i, c_i, $i = 1, 2, 3$, such that

1) $\max\{b_1, c_1\} \le a_1$, $\max\{a_2, c_2\} \le b_2$, $\max\{a_3, b_3\} \le c_3$;
2) at least one of the inequalities $c_1 \le b_1$, $a_2 \le c_2$, $b_3 \le c_3$ holds for the matrices of the first kind;
3) the inequality $a_2 + c_2 \ge b_2$ and at least one of the inequalities $a_3 \le b_3$, $c_1 \le b_1$ hold for the matrices of the second kind.

Here, at the expense of redenoting the coordinates and the vectors, one can achieve the situation in which the three (for matrices of the first kind) or two (for matrices of the second kind) alternative inequalities are replaced by the single inequality $c_1 \le b_1$.

3. THREE-DIMENSIONAL VAHLEN THEOREM

For the above results related to the three-dimensional Vahlen's theorem and also for some its refinements, see [10]–[12].

Theorem 1 (Avdeeva, Bykovskii, 2006). *Let the nodes $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$ form a Minkowski basis of a lattice Γ. Then*

$$|\gamma_1^{(1)}\gamma_2^{(1)}\gamma_3^{(1)}| + |\gamma_1^{(2)}\gamma_2^{(2)}\gamma_3^{(2)}| + |\gamma_1^{(3)}\gamma_2^{(3)}\gamma_3^{(3)}| \le \det\Gamma. \tag{3}$$

Proof. For the matrix M whose columns are the vectors $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$, we claim that $\Delta(M) \ge 0$, where

$$\Delta(M) = \det M - |\gamma_1^{(1)}\gamma_2^{(1)}\gamma_3^{(1)}| - |\gamma_1^{(2)}\gamma_2^{(2)}\gamma_3^{(2)}| - |\gamma_1^{(3)}\gamma_2^{(3)}\gamma_3^{(3)}|.$$

The quantity $\Delta(M)$ is preserved under the action of the transformations in the group G_3. Therefore, it suffices to establish the bound $\Delta(M) \ge 0$ for bases whose matrices are of the form (2) and satisfy the inequality $c_1 \le b_1 \le a_1$. Since $b_2c_3 \ge a_2a_3$, it follows that the coefficient at a_1 in the expansion ($\varepsilon = -1$ for the matrices of the first kind and $\varepsilon = 1$ for the matrices of the second kind)

$$\Delta(M) = a_1(b_2c_3 + c_2b_3 - a_2a_3) + b_1(a_2c_3 + c_2a_3 - b_2b_3) + \varepsilon c_1(a_2b_3 - b_2a_3) - c_1c_2c_3$$

is nonnegative, and therefore it suffices to prove the inequality $\Delta(M) \ge 0$ for $a_1 = b_1$. Then

$$\Delta(M) = b_1(b_2(c_3 - b_3) + a_2(c_3 - a_3) + c_2(a_3 + b_3)) + \varepsilon c_1(a_2b_3 - b_2a_3) - c_1c_2c_3,$$

and the coefficient at b_1 is nonnegative. Thus, it suffices to restrict ourselves to the case in which $a_1 = b_1 = c_1$. For the matrices of the first kind ($\varepsilon = -1$),

$$\Delta(M) = c_1(a_3(b_2 - a_2) + a_2(c_3 - b_3) + (b_2 - c_2)(c_3 - b_3) + a_3c_2) \geq 0,$$

because all summands are nonnegative. For the matrices of the second kind ($\varepsilon = 1$),

$$\Delta(M) = c_1((b_2 - c_2)(c_3 - a_3) + a_2(c_3 - a_3) + b_3(a_2 + c_2 - b_2)) \geq 0,$$

because the additional condition $a_2 + c_2 \geq b_2$ holds. □

4. GENERALIZATION OF THE THREE-DIMENSIONAL VAHLEN THEOREM

The result of Theorem 1 admits the following refinement.

Theorem 2. *Inequality* (3) *holds for an any minimal system of three vectors* $\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}$.

Proof. With regard to Theorem 1, it is sufficient to carry out the proof for a degenerate minimal triple. Suppose that $(\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})$ is a degenerate system of three vectors and their signs are chosen in such a way that $\gamma^{(1)} + \gamma^{(2)} + \gamma^{(3)} = 0$. Then the matrix of the degenerate minimal triple can be reduced by elementary transformations to the form (see [7, Article 109])

$$\begin{pmatrix} a_1 & -b_1 & -c_1 \\ -a_2 & b_2 & -c_2 \\ -a_3 & -b_3 & c_3 \end{pmatrix} = \begin{pmatrix} b_1 + c_1 & -b_1 & -c_1 \\ -a_2 & a_2 + c_2 & -c_2 \\ -a_3 & -b_3 & a_3 + b_3 \end{pmatrix}$$

with nonnegative a_i, b_i, c_i, $i = 1, 2, 3$. The interior of the parallelepiped $[-a_1, a_1] \times [-b_2, b_2] \times [-c_3, c_3]$ contains no points of the lattice Γ that differ from the origin. By Minkowski's theorem on convex bodies, $\det \Gamma \geq a_1b_2c_3$. Hence

$$\begin{aligned} |\gamma_1^{(1)}\gamma_2^{(1)}\gamma_3^{(1)}| + |\gamma_1^{(2)}\gamma_2^{(2)}\gamma_3^{(2)}| + |\gamma_1^{(3)}\gamma_2^{(3)}\gamma_3^{(3)}| &= (b_1 + c_1)a_2a_3 + (a_2 + c_2)b_1b_3 + (a_3 + b_3)c_1c_2 \\ &\leq (b_1 + c_1)(a_2 + c_2)(a_3 + b_3) = a_1b_2c_3 \leq \det \Gamma. \end{aligned}$$ □

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