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Coefficient Rings of Tate Formal Groups Determining Krichever Genera

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On the anniversary of Vladimir Petrovich Platonov
with admiration for his outstanding results

Abstract—The paper is devoted to problems at the intersection of formal group theory, the theory of Hirzebruch genera, and the theory of elliptic functions. In the focus of our interest are Tate formal groups corresponding to the general five-parametric model of the elliptic curve as well as formal groups corresponding to the general four-parametric Krichever genus. We describe coefficient rings of formal groups whose exponentials are determined by elliptic functions of levels 2 and 3.

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1. INTRODUCTION

All rings considered in this paper will be supposed to be commutative rings with unity. We will use the following standard notation: $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the ring of rational numbers, $\mathbb{Z}_p$ is the ring of integer p-adic numbers, $\mathbb{F}_p$ is the prime field of order p, and Ω_{U} is the cobordism ring of stably complex manifolds.

The coefficient ring $\mathcal{R}_{\mathrm{U}}$ of the universal formal group $\mathcal{F}_{\mathrm{U}}(u,v) = u + v + \sum \alpha_{i,j} u^i v^j$, where $i + j \geq 2$, $i > 0$, and $j > 0$, is given by $\mathbb{Z}[\alpha_{i,j}]/J$, where J is the associativity ideal (see Section 3 below).

According to Lazard's theorem [14], the isomorphism $\mathcal{R}_{\mathrm{U}} = \mathbb{Z}[e_k\colon k = 1, 2, \ldots]$ holds (see [7] for a new proof). The fact that there is no canonical choice of multiplicative generators e_k, $k = 1, 2, \ldots$, gives rise to nontrivial problems of classifying formal groups over a given ring R up to a change of coordinates u and v over R (see the case $R = \mathbb{Z}$ in [12]).

Given a ring R, there is an important problem of constructing formal groups over R such that their coefficient ring is explicitly described in terms of multiplicative generators of the ring R. Each solution of this problem contains information on the coefficients $\alpha_{i,j}$ as polynomials in e_k.

According to the Milnor–Novikov theorem, the ring Ω_{U} is isomorphic to the ring $\mathbb{Z}[a_k\colon k = 1, 2, \ldots]$. In complex cobordism theory $\mathrm{U}^*(\cdot)$, the remarkable formal group of geometric cobordisms $\mathcal{F}_{\mathrm{U}}(u,v)$ (see [15]) over the ring Ω_{U} is defined. The homomorphism $\mathcal{R}_{\mathrm{U}} \to \Omega_{\mathrm{U}}$ classifying this group is an isomorphism (see [19]). Using this result, we will identify the rings $\mathcal{R}_{\mathrm{U}}$ and Ω_{U}.

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The connection between formal groups and cobordism theory has led to a deep interpenetration of ideas, methods, and results of these theories (see [2, 5, 11]).

One of the best known results of algebraic topology is the solution of the Milnor–Hirzebruch problem on relations between the Chern numbers of stably complex manifolds, which was obtained independently by Stong and Hattori (see [8, 20]).

Let us describe a result in formal group theory that is equivalent to the Stong–Hattori theorem on the characteristic numbers of stably complex manifolds.

Consider the ring $\mathcal{B} = \mathbb{Z}[a, b_1, \dots, b_n, \dots]$ and the series $b(x) = x + b_1x^2 + \dots + b_nx^{n+1} + \dots$. Using the formal group $u + v + auv$ in complex K-theory, let us introduce a formal group $F(u,v)$ over $\mathcal{B}$ by applying the relation

$$b(x + y + axy) = F(b(x), b(y)),$$

that is, by applying the general change $u = b(x)$, $v = b(y)$ of the coordinates u and v. The coefficients of the formal group $F(u,v)$ can be expressed explicitly as polynomials in $a, b_1, \dots, b_n, \dots$. The ring homomorphism $h\colon \Omega_{\mathrm{U}} \to \mathcal{B}$ classifying this group allows one to represent the image of the coefficients $\alpha_{i,j}$ of the universal formal group as polynomials in $a, b_1, \dots, b_n, \dots$ and to prove directly that h is a monomorphism onto a direct summand. This result is equivalent to Stong and Hattori's result stating that the characteristic numbers with values in complex K-theory describe all divisibility relations for the classical characteristic numbers of stably complex manifolds.

Let us formulate the results that form the basis of the present work (in parentheses we indicate the numbers of sections in which the results are presented in the required form).

In [3] the addition law for the Tate formal group $\mathcal{F}_{\mathrm{T}}(u,v)$ over the ring $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ and its exponential $f_{\mathrm{T}}(x)$ are described in terms of the Weierstrass $\wp$-functions (see Section 5). Thus, a five-parametric Hirzebruch genus $L_{f_{\mathrm{T}}}\colon \Omega_{\mathrm{U}} \to \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ is obtained, and in its applications one can use the classical theory of elliptic functions. We call this genus the Tate genus.

In [13] I.M. Krichever introduced the Hirzebruch genus in terms of the elliptic Baker–Akhiezer function (see Section 6), and it allowed him to obtain important algebraic–topological applications on the basis of analytic properties of this function. Note that all famous Hirzebruch genera turned out to be specializations of the Krichever genus.

In [3] all Tate genera that can be realized as Krichever genera are described.

In [1] the formal group of the form

$$F(u,v) = \frac{u^2A(v) - v^2A(u)}{uB(v) - vB(u)}$$

was introduced, and it was shown that the exponential of the universal formal group $\mathcal{F}_{\mathrm{B}}(u,v)$ of this form determines the Krichever genus (see Section 7).

In [7] the coefficient ring $\mathcal{R}_{\mathrm{B}}$ of the formal group $\mathcal{F}_{\mathrm{B}}(u,v)$ (see Section 7), as well as the coefficient rings of its important specializations, is described.

In Section 9 all specializations of the Tate formal group that are simultaneously specializations of $\mathcal{F}_{\mathrm{B}}(u,v)$ are described.

Denote by $\mathcal{S}_2$ the coefficient ring of the formal group $\mathscr{F}_2(u,v)$ that is a specialization $\{A(u) = 1\}$ of $\mathcal{F}_{\mathrm{B}}(u,v)$, and denote by $\mathcal{R}_2$ the coefficient ring of the specialization $\{\mu_k = 0,\ k = 1, 3, 6\}$ of the formal group $\mathcal{F}_{\mathrm{T}}(u,v)$.

Let $\mathcal{S}_3$ be the coefficient ring of the formal group $\mathscr{F}_3(u,v)$ that is a specialization $\{B(u) = A(u)^2\}$ of $\mathcal{F}_{\mathrm{B}}(u,v)$, and let $\mathcal{R}_3$ be the coefficient ring of the specialization $\{\mu_2 = -\mu_1^2,\ \mu_4 = -\mu_1\mu_3,\ 3\mu_6 = -\mu_3^2\}$ of $\mathcal{F}_{\mathrm{T}}(u,v)$.

Note that in [18] Von Oehsen found and expression for the exponential of the formal group $\mathscr{F}_3(u,v)$ in terms of Jacobi polynomials.

The main results of our study are as follows (see Sections 11 and 12):

The rings $\mathcal{S}_k$, $k = 2, 3$, are torsion-free, and the formal groups $\mathscr{F}_k(u, v)$ are determined by two-parametric elliptic Hirzebruch genera of level 2 and 3, respectively (see [13, 10]). Note that for $k = 2$ this is the famous Ochanine–Witten genus [17, 22].

There exist classifying homomorphisms $h_k\colon \mathcal{S}_k \to \mathcal{R}_k$, $k = 2, 3$, which are isomorphisms.

These results imply important corollaries in formal group theory and algebraic topology, because

- the ring $\mathcal{S}_2$, which is multiplicatively generated by e_n with $n = 2^r$, $r \geq 1$, is a subring in $\mathbb{Z}[\mu_2, \mu_4]$;
- the ring $\mathcal{S}_3$, which is multiplicatively generated by e_n with $n = 3^r$, $r \geq 0$, is a subring in $\mathbb{Z}[\mu_1, \mu_3, \mu_6]/\{\mu_3^2 = -3\mu_6\}$.

2. FORMAL GROUPS

A *one-dimensional formal group law* (or, briefly, a *formal group*) over a ring R is a formal series

$$F(u, v) = u + v + \sum_{i,j \geq 1} a_{i,j} u^i v^j \in R[[u, v]] \tag{2.1}$$

satisfying the associativity condition

$$F(u, F(v, w)) = F(F(u, v), w)$$

and the commutativity condition

$$F(u, v) = F(v, u).$$

Note that if the ring R has no nilpotent elements, then the associativity condition implies the commutativity condition (see [14]).

The *exponential* of a formal group $F(u, v)$ over R is a formal series $f(x) \in R \otimes \mathbb{Q}[[x]]$ determined uniquely by the addition law

$$f(x + y) = F(f(x), f(y)), \qquad f(0) = 0, \quad f'(0) = 1. \tag{2.2}$$

Thus, the substitution $u = f(x)$, $v = f(y)$ linearizes any law $F(u, v)$ over $R \otimes \mathbb{Q}$. The series $g(u)$ functionally inverse to $f(x)$ is called the *logarithm*.

If R is torsion-free, then the formal group $F(u, v)$ over R is uniquely recovered from its exponential $f(x)$ because of the relation

$$\frac{\partial}{\partial v} F(u, v)\bigg|_{v=0} = \frac{1}{g'(u)}. \tag{2.3}$$

It is said that $F(u, v)$ is linearized over R if the exponential $f(x)$ is defined over R. For example, if $F(u, u) = 0$, then the ring R is an algebra over the residue field $\mathbb{F}_2$ and there exists a series $f(x) \in R[[x]]$ such that equalities (2.2) hold (see [14, 9]).

3. UNIVERSAL FORMAL GROUPS

The formal group $\mathcal{F}_{\mathrm{U}}(u, v) = u + v + \sum \alpha_{i,j} u^i v^j$ over a ring $\mathcal{R}_{\mathrm{U}}$ is called a *universal formal group* if for any formal group $F(u, v)$ over any ring R there exists a unique homomorphism $h_F\colon \mathcal{R}_{\mathrm{U}} \to R$ such that $F(u, v) = u + v + \sum h_F(\alpha_{i,j}) u^i v^j$. The homomorphism h_F is said to be *classifying*.

Further we suppose that the ring homomorphism h_F is graded, that is, the ring R is graded and $\deg a_{i,j} = -2(i + j - 1)$.

In the ring R one can select a subring R_F of coefficients of the formal group $F(u, v)$. By definition, the ring R_F is multiplicatively generated by $a_{i,j} = h_F(\alpha_{i,j})$. In the case when $R = R_F$, the formal group $F(u, v)$ over R is said to be *generating*.

We will speak of a *formal group of a given form* if additional conditions are imposed on its law (along with associativity and commutativity). We will speak of a *universal formal group of a given form* if there exists a formal group of a given form $\mathcal{F}_{\mathrm{V}}(u,v)$ over $\mathcal{R}_{\mathrm{V}}$ such that for any formal group $F(u,v)$ of this form over R there exists a unique homomorphism $h_F^{\mathrm{V}}\colon \mathcal{R}_{\mathrm{V}} \to R$ such that $F(u,v) = u + v + \sum h_F^{\mathrm{V}}(h_{\mathrm{V}}(\alpha_{i,j}))u^i v^j$, where h_{V} is the classifying homomorphism for $\mathcal{F}_{\mathrm{V}}$. The homomorphism h_F^{V} will also be called *classifying*.

We will say that a formal group $F(u,v)$ over R is a *specialization* of a formal group $\mathcal{F}(u,v)$ over $\mathcal{R}$ if there exists a ring homomorphism $h_F^{\mathcal{F}}\colon \mathcal{R} \to R$ such that the ring homomorphism $h_F\colon \mathcal{R}_{\mathrm{U}} \to R$ can be decomposed as $h_F^{\mathcal{F}} h_{\mathcal{F}}$. In particular, each formal group of a given form is a specialization of a universal formal group of this form. Thus the ideal $I_F = \operatorname{Ker} h_F^{\mathcal{F}} \subset \mathcal{R}$ is defined. We will call it the *specialization ideal of the formal group* $F(u,v)$.

Let us recall the construction of the formal group $\mathcal{F}_{\mathrm{U}}(u,v)$. To this end, consider the ring $\mathcal{U} = \mathbb{Z}[\beta_{i,j}\colon i \le j \in \mathbb{N}]$ and the formal series $\widehat{F}(u,v) = u + v + \sum \beta_{i,j} u^i v^j$ with $\beta_{j,i} = \beta_{i,j}$. Set $\widehat{F}(\widehat{F}(u,v),w) - \widehat{F}(u,\widehat{F}(v,w)) = \sum r_{i,j,k} u^i v^j w^k$. Then $\mathcal{R}_{\mathrm{U}} = \mathcal{U}/J$, where $J \subset \mathcal{U}$ is the associativity ideal with generators $r_{i,j,k}$. For the canonical projection $\pi\colon \mathcal{U} \to \mathcal{R}_{\mathrm{U}}$ we obtain $\mathcal{F}_{\mathrm{U}}(u,v) = u + v + \sum \alpha_{i,j} u^i v^j$, where $\alpha_{i,j} = \pi(\beta_{i,j})$.

The classifying ring homomorphism $h_F\colon \mathcal{R}_{\mathrm{U}} \to R$ is determined by the homomorphism $\widehat{h}_F\colon \mathcal{U} \to R$, $\widehat{h}_F(\beta_{i,j}) = a_{i,j}$. The associativity condition implies that the homomorphism $\widehat{h}_F$ factors through $\mathcal{R}_{\mathrm{U}}$:

$$\mathcal{U} \to \mathcal{R}_{\mathrm{U}} \to R.$$

By Lazard's theorem (see [14] as well as [7]), the ring $\mathcal{R}_{\mathrm{U}}$ is isomorphic to $\mathbb{Z}[e_k\colon k = 1,2,\dots]$, where $\deg e_k = -2k$. The choice of generators e_k is not canonical. If the ring homomorphism $h_F\colon \mathcal{R}_{\mathrm{U}} \to R$ that classifies the formal group $F(u,v)$ is an epimorphism, then one can choose generators $e_1,\dots,e_n,\dots$ such that the set of nonzero elements $h_F(e_n)$ is a minimal set of multiplicative generators of the ring R. Further we will need the following two lemmas, each of which allows one to choose such a set of multiplicative generators.

Lemma 3.1 (corollary to Lazard's theorem). *Let a formal group $F(u,v)$ over the ring $R = \mathbb{Z}[\beta_k\colon k = 1,2,\dots]/J$, where $\deg\beta_k = -2k$ and J is a graded ideal, be generating. Then in the ring $\mathcal{R}_{\mathrm{U}}$ one can choose generators e_k such that $h_F(e_k) = \beta_k$.*

We present the construction from [7].

For natural numbers $m_1,\dots,m_k$ we denote by $(m_1,\dots,m_k)$ their greatest common divisor. Using the Euclidean algorithm, one can always find integers $\lambda_1,\dots,\lambda_k$ satisfying the equality

$$\lambda_1 m_1 + \ldots + \lambda_k m_k = (m_1,\dots,m_k).$$

For $n \ge 2$ define the number

$$d(n) = \left(\binom{n}{1},\binom{n}{2},\dots,\binom{n}{n-1}\right) = \begin{cases} p & \text{for } n = p^k, \text{ where } p \text{ is a prime number,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

For each $n \ge 1$ we fix a set of numbers $(\lambda_1,\dots,\lambda_n)$ satisfying the equality

$$\lambda_1\binom{n+1}{1} + \lambda_2\binom{n+1}{2} + \ldots + \lambda_n\binom{n+1}{n} = d(n+1). \tag{3.2}$$

For any formal group $F(u,v)$ over a ring R, this allows us to introduce elements $\epsilon_n \in R$ using the formula

$$\epsilon_n = \lambda_1 a_{1,n} + \lambda_2 a_{2,n-1} + \ldots + \lambda_n a_{n,1}. \tag{3.3}$$

Note that the set of numbers $(\lambda_1,\dots,\lambda_n)$ satisfying (3.2) is defined ambiguously.

Remark 3.2. If $n+1$ is a prime, one can set $(\lambda_1, \dots, \lambda_n) = (1, 0, \dots, 0)$ and $\epsilon_n = a_{1,n}$.

Lemma 3.3 (see [7, Remark 4.4])**.** *The generators e_k of the ring $\mathcal{R}_{\mathrm{U}}$ can be defined by formula* (3.3) *for the formal group $\mathcal{F}_{\mathrm{U}}(u,v)$. At the same time for any formal group $F(u,v)$ we get*

$$h_F(e_k) = \epsilon_k.$$

For $a \in R$, by $\widehat{a}$ we denote the element $\widehat{a} \in \widehat{R} = R/I^2$, where I^2 is the ideal in R generated by all elements of the form $a_{i_1,j_1} a_{i_2,j_2}$ with indices i_1, i_2, j_1, and j_2 each of which is not less than 1.

Definition 3.4. For a formal group $F(u,v)$ over a ring R, we define a function $\rho_F\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ such that $\rho_F(n)$ is the order of the element $\widehat{\epsilon}_n$ in the ring $\widehat{R}$.

Note that the value of $\rho_F(n)$ is independent of the choice of the set $(\lambda_1, \dots, \lambda_n)$ in (3.2).

In the case of a generating formal group $F(u,v)$ over R, the ring $R = R_F$ is multiplicatively generated by the set of all elements $\epsilon_k = h_F(e_k)$ such that $\rho_F(k) > 1$.

Denote by $\mathcal{F}_{\mathrm{U},2}(u,v)$ the universal formal group for the groups whose law $F(u,v)$ satisfies the condition $F(u,u) = 0$.

Theorem 3.5 [14, 9]**.** 1. *The formal group $\mathcal{F}_{\mathrm{U},2}(u,v)$ is defined over the ring $\mathcal{R}_{\mathrm{U},2} = \mathbb{F}_2[\gamma_2, \gamma_4, \dots, \gamma_n, \dots]$, where $n \neq 2^k - 1$ for $k \geq 1$.*

2. *The formal group $\mathcal{F}_{\mathrm{U},2}(u,v)$ is linearized over $\mathcal{R}_{\mathrm{U},2}$ by the exponential*

$$f(x) = x + \sum_{n \neq 2^k - 1} \gamma_n x^{n+1}.$$

4. HIRZEBRUCH GENUS

For the series $f(x) = x + \sum_{k=1}^{\infty} f_k x^{k+1}/(k+1)! \in R \otimes \mathbb{Q}[[x]]$, where $f_k \in R$, we set

$$L_{f,m} = L_f(\sigma_1, \dots, \sigma_m) = \prod_{i=1}^{m} \frac{x_i}{f(x_i)},$$

where σ_k is the kth elementary symmetric function of $x_1, \dots, x_m$. For $m = 3$ we have

$$L_f(\sigma_1, \sigma_2, \sigma_3) = 1 - \frac{1}{2} f_1 \sigma_1 + \left(\frac{1}{4} f_1^2 - \frac{1}{6} f_2\right)\sigma_1^2 - \left(\frac{1}{4} f_1^2 - \frac{1}{3} f_2\right)\sigma_2$$
$$- \left(\frac{1}{8} f_1^3 - \frac{1}{6} f_1 f_2 + \frac{1}{24} f_3\right)\sigma_1^3 + \left(\frac{1}{4} f_1^3 - \frac{5}{12} f_1 f_2 + \frac{1}{8} f_3\right)\sigma_1\sigma_2 - \left(\frac{1}{8} f_1^3 - \frac{1}{4} f_1 f_2 + \frac{1}{8} f_3\right)\sigma_3 + \dots.$$

The *complex Hirzebruch characteristic class* of a complex m-dimensional vector bundle $\xi \to B$ is the cohomology class

$$L_{f,m}(\xi) = L_f(c_1(\xi), \dots, c_m(\xi)) \in H^*(B; R \otimes \mathbb{Q}),$$

where $c_k(\xi)$ is the kth Chern class of the bundle ξ.

In the important special case of $c_1(\xi) = 0$, we get

$$L_f(0, c_2(\xi), \dots, c_m(\xi)) = 1 - \left(\frac{1}{4} f_1^2 - \frac{1}{3} f_2\right) c_2(\xi) - \left(\frac{1}{8} f_1^3 - \frac{1}{4} f_1 f_2 + \frac{1}{8} f_3\right) c_3(\xi)$$
$$+ \left(\frac{1}{16} f_1^4 - \frac{1}{6} f_1^2 f_2 + \frac{1}{12} f_2^2 + \frac{1}{24} f_1 f_3 - \frac{1}{60} f_4\right) c_2(\xi)^2$$
$$+ \left(-\frac{1}{16} f_1^4 + \frac{1}{6} f_1^2 f_2 - \frac{1}{18} f_2^2 - \frac{1}{12} f_1 f_3 + \frac{1}{30} f_4\right) c_4(\xi) + \dots.$$

A smooth closed manifold M^{2n} is called *stably complex* if for some integer $m \geq n$ there is a complex m-dimensional vector bundle $\xi \to M^{2n}$ and an isomorphism of real vector bundles

$$\rho\colon\ \tau(M^{2n}) \oplus \mathbb{R}^{2(m-n)} \to \xi$$

over M^{2n} is fixed. The fundamental cycle $\langle M^{2n}\rangle \in H_{2n}(M^{2n}, \mathbb{Z})$ corresponding to the canonical orientation of the bundle ξ is defined.

The *complex Hirzebruch genus $L_f(M^{2n})$ of a stably complex manifold M^{2n}* is determined by

$$L_f(M^{2n}) = (L_{f,n}(\xi), \langle M^{2n}\rangle) \in R \otimes \mathbb{Q}.$$

The correspondence $[M^{2n}] \to L_f(M^{2n})$ defines a ring homomorphism

$$L_f\colon\ \Omega_{\mathrm{U}} \to R \otimes \mathbb{Q},$$

which is called the *Hirzebruch genus of the series f*.

The Hirzebruch genus is said to be *R-integer* if it determines a ring homomorphism $\Omega_{\mathrm{U}} \to R$.

The universality of the formal group of complex cobordisms implies that each formal group $F(u,v)$ over a ring R determines an R-integer Hirzebruch genus $L_f\colon \Omega_{\mathrm{U}} \to R$, where $f(x) \in R \otimes \mathbb{Q}[[x]]$ is the exponential of this group.

Theorem 4.1 (A.S. Mishchenko [15]). *The logarithm $g(u)$ of the formal group of geometric cobordisms $\mathcal{F}_{\mathrm{U}}(u,v)$ satisfies the relation*

$$g(u) = u + \sum_{n=1}^{\infty} [\mathbb{C}\mathrm{P}^n] \frac{u^{n+1}}{n+1}.$$

Corollary 4.2 [16]. *For any Hirzebruch genus $L_f\colon \Omega_{\mathrm{U}} \to R \otimes \mathbb{Q}$, the formula*

$$u + \sum_{n=1}^{\infty} L_f(\mathbb{C}\mathrm{P}^n) \frac{u^{n+1}}{n+1} = g_f(u)$$

holds, where $g_f(f(x)) = x$. If the ring R is torsion-free and $L\colon \Omega_{\mathrm{U}} \to R$ is a ring homomorphism, then $L = L_f$, where $f(x)$ is a series such that

$$u + \sum L[\mathbb{C}\mathrm{P}^n] \frac{u^{n+1}}{n+1} = g_f(u).$$

5. TATE FORMAL GROUP

For a detailed description of the universal Tate formal group, see [3].

The *general Weierstrass model for the elliptic curve* in homogeneous coordinates $(X:Y:Z)$ is given by the equation

$$Y^2 Z + \mu_1 XYZ + \mu_3 YZ^2 = X^3 + \mu_2 X^2 Z + \mu_4 XZ^2 + \mu_6 Z^3, \tag{5.1}$$

which depends on five parameters $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_6)$.

A linear change of coordinates

$$(X:Y:Z) \mapsto \Big(X + \frac{\nu_2}{2} Z : 2Y + \mu_1 X + \mu_3 Z : Z\Big), \qquad \nu_2 = \frac{\mu_1^2 + 4\mu_2}{6},$$

brings it into the *standard Weierstrass model of the elliptic curve*, which is given in the neighborhood $(X:Y:Z)\colon Z \neq 0$ in Weierstrass coordinates $(x:y:1)$ by the equation

$$y^2 = 4x^3 - g_2 x - g_3, \tag{5.2}$$

where

$$g_2 = 3\nu_2^2 - 2\mu_1\mu_3 - 4\mu_4, \qquad g_3 = -\nu_2^3 + \nu_2\mu_1\mu_3 - \mu_3^2 + 2\nu_2\mu_4 - 4\mu_6. \tag{5.3}$$

The map $t \mapsto (x, y) = (\wp(t), \wp'(t))$, where $\wp(t)$ is the Weierstrass $\wp$-function with parameters g_2 and g_3, gives a uniformization of the curve (5.2).

In the neighborhood $(X : Y : Z)\colon Y \neq 0$ in Tate coordinates $(u : -1 : s)$ (see [3, 21]) the equation of the elliptic curve (5.1) takes the form

$$s = u^3 + \mu_1 us + \mu_2 u^2 s + \mu_3 s^2 + \mu_4 us^2 + \mu_6 s^3. \tag{5.4}$$

It can be seen directly from (5.4) that the coordinate s can be represented as a series $s(u) \in \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6][[u]]$. We have

$$s(u) = u^3 + \mu_1 u^4 + (\mu_2 + \mu_1^2)u^5 + (\mu_3 + 2\mu_1\mu_2 + \mu_1^3)u^6 + \dots .$$

Thus, the map $u \mapsto (u, s(u))$ determines a uniformization of the curve (5.4).

The classical geometric construction of the group structure on an elliptic curve in coordinates $(u, s(u))$ in the vicinity of $(0, 0)$ determines a formal group $\mathcal{F}_{\mathrm{T}}(u, v)$ over the ring $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$, which will be called the *Tate formal group.*

Theorem 5.1 (see [3]). *The addition law in the formal group $\mathcal{F}_{\mathrm{T}}(u, v)$ has the form*

$$\left(u + v - uv\frac{\mu_1 + \mu_3 m + (\mu_4 + 2\mu_6 m)k}{1 - \mu_3 k - \mu_6 k^2}\right)\frac{1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3}{(1 + \mu_2 n + \mu_4 n^2 + \mu_6 n^3)(1 - \mu_3 k - \mu_6 k^2)}, \tag{5.5}$$

where

$$m = \frac{s(u) - s(v)}{u - v}, \qquad k = \frac{us(v) - vs(u)}{u - v}, \qquad n = m + uv\frac{1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3}{1 - \mu_3 k - \mu_6 k^2}.$$

Modulo the ideal generated by $\mu_i\mu_j$, we have

$$\begin{aligned} \mathcal{F}_{\mathrm{T}}(u, v) \equiv u + v - uv\Big(&\mu_1 + \mu_2(u + v) + \mu_3(2u^2 + 3uv + 2v^2) \\ &+ 2\mu_4(u + v)(u^2 + uv + v^2) + 3\mu_6(u + v)(u^2 + uv + v^2)^2\Big). \end{aligned} \tag{5.6}$$

Thus, $2\mu_4$ and $3\mu_6$ lie in the coefficient ring $\mathcal{R}_{\mathrm{T}}$, but μ_4 and μ_6 do not. Therefore, the formal group $\mathcal{F}_{\mathrm{T}}(u, v)$ over the ring $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ is not generating, while $\mathcal{R}_{\mathrm{T}} \otimes \mathbb{Z}_p = \mathbb{Z}_p[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ for any prime $p > 3$.

The problem of describing the coefficient ring $\mathcal{R}_{\mathrm{T}} \subset \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ of the formal group $\mathcal{F}_{\mathrm{T}}(u, v)$ arises naturally. It is clear that the ring $\mathcal{R}_{\mathrm{T}}$ coincides with the image of the homomorphism classifying the formal group $\mathcal{F}_{\mathrm{T}}(u, v)$. Below, this problem will be solved for formal groups that play an important role in algebraic topology (see Theorem 9.1).

Corollary 5.2 [3].

$$\frac{\partial}{\partial v}\mathcal{F}_{\mathrm{T}}(u, v)\bigg|_{v=0} = 1 - \mu_1 u - \mu_2 u^2 - 2\mu_3 s(u) - 2\mu_4 us(u) - 3\mu_6 s(u)^2.$$

Theorem 5.3 [3]. *The exponential of the Tate formal group $\mathcal{F}_{\mathrm{T}}(u, v)$ is determined by the series $f_{\mathrm{T}}(t) \in \mathbb{Q}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6][[t]]$ that is the expansion at $t = 0$ of*

$$-2\frac{\wp(t) - \frac{1}{12}(4\mu_2 + \mu_1^2)}{\wp'(t) - \mu_1\wp(t) + \frac{1}{12}\mu_1(4\mu_2 + \mu_1^2) - \mu_3}, \tag{5.7}$$

where the expressions for the parameters g_2 and g_3 are given by (5.3).

Corollary 5.4.

$$f_{\mathrm{T}}(t) = t - \mu_1 \frac{t^2}{2} + \left(\mu_1^2 - 2\mu_2\right)\frac{t^3}{3!} - \left(\mu_1^3 - 8\mu_1\mu_2 + 12\mu_3\right)\frac{t^4}{4!} + \left(\mu_1^4 - 22\mu_1^2\mu_2 + 16\mu_2^2 + 36\mu_1\mu_3 - 48\mu_4\right)\frac{t^5}{5!} + \dots .$$

6. KRICHEVER GENUS

Let Λ be a lattice with generators $2\omega_1, 2\omega_2 \in \mathbb{C}$, where $\operatorname{Im}(\omega_2/\omega_1) > 0$.

The *Baker–Akhiezer function* is defined by the expression

$$\Phi(x,z) = \frac{\sigma(z-x)}{\sigma(x)\sigma(z)} e^{\zeta(z)x}, \tag{6.1}$$

where $\sigma(x)$ and $\zeta(x)$ are the Weierstrass functions for the elliptic curve $\mathbb{C}/\Lambda$.

The following properties uniquely determine $\Phi(x,z)$ as a function of x:

- for $x = 0$ it has a simple pole;
- for $x = z$ it has a simple zero;
- it has the following properties under shifts by periods:

$$\Phi(x + 2\omega_k, z) = \Phi(x,z)e^{2\zeta(z)\omega_k - 2\zeta(\omega_k)z}, \qquad k = 1, 2. \tag{6.2}$$

In [13] Krichever introduced the Hirzebruch genus L_f, which is now called the *Krichever genus*, where

$$f(x) = f_{\mathrm{Kr}}(x) = \frac{\exp(\alpha x)}{\Phi(x,z)} = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} \exp(\alpha x - \zeta(z)x), \tag{6.3}$$

and showed that it has important algebraic–topological properties. Note that the series $f_{\mathrm{Kr}}(x)$ depends on four parameters: z, α, ω_1, and ω_2. The parameters ω_1 and ω_2 of the lattice Λ can be replaced by the parameters of the elliptic curve (5.2) uniformized by the Weierstrass $\wp$-functions corresponding to the lattice Λ. We have $f_{\mathrm{Kr}}(x) \in \mathbb{Q}[\alpha, \wp(z), \wp'(z), g_2][[x]]$ (see [3, 6]). Namely,

$$f_{\mathrm{Kr}}(x) = x + \alpha x^2 + \left(\alpha^2 + \wp(z)\right)\frac{x^3}{2} + \left(\alpha^3 + 3\alpha\wp(z) - \wp'(z)\right)\frac{x^4}{3!} + \left(\alpha^4 + 6\alpha^2\wp(z) - 4\alpha\wp'(z) + 9\wp(z)^2 - \frac{3}{5}g_2\right)\frac{x^5}{4!} + \dots .$$

The notion of elliptic genus of level N was introduced by F. Hirzebruch in [10] in connection with a well-known problem in algebraic topology. For a natural N, a lattice Λ, and $z \in \mathbb{C}$, consider an elliptic function $g(x)$ with divisor $N\cdot 0 - N\cdot z$ and condition that the coefficient of x^N in its series expansion in the vicinity of $x = 0$ is equal to 1. From the theory of elliptic functions it follows that such a function $g(x)$ exists and is unique for given (N, Λ, z) such that $z \notin \Lambda$ and $Nz \in \Lambda$, that is, $N > 1$. Set $f_N(x) = g(x)^{1/N}$ with the condition that $f_N'(0) = 1$. The Hirzebruch genus for the series $f_N(x)$ is called the *elliptic genus of level* N.

The elliptic genus of level 2 is defined by the Jacobi elliptic sine and is the *Ochanine–Witten genus.*

Krichever in [13] realized $f_N(x)$ in terms of the Baker–Akhiezer function (6.1); that is, he showed that the elliptic genus of level N is a specialization of the Krichever genus and is determined by the conditions

$$z = 2\frac{n}{N}\omega_1 + 2\frac{m}{N}\omega_2, \qquad \alpha = -2\frac{n}{N}\zeta(\omega_1) - 2\frac{m}{N}\zeta(\omega_2) + \zeta(z),$$

$n, m \in \{0, 1, \dots, N-1\}$, $(n,m) \neq (0,0)$. It follows from (6.2) that for shifts by the period $2\omega_k$ the function $f_N(x)$ is multiplied by an Nth root of 1.

7. BUCHSTABER FORMAL GROUP

A formal group $F(u,v)$ over a ring R is called a *Buchstaber formal group* if it can be presented in the form

$$F(u,v)=\frac{u^2A(v)-v^2A(u)}{uB(v)-vB(u)}, \tag{7.1}$$

where

$$A(u)=1+\sum_{i=1}^{\infty}a_iu^i,\quad a_i\in R,\qquad B(u)=1+\sum_{i=1}^{\infty}b_iu^i,\quad b_i\in R.$$

Formal groups of the form (7.1) were introduced in [1].

Let us describe the construction of the universal Buchstaber formal group.

A formal group of the form (7.1) over a ring R is determined by its coefficients $a_k\in R$, $k\neq 2$, and $b_m\in R$, $m\neq 1$. Denote by $\mathcal{F}_{\mathrm{B}}(u,v)$ the formal group (7.1) over the ring $\mathcal{R}_{\mathrm{B}}=\mathbb{Z}[a_k,\, k\neq 2,\, b_m,\, m\neq 1]/J$, where J is the associativity ideal. Its coefficients $a_{i,j}$ (see (2.1)) satisfy the relations (see [7, formulas (19)])

$$\begin{aligned} &a_{1,1}\equiv a_1,\\ &a_{i,1}\equiv a_{1,i}\equiv b_i,\qquad i>1,\\ &a_{i,j}\equiv a_{j,i}\equiv 2b_{i+j-1}-a_{i+j-1},\qquad i,j>1 \end{aligned} \tag{7.2}$$

(hereafter all congruences are considered modulo the ideal I^2), so the formal group $\mathcal{F}_{\mathrm{B}}(u,v)$ over $\mathcal{R}_{\mathrm{B}}$ is generating. Therefore, for any formal group of the form (7.1) over R, the classifying homomorphism $h\colon\mathcal{R}_{\mathrm{U}}\to R$ splits into a composition with a ring homomorphism $\mathcal{R}_{\mathrm{B}}\to R$; i.e., the formal group $\mathcal{F}_{\mathrm{B}}(u,v)$ over the ring $\mathcal{R}_{\mathrm{B}}$ is a universal formal group of the form (7.1).

Theorem 7.1 [7]**.** *The ring $\mathcal{R}_{\mathrm{B}}$ is multiplicatively generated by $h_{\mathrm{B}}(e_n)$, where $n=1,2,3,4$, $n=p^r$ with $r\geq 1$ and p ranging over all primes, and $n=2^k-2$ with $k>2$. In addition,*

$$\rho_{\mathrm{B}}(n)=\begin{cases}\infty & \text{for } n=1,2,3,4,\\ p & \text{for } n=p^r,\ n\neq 2,3,4,\ r\geq 1,\\ 2 & \text{for } n=2^k-2,\ k\geq 3,\\ 1 & \text{in all other cases}\end{cases}$$

(*see Definition* 3.4)*. The ring $\mathcal{R}_{\mathrm{B}}$ has only* 2*-torsion. In the ring $\mathcal{R}_{\mathrm{U}}$ there are multiplicative generators e_{2^k-2}, $k>2$, such that $h_{\mathrm{B}}(e_{2^k-2})$ generate the ideal of elements of order* 2 *in $\mathcal{R}_{\mathrm{B}}$.*

In [1] it was shown that the exponential of the formal group $\mathcal{F}_{\mathrm{B}}(u,v)$ is given by (6.3). Thus, the universal formal group of the form (7.1) determines the Krichever genus.

Further we will describe the universal formal group of the form (7.1) with the condition $F(u,u)=0$ over a ring with 2-torsion and show that it determines a new Hirzebruch genus.

8. COEFFICIENT RING OF THE BUCHSTABER FORMAL GROUP WITH THE CONDITION $A(u)=B(u)$

Theorem 8.1 [7, Theorem 7.1]**.** *The coefficient ring $\mathcal{S}_1$ of the universal formal group of the form*

$$\mathscr{F}_1(u,v)=\frac{u^2A(v)-v^2A(u)}{uA(v)-vA(u)}, \tag{8.1}$$

where

$$A(u) = 1 + \sum_{i=1}^{\infty} a_i u^i, \qquad a_i \in R,$$

is multiplicatively generated by ϵ_n, *where* $n = 1$ *and* $n = 2^r - 2$, $r \geq 2$. *Moreover,*

$$\rho_1(n) = \begin{cases} \infty & \text{for } n = 1, 2, \\ 2 & \text{for } n = 2^k - 2, \ k \geq 3, \\ 1 & \text{in all other cases.} \end{cases}$$

In dimensions $n = 2^k - 2$, $k \geq 3$, *the generators* e_n *of the ring* $\mathcal{R}_{\mathrm{U}}$ *can be chosen so that the equalities* $2h_1(e_n) = 0$ *hold.*

Theorem 8.2. *The coefficient ring* $\mathcal{S}_1$ *of the universal formal group of the form* (8.1) *has the form* $\mathbb{Z}[a_1, a_2, a_6, a_{14}, \ldots]/J$, *where the ideal* J *is generated by the relations*

$$2a_{2^k-2} = 0, \qquad k \geq 3, \tag{8.2}$$

$$a_1 a_{2^k-2} = 0, \qquad k \geq 3. \tag{8.3}$$

In particular, under the additional condition $a_1 = 0$, *the ideal* J *is generated by* (8.2).

Proof. Set $\mathscr{N}(u, v, w) = \mathscr{F}_1(\mathscr{F}_1(u, v), w) - \mathscr{F}_1(u, \mathscr{F}_1(v, w))$. As the coefficient of vw in the series $\mathscr{N}(u, v, w)$ is equal to

$$A(u)\left(\frac{2}{u}(A(u) - 1) - a_1 - A'(u)\right)$$

and $A(0) = 1$, the equality

$$A'(u) = \frac{2}{u}(A(u) - 1) - a_1 \tag{8.4}$$

holds in the ring $\mathcal{S}_1$. Substituting it into the expression for the coefficient of vw^2 in the series $\mathscr{N}(u, v, w)$, we arrive at the relation

$$2\frac{A(u)}{u^2}\big(A(u) - 1 - a_1 u - a_2 u^2\big) = 0,$$

which implies that

$$2A(u) = 2(1 + a_1 u + a_2 u^2). \tag{8.5}$$

Hence, in the ring $\mathcal{S}_1$ the relations $2a_k = 0$ ($k \geq 3$) and, in particular, relation (8.2) hold.

From (8.4) and (8.5) it follows that

$$A'(u) = a_1 + 2a_2 u. \tag{8.6}$$

Consequently, $a_{2l+1} = 0$, $l \geq 1$. From formula (8.6) and the expression for the coefficient of $v^2 w$ in the series $\mathscr{N}(u, v, w)$ we obtain the equality

$$u^2 \frac{A''(u)}{2} = A(u) - 1 - a_1 u. \tag{8.7}$$

Consequently, $a_{4s} = 0$, $s \geq 1$. From (8.6) and (8.7) we find that the expression for the coefficient of $v^2 w^2$ in the equation $\mathscr{N}(u, v, w) = 0$ gives the equality

$$\frac{A(u)}{u}\big(2 - 8A(u) + 3a_1 u\big)\big(1 + a_1 u + a_2 u^2 - A(u)\big) = 0. \tag{8.8}$$

In the resulting equation the last factor has the form $-(a_3u^3 + a_4u^4 + \ldots)$. Therefore, relations (8.2) imply that formula (8.8) is equivalent to the equality

$$a_1(a_3u^3 + a_4u^4 + \ldots) = 0,$$

from which relations (8.3) follow.

If the associativity ideal contains a relation that does not follow from (8.2) and (8.3), then such a relation cannot contain a_1 (a_1 and a_2 enter the ring $\mathcal{S}_1$ freely, so the relation cannot contain terms that depend only on a_1 and a_2, and, in addition, $a_1a_k = 0$ for $k > 2$). Since the ring $\mathcal{S}_1$ contains the subring $\mathbb{F}_2[a_2, a_6, a_{14}, \ldots]$ (see Corollary 15.2), by Theorem 8.1 the relation should be of the form $2P(a_2, a_6, a_{14}, \ldots) = 0$. From (8.2) it follows that such a relation can be brought to the form $2a_2^m = 0$, which is impossible due to the fact that a_2 enters the ring freely. $\square$

9. SPECIALIZATIONS OF THE TATE FORMAL GROUP THAT GIVE BUCHSTABER FORMAL GROUPS

In this section we describe all those Tate formal groups over rings without zero divisors that define Buchstaber formal groups. In Lemma 9.3 we describe an ideal $\mathcal{J}$ in the ring $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ such that if a specialization of a Tate formal group is a Buchstaber formal group, then its specialization ideal $\mathcal{I}$ contains $\mathcal{J}$. In Lemma 9.5 we find all minimal prime ideals $\mathcal{I}_k$, $k = 1, 2, 3, 4$, containing $\mathcal{J}$. In Lemmas 9.6–9.9 we show that Tate formal groups over the rings $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]/\mathcal{I}_k$ are Buchstaber formal groups. Thus, in this section we prove the following theorem:

Theorem 9.1 (cf. [3])**.** *A specialization of a Tate formal group over a ring without zero divisors gives a Buchstaber formal group if and only if its specialization ideal $\mathcal{I}$ contains at least one of the ideals*

$$\begin{aligned}
\mathcal{I}_1 &= \{\mu_3 = 0,\ \mu_4 = 0,\ \mu_6 = 0\},\\
\mathcal{I}_2 &= \{\mu_1 = 0,\ \mu_3 = 0,\ \mu_6 = 0\},\\
\mathcal{I}_3 &= \{\mu_1^2 + \mu_2 = 0,\ \mu_1\mu_3 + \mu_4 = 0,\ \mu_3^2 + 3\mu_6 = 0\},\\
\mathcal{I}_4 &= \{2 = 0,\ \mu_1 = 0,\ \mu_3 = 0\}.
\end{aligned}$$

Lemma 9.2. *Let a specialization of a Tate formal group be a Buchstaber formal group. Then*

$$\begin{aligned}
A(u) &= 1 - \mu_1u + a_2u^2 - \mu_3s(u) + (\mu_2\mu_3 - \mu_1\mu_4)u^2s(u)\\
&\quad + (\mu_3^2 + 3\mu_6)s(u)^2 + (\mu_3\mu_4 - 3\mu_1\mu_6)us(u)^2, \qquad (9.1)\\
B(u) &= 1 + b_1u - \mu_2u^2 - 2\mu_3s(u) - 2\mu_4us(u) - 3\mu_6s(u)^2.
\end{aligned}$$

Proof. According to Corollary 5.2,

$$\frac{\partial}{\partial v}\mathcal{F}_{\mathrm{T}}(u,v)\bigg|_{v=0} = 1 - \mu_1u - \mu_2u^2 - 2\mu_3s(u) - 2\mu_4us(u) - 3\mu_6s(u)^2.$$

On the other hand,

$$\frac{\partial}{\partial v}\mathcal{F}_{\mathrm{B}}(u,v)\bigg|_{v=0} = B(u) - b_1u + a_1u. \qquad (9.2)$$

Thus, under the conditions of the lemma we obtain $a_1 = -\mu_1$ and

$$B(u) = 1 + b_1u - \mu_2u^2 - 2\mu_3s(u) - 2\mu_4us(u) - 3\mu_6s(u)^2.$$

For a formal group $F(u,v)$ consider the expression

$$\left(-\frac{u}{2}\frac{\partial^2}{\partial v^2}F(u,v)+\left(\frac{\partial}{\partial v}F(u,v)\right)^2-ku\frac{\partial}{\partial v}F(u,v)\right)\Big|_{v=0}, \tag{9.3}$$

where $k=\frac{\partial^2}{\partial u\,\partial v}F(u,v)\big|_{u=0,\,v=0}$. Since the operator $\frac{1}{2}\frac{d^2}{dv^2}$ defines a homomorphism $\mathbb{Z}[[v]]\to\mathbb{Z}[[v]]$, expression (9.3) for the series $F(u,v)\in R[[u,v]]$ determines a series in $R[[u]]$. For $F(u,v)=\mathcal{F}_{\mathrm{B}}(u,v)$ this series is equal to

$$A(u)-a_2u^2+b_2u^2.$$

For $F(u,v)=\mathcal{F}_{\mathrm{T}}(u,v)$, from the form of (5.5), using the conditions $s(0)=0$, $s'(0)=0$, and $s''(0)=0$, we find that this series is equal to

$$1-\mu_1u-\mu_2u^2-\mu_3s(u)+\mu_4u\big(u^3-s(u)+\mu_2u^2s(u)+2\mu_3s(u)^2+\mu_4us(u)^2+\mu_6s(u)^3\big)$$
$$+\mu_2\mu_3u^2s(u)+\mu_3^2s(u)^2+3\mu_6s(u)\big(u^3+\mu_2u^2s(u)+\mu_3s(u)^2+\mu_4us(u)^2+\mu_6s(u)^3\big).$$

In view of relation (5.4),

$$s(u)=u^3+\mu_1us(u)+\mu_2u^2s(u)+\mu_3s(u)^2+\mu_4us(u)^2+\mu_6s(u)^3,$$

this series takes the form

$$1-\mu_1u-\mu_2u^2-\mu_3s(u)+(\mu_2\mu_3-\mu_1\mu_4)u^2s(u)+(\mu_3^2+3\mu_6)s(u)^2+(\mu_3\mu_4-3\mu_1\mu_6)us(u)^2.$$

Thus, under the conditions of the lemma, we obtain the expression for the series $A(u)$. □

Lemma 9.3. *Denote by $\mathcal{J}$ the ideal generated by the relations*

$$\mu_2\mu_3-\mu_1\mu_4=0,\qquad 2(\mu_3^2+3\mu_6)=0,$$
$$\mu_1(\mu_3^2+3\mu_6)=0,\qquad \mu_3(\mu_1\mu_3+\mu_4)=0,$$
$$4\mu_6(\mu_1^2+\mu_2)=0,\qquad \mu_3(\mu_3^2+3\mu_6)=0,$$
$$2\mu_6(\mu_1^2\mu_2+\mu_2^2+\mu_1\mu_3+\mu_4)=0,\qquad 2\mu_6(\mu_1^2+\mu_2)^2=0.$$

Let a specialization of a Tate formal group be a Buchstaber formal group. Then the specialization ideal $\mathcal{I}$ contains the ideal $\mathcal{J}$.

Remark 9.4. In other words, any specialization of a Tate formal group that gives a Buchstaber formal group is a specialization of a Tate formal group over the ring $\mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6]/\mathcal{J}$, where the ideal $\mathcal{J}$ is given by the relations of Lemma 9.3.

Proof of Lemma 9.3. Consider a formal series $F(u,v)$ of the form (7.1), where $A(u)$ and $B(u)$ are defined by relations (9.1) for $s(u)$ given by (5.4). Consider the series $\mathcal{F}_{\mathrm{T}}(u,v)-F(u,v)=\sum_{i,j}\gamma_{i,j}u^iv^j$, where $\mathcal{F}_{\mathrm{T}}(u,v)$ is given by (5.5). Lemma 9.2 implies that for the formal group $\mathcal{F}_{\mathrm{T}}(u,v)$ to be a Buchstaber formal group, it is necessary that $\gamma_{i,j}=0$ for all i and j.

Using the recursion for the coefficients of the series $s(u)$ given by equation (5.4), we obtain the expansion

$$s(u)=u^3+\mu_1u^4+\big(\mu_1^2+\mu_2\big)u^5+\big(\mu_1^3+2\mu_1\mu_2+\mu_3\big)u^6+\big(\mu_1^4+3\mu_1^2\mu_2+\mu_2^2+3\mu_1\mu_3+\mu_4\big)u^7$$
$$+\big(\mu_1^5+4\mu_1^3\mu_2+3\mu_1\mu_2^2+6\mu_1^2\mu_3+3\mu_2\mu_3+3\mu_1\mu_4\big)u^8$$
$$+\big(\mu_1^6+5\mu_1^4\mu_2+6\mu_1^2\mu_2^2+\mu_2^3+10\mu_1^3\mu_3+12\mu_1\mu_2\mu_3+2\mu_3^2+6\mu_1^2\mu_4+3\mu_2\mu_4+\mu_6\big)u^9+\dots. \tag{9.4}$$

Substituting the coefficients of the series $s(u)$ from (9.4) into $\gamma_{i,j} = 0$ for $i + j \le 11$, we obtain the assertion of the lemma. $\square$

Lemma 9.5. *Let $\mathcal{I}$ be a prime ideal containing $\mathcal{J}$. Then $\mathcal{I} \supset \mathcal{I}_k$ for one of the prime ideals $\mathcal{I}_k$ (see Theorem 9.1).*

Proof. Consider three cases:

1. Let $\mu_1 = 0$ and $\mu_3 = 0$. The relations of Lemma 9.3 in the ring R without zero divisors then take the form

$$6\mu_6 = 0, \qquad 4\mu_2\mu_6 = 0, \qquad 2\mu_6(\mu_2^2 + \mu_4) = 0, \qquad 2\mu_2^2\mu_6 = 0,$$

which implies that either $\mu_6 = 0$ and $\mathcal{I} \supset \mathcal{I}_2$, or $2 = 0$ and $\mathcal{I} \supset \mathcal{I}_4$, or $\mu_2 = \mu_4 = 3 = 0$ and $\mathcal{I} \supset \mathcal{I}_3$.

2. Let $\mu_1 \ne 0$ and $\mu_3 = 0$. The relations of Lemma 9.3 in the ring R without zero divisors take the form

$$\mu_4 = 0, \qquad 3\mu_6 = 0, \qquad \mu_6(\mu_1^2 + \mu_2) = 0,$$

which implies that either $\mu_6 = 0$ and $\mathcal{I} \supset \mathcal{I}_1$ or $\mu_1^2 + \mu_2 = 0$ and $\mathcal{I} \supset \mathcal{I}_3$.

3. Let $\mu_3 \ne 0$. Among the relations of Lemma 9.3 in the ring R without zero divisors, there are relations

$$\mu_2\mu_3 - \mu_1\mu_4 = 0, \qquad \mu_1\mu_3 + \mu_4 = 0, \qquad \mu_3^2 + 3\mu_6 = 0;$$

from the two first relations we obtain $\mu_1^2 + \mu_2 = 0$, and so $\mathcal{I} \supset \mathcal{I}_3$. $\square$

Lemma 9.6. *The specialization of a Tate formal group with specialization ideal $\mathcal{I}_1$ is defined over the ring $\mathbb{Z}[\mu_1, \mu_2]$ and has the form*

$$\mathcal{F}_1(u, v) = \frac{u + v - \mu_1 uv}{1 + \mu_2 uv}. \tag{9.5}$$

It can be presented in the form (7.1) with $A(u) = B(u) = 1 - \mu_1 u - \mu_2 u^2$. This formal group determines the two-parametric Todd genus.

Lemma 9.7. *The specialization of a Tate formal group with specialization ideal $\mathcal{I}_2$ is defined over the ring $\mathbb{Z}[\mu_2, \mu_4]$ and has the form*

$$\mathcal{F}_2(u, v) = \frac{u\sqrt{1 - 2\delta v^2 + \varepsilon v^4} + v\sqrt{1 - 2\delta u^2 + \varepsilon u^4}}{1 - \varepsilon u^2 v^2}, \tag{9.6}$$

where $\delta = \mu_2$ and $\varepsilon = \mu_2^2 - 4\mu_4$. It can be presented in the form (7.1) with $A(u) = 1$ and $B(u) = \sqrt{(1 - \mu_2 u^2)^2 - 4\mu_4 u^4}$. Note that $B(u) = 1 - \mu_2 u^2 - 2\mu_4 u^4 - 2\mu_2\mu_4 u^6 + \ldots \in \mathbb{Z}[\mu_2, \mu_4][[u]]$. This formal group determines the Ochanine–Witten elliptic genus.

Lemma 9.8. *The specialization of a Tate formal group with specialization ideal $\mathcal{I}_3$ is defined over the ring $\mathbb{Z}[\mu_1, \mu_3, \mu_6]/\{3\mu_6 = -\mu_3^2\}$ and has the form*

$$\mathcal{F}_3(u, v) = \frac{u^2(1 - \mu_1 v - \mu_3 s(v)) - v^2(1 - \mu_1 u - \mu_3 s(u))}{u(1 - \mu_1 v - \mu_3 s(v))^2 - v(1 - \mu_1 u - \mu_3 s(u))^2}, \tag{9.7}$$

where $s(u) = u^3 + \ldots$ is determined by the equation

$$s(u) = u^3 + \mu_1 u s(u) - \mu_1^2 u^2 s(u) + \mu_3 s(u)^2 - \mu_1\mu_3 u s(u)^2 + \mu_6 s(u)^3.$$

It can be presented in the form (7.1) with $A(u) = 1 - \mu_1 u - \mu_3 s(u)$ and $B(u) = A(u)^2$. This formal group determines the elliptic genus of level 3.

Proof of Lemmas 9.6–9.8. It was shown in [3] that a Tate formal group under the condition that the relations of at least one of the ideals $\mathcal{I}_1$, $\mathcal{I}_2$, or $\mathcal{I}_3$ hold is a Krichever formal group; that is, it can be presented as a specialization of the *universal Krichever formal group*

$$\mathcal{F}_{\mathrm{Kr}}(u,v)=ub(v)+vb(u)-b'(0)uv+\frac{b(u)\beta(u)-b(v)\beta(v)}{(ub(v)-vb(u))}u^2v^2, \tag{9.8}$$

where $b(u)=1+\sum b_iu^i$, $\beta(u)=(b'(u)-b'(0))/(2u)$, $b_1=\chi_1$, $b_{2i}=\chi_{2i}$, and $b_{2i+1}=2\chi_{2i+1}$, over the ring $\mathcal{B}=\mathbb{Z}[\chi_k\colon k=1,2,\dots]/J$, where J is the associativity ideal. The formal group (9.8) is a Buchstaber formal group with $B(u)=b(u)$ and

$$A(u)=b(u)^2-b'(0)ub(u)-u^2b(u)\beta(u).$$

Consequently, provided that the relations of at least one of the ideals $\mathcal{I}_1$, $\mathcal{I}_2$, or $\mathcal{I}_3$ hold, the Tate formal group is a Buchstaber formal group. By Lemma 9.2, selecting values for a_2 and b_1, we obtain the expressions for $A(u)$ and $B(u)$ indicated in the lemmas. In the case of Lemma 9.7, we obtain $B(u)=1-\mu_2u^2-2\mu_4us(u)$ and additionally use relation (5.4), which takes the form $s(u)=u^3+\mu_2u^2s(u)+\mu_4us(u)^2$, to obtain the relation indicated for $B(u)^2$.

For Lemma 9.6 we obtain the expression for the formal group that determines the two-parametric Todd genus:

$$\mathcal{F}_1(u,v)=\frac{u^2(1-\mu_1v)-v^2(1-\mu_1u)}{u(1-\mu_2v^2)-v(1-\mu_2u^2)}.$$

The equality of this expression and its better known form (9.5) is verified by direct calculation.

For Lemma 9.7, by substituting the values of δ and ε from the lemma, we obtain an expression for the formal group that determines the Ochanine–Witten elliptic genus:

$$\mathcal{F}_1(u,v)=\frac{u^2-v^2}{u\sqrt{1-2\delta v^2+\varepsilon v^4}-v\sqrt{1-2\delta u^2+\varepsilon u^4}}.$$

The equality of this expression and its better known form (9.6) is verified by direct calculation.

For Lemma 9.8 we obtain the expression (9.7). As follows from [4], the Tate formal group with relations $\mathcal{I}_3$ determines the elliptic genus of level 3. $\square$

Lemma 9.9. *The specialization of a Tate formal group with specialization ideal* $\mathcal{I}_4$ *is defined over the ring* $\mathbb{F}_2[\mu_2,\mu_4,\mu_6]$ *and has the form*

$$\mathcal{F}_4(u,v)=\frac{u^2(1+\mu_6s(v)^2)+v^2(1+\mu_6s(u)^2)}{u(1+\mu_2v^2+\mu_6s(v)^2)+v(1+\mu_2u^2+\mu_6s(u)^2)}, \tag{9.9}$$

where $s(u)=u^3+\dots$ *is determined by the equation* $s(u)=u^3+\mu_2u^2s(u)+\mu_4us(u)^2+\mu_6s(u)^3$. *It can be presented in the form* (7.1) *with* $A(u)=B(u)=1+\mu_2u^2+\mu_6s(u)^2$.

Proof. In Lemma 9.2, under the conditions of Lemma 9.9, we obtain

$$A(u)=1+a_2u^2+\mu_6s(u)^2,\qquad B(u)=1+b_1u+\mu_2u^2+\mu_6s(u)^2;$$

hence, by setting $b_1=0$ and $a_2=\mu_2$, we obtain $A(u)=B(u)=1+\mu_2u^2+\mu_6s(u)^2$.

So we get the expression (9.9). Now let us prove that this formal group is a specialization of a Tate formal group. We have

$$\mathcal{F}_4(u,v)=u+v+\frac{(uv(u^2+v^2))\Big(\mu_2+\mu_6\frac{s(u)^2+s(v)^2}{u^2+v^2}\Big)}{u+v+\mu_2uv(u+v)+\mu_6(us(v)^2+s(u)^2v)}.$$

According to [3], for $\mu_1 = 0$ and $\mu_3 = 0$, the Tate formal group over the ring $\mathbb{Z}[\mu_2, \mu_4, \mu_6]$ takes the form

$$F_{\mathrm{T}}(u,v) = u + v + k\frac{\mu_2 + 2\mu_4 m + 3\mu_6 m^2}{1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3},$$

where expressions for m and k are given in Theorem 5.1, and equation (5.4) takes the form

$$s(u) = u^3 + \mu_2 u^2 s(u) + \mu_4 u s(u)^2 + \mu_6 s(u)^3. \tag{9.10}$$

Over $\mathbb{F}_2[\mu_2, \mu_4, \mu_6]$ we obtain

$$F_{\mathrm{T}}(u,v) = u + v + \frac{(us(v) + vs(u))\Big(\mu_2 + \mu_6 \frac{s(u)^2 + s(v)^2}{u^2+v^2}\Big)}{u + v + \mu_2(s(u) + s(v)) + \mu_4 \frac{s(u)^2+s(v)^2}{u+v} + \mu_6 \frac{(s(u)+s(v))^3}{u^2+v^2}}.$$

Thus, the condition $\mathcal{F}_4(u,v) = F_{\mathrm{T}}(u,v)$ is equivalent to the relation

$$uv(u^2+v^2)\bigg(u + v + \mu_2(s(u)+s(v)) + \mu_4\frac{s(u)^2+s(v)^2}{u+v} + \mu_6\frac{(s(u)+s(v))^3}{u^2+v^2}\bigg)$$
$$= (us(v) + vs(u))\Big(u + v + \mu_2 uv(u+v) + \mu_6(us(v)^2 + s(u)^2 v)\Big).$$

Excluding μ_4 from this equation and using relation (9.10), we obtain the identical equality. □

10. COEFFICIENT RING OF THE FORMAL GROUP DETERMINING THE TWO-PARAMETRIC TODD GENUS

Consider the formal group (9.5),

$$\mathcal{F}_1(u,v) = \frac{u + v - \mu_1 uv}{1 + \mu_2 uv}.$$

It is defined over the ring $\mathbb{Z}[\mu_1, \mu_2]$. Since

$$\mathcal{F}_1(u,v) = u + v - \mu_1 uv - \mu_2 uv(u+v) + \dots, \tag{10.1}$$

the formal group (9.5) over the ring $\mathbb{Z}[\mu_1, \mu_2]$ is generating.

The formal group (9.5) is a specialization of the Buchstaber formal group with $A(u) = B(u) = 1 - \mu_1 u - \mu_2 u^2$.

Corollary 10.1 (to Theorem 8.1)**.** *Over a ring R without 2-torsion, the universal formal group of the form* (8.1) *is the formal group* (9.5) *over the ring* $\mathbb{Z}[\mu_1, \mu_2]$.

Proof. Over rings without 2-torsion, in Theorem 8.1 we obtain $h_1(e_n) = 0$ for $n = 2^k - 2$, $k \geq 3$; i.e., the coefficient ring $\widehat{\mathcal{S}}_1$ of the universal formal group of the form (8.1) over rings R without 2-torsion is generated by the two elements ϵ_1 and ϵ_2. Consider the ring homomorphism $h_1^{\mathrm{V}}\colon \widehat{\mathcal{S}}_1 \to \mathbb{Z}[\mu_1, \mu_2]$ classifying the formal group (9.5) as a group of the form (8.1) over the ring R without 2-torsion. From (3.3) we obtain $\epsilon_1 = a_{1,1}$ and $\epsilon_2 = a_{1,2}$, and from (10.1) we get $h_1^{\mathrm{V}}(\epsilon_1) = -\mu_1$ and $h_1^{\mathrm{V}}(\epsilon_2) = -\mu_2$.

Therefore, h_1^{V} gives the isomorphism $\widehat{\mathcal{S}}_1 \cong \mathbb{Z}[\mu_1, \mu_2]$. □

11. COEFFICIENT RING OF THE FORMAL GROUP DETERMINING THE ELLIPTIC GENUS OF LEVEL 2 (OCHANINE–WITTEN GENUS)

The formal group (9.6) over the ring $\mathbb{Z}[\mu_2,\mu_4]$ can be presented in the form

$$\mathcal{F}_2(u,v) = \frac{u^2-v^2}{u\sqrt{(1-\mu_2v^2)^2-4\mu_4v^4} - v\sqrt{(1-\mu_2u^2)^2-4\mu_4u^4}}.$$

It is a specialization of the Buchstaber formal group with $A(u)=1$ and the series

$$B(u) = 1-\mu_2u^2-2\mu_4u^4-2\mu_2\mu_4u^6-2\mu_4(\mu_2^2+\mu_4)u^8+\ldots\in\mathbb{Z}[\mu_2,\mu_4][[u]] \tag{11.1}$$

determined by the condition $B(u)^2=(1-\mu_2u^2)^2-4\mu_4u^4$. Since

$$\mathcal{F}_2(u,v) = u+v-\mu_2uv(u+v)+\mu_2^2u^2v^2(u+v)-2\mu_4uv(u+v)(u^2+uv+v^2)+\ldots, \tag{11.2}$$

the formal group (9.6) over the ring $\mathbb{Z}[\mu_2,\mu_4]$ is not generating. Its coefficient ring $\mathcal{R}_2$ is a subring in $\mathbb{Z}[\mu_2,\mu_4]$. Further we describe the ring $\mathcal{R}_2$.

Consider the universal formal group of the form

$$\mathscr{F}_2(u,v) = \frac{u^2-v^2}{uB(v)-vB(u)}, \tag{11.3}$$

where

$$B(x) = 1+\sum_{i=2}^{\infty} b_ix^i, \qquad b_i\in R.$$

Denote by $\mathcal{S}_2$ its coefficient ring.

Lemma 11.1.

$$\mathcal{S}_2 = \mathbb{Z}[b_2,b_3,b_4,\ldots]/J,$$

where J is the associativity ideal.

Proof. The formal group of the form (11.3) over the ring R is determined uniquely by its coefficients $b_k\in R$. In the case of the formal group (11.3), from (7.2) we obtain

$$a_{i,1}\equiv a_{1,i}\equiv b_i \mod I^2, \qquad i>1;$$

therefore, the formal group (11.3) over the ring $\mathbb{Z}[b_2,b_3,b_4,\ldots]/J$ is generating. $\square$

Theorem 11.2 [7, Theorem 8.2]. *The ring $\mathcal{S}_2$ is multiplicatively generated by the elements ϵ_n, where $n=2^k$, $k\geq 1$. Moreover,*

$$\rho_2(n) = \begin{cases} \infty & \text{for } n=2,4,\\ 2 & \text{for } n=2^k,\ k\geq 3,\\ 1 & \text{in all other cases.}\end{cases}$$

The ring $\mathcal{S}_2$ has no torsion.

Theorem 11.3. *The ring homomorphism $h_2^{\mathrm{V}}\colon \mathcal{S}_2\to\mathcal{R}_2$ classifying the formal group* (9.6) *as a group of the form* (11.3) *is an isomorphism.*

Proof. Since $\mathcal{R}_2\subset\mathbb{Z}[\mu_2,\mu_4]$ is a ring homomorphism, h_2^{V} is an epimorphism. Let us show that $h_2^{\mathrm{V}}\colon\mathcal{S}_2\to\mathbb{Z}[\mu_2,\mu_4]$ is a monomorphism. From Theorem 11.2 we find that the ring $\mathcal{S}_2\otimes\mathbb{Q}$ is multiplicatively generated by the elements ϵ_2 and ϵ_4. From Remark 3.2 and formula (11.2) we get $h_2^{\mathrm{V}}(\epsilon_2)=-\mu_2$ and $h_2^{\mathrm{V}}(\epsilon_4)=-2\mu_4$; therefore, $\mathcal{S}_2\otimes\mathbb{Q}\to\mathbb{Q}[\mu_2,\mu_4]$ is an isomorphism. Because $\mathcal{S}_2$ has no torsion, we find that $\mathcal{S}_2\to\mathbb{Z}[\mu_2,\mu_4]$ is a monomorphism, which proves the theorem. $\square$

Let us introduce the generators e_k in $\mathcal{R}_{\mathrm{U}}$ by Lemma 3.1 for $\beta_k=b_k$.

Corollary 11.4. *The value $h_2(e_k)$ is equal to the coefficient of u^k in* (11.1). *In particular, $h_2(e_2)=-\mu_2$, $h_2(e_4)=-2\mu_4$, and $h_2(e_{2k+1})=0$.*

12. COEFFICIENT RING OF THE FORMAL GROUP DETERMINING THE ELLIPTIC GENUS OF LEVEL 3

Consider the formal group (9.7),

$$\mathcal{F}_3(u,v)=\frac{u^2(1-\mu_1 v-\mu_3 s(v))-v^2(1-\mu_1 u-\mu_3 s(u))}{u(1-\mu_1 v-\mu_3 s(v))^2-v(1-\mu_1 u-\mu_3 s(u))^2}.$$

It is defined over the ring $\mathbb{Z}[\mu_1,\mu_3,\mu_6]/\{3\mu_6=-\mu_3^2\}$. Here $s(u)=u^3+\dots$ is determined by the equation

$$s(u)=u^3+\mu_1 us(u)-\mu_1^2u^2s(u)+\mu_3 s(u)^2-\mu_1\mu_3 us(u)^2+\mu_6 s(u)^3. \tag{12.1}$$

The formal group (9.7) is a specialization of the Buchstaber formal group (7.1) with $A(u)=1-\mu_1 u-\mu_3 s(u)$ and $B(u)=A(u)^2$.

The initial coefficients of the series expansion of the formal group show that over the ring $\mathbb{Z}[\mu_1,\mu_3,\mu_6]/\{3\mu_6=-\mu_3^2\}$ the formal group is not generating; namely, μ_6 does not belong to its coefficient ring $\mathcal{R}_3$. Further we will describe the ring $\mathcal{R}_3$.

Consider the universal formal group of the form

$$\mathscr{F}_3(u,v)=\frac{u^2C(v)-v^2C(u)}{uC(v)^2-vC(u)^2}, \tag{12.2}$$

where

$$C(u)=1+\sum_{i=1}^{\infty}c_iu^i,\qquad c_2=0,\quad c_i\in R. \tag{12.3}$$

Denote by $\mathcal{S}_3$ its coefficient ring.

Lemma 12.1.

$$\mathcal{S}_3=\mathbb{Z}[c_1,c_3,c_4,\dots]/J,$$

where J is the associativity ideal.

Proof. The formal group of the form (12.2) over the ring R is uniquely determined by its coefficients $c_k\in R$, $k\neq 2$. In the case of the formal group (12.2), from (7.2) we get

$$a_{1,1}\equiv c_1,\qquad a_{i,1}\equiv 2c_i\quad (i>1),\qquad a_{i,j}\equiv 3c_{i+j-1}\quad (i,j>1)\mod I^2, \tag{12.4}$$

and, since $c_2=0$, the formal group (12.2) over the ring $\mathbb{Z}[c_1,c_3,c_4,\dots]/J$ is generating. □

Theorem 12.2. *The ring $\mathcal{S}_3$ is multiplicatively generated by c_{3^r}, where $r\ge 0$. Moreover,*

$$\rho_3(n)=\begin{cases}\infty & \text{for } n=1,3,\\ 3 & \text{for } n=3^r,\ r\ge 2,\\ 1 & \text{in all other cases.}\end{cases}$$

Proof. For the Lazard ring $\mathcal{R}_{\mathrm{U}}$, the generating elements e_n determined by Lemma 3.3 satisfy the conditions (see [7, formula (63)])

$$\alpha_{i,j}\equiv\binom{n+1}{i}\frac{e_n}{d(n+1)}\mod I^2,\qquad i+j=n+1,$$

where $d(n)$ is defined in (3.1). Hence, taking into account the congruence (12.4), we get a system of congruences $\bmod I^2$:

$$\begin{aligned} h_3(e_1) &\equiv c_1, \\ \frac{n+1}{d(n+1)} h_3(e_n) &\equiv 2c_n, \qquad n>1, \\ \binom{n+1}{i}\frac{h_3(e_n)}{d(n+1)} &\equiv 3c_n, \qquad 2\le i\le n-1, \quad n\ge 3. \end{aligned} \tag{12.5}$$

In particular, when $n=3$, we obtain the system

$$2h_3(e_3)\equiv 2c_3, \qquad 3h_3(e_3)\equiv 3c_3,$$

which implies that $h_3(e_3)\equiv c_3$. For $n=4$ we obtain the system

$$h_3(e_4)\equiv 2c_4, \qquad 2h_3(e_4)\equiv 3c_4,$$

which implies that $h_3(e_4)\equiv c_4\equiv 0$. Now suppose that $n\ge 5$. We define the value

$$\begin{aligned} D(n) &= \left(\binom{n+1}{3}-\binom{n+1}{2}, \binom{n+1}{4}-\binom{n+1}{3}, \dots, \binom{n+1}{n-1}-\binom{n+1}{n-2}\right) \\ &= \left(\binom{n}{3}-\binom{n}{1}, \binom{n}{4}-\binom{n}{2}, \dots, \binom{n}{n-1}-\binom{n}{n-3}\right). \end{aligned}$$

Then (see [7, Theorem 7.9])

$$\frac{D(n)}{d(n+1)} = \begin{cases} d(n) & \text{for } n\ne 2^k-2, \\ 2 & \text{for } n=2^k-2. \end{cases} \tag{12.6}$$

Subtracting the congruences of system (12.5) from one another, we arrive at an equivalent system

$$\frac{n+1}{d(n+1)} h_3(e_n) \equiv 2c_n, \qquad n\ge 3, \tag{12.7}$$

$$\frac{1}{d(n+1)}\left(\binom{n+1}{2}-\binom{n+1}{1}\right) h_3(e_n) \equiv c_n, \qquad n\ge 3, \tag{12.8}$$

$$\frac{1}{d(n+1)}\left(\binom{n+1}{i+1}-\binom{n+1}{i}\right) h_3(e_n) \equiv 0, \qquad 2\le i\le n-1, \quad n\ge 3. \tag{12.9}$$

Subtracting the doubled congruence (12.8) from (12.7), we additionally find that

$$\frac{(n+1)(n-3)}{d(n+1)} h_3(e_n) \equiv 0. \tag{12.10}$$

It follows from the congruences (12.9) that

$$\frac{D(n)}{d(n+1)} h_3(e_n) \equiv 0. \tag{12.11}$$

Hence, according to (12.6), we have $h_3(e_n)\equiv 0$, except possibly the cases when $n=p^r$ and $n=2^{r_0}-2$, where p is prime, $r\ge 1$, and $r_0\ge 3$.

In the first case the congruence (12.11) means that $ph_3(e_{p^r}) \equiv 0$. If $p \neq 3$, then the number $(n+1)(n-3)$ is not divisible by p. Therefore, (12.10) and the condition that $ph_3(e_{p^r}) \equiv 0$ imply that $h_3(e_{p^r}) \equiv 0$. If $p = 3$, then the congruence (12.10) is a corollary to the fact that $3h_3(e_{3^r}) \equiv 0$.

In the remaining case, when $n = 2^{r_0} - 2$, $r_0 \geq 3$, the congruence (12.11) means that $2h_3(e_n) \equiv 0$. Considering the congruence (12.10) modulo 2, we obtain $h_3(e_n) \equiv 0$. □

Corollary 12.3. *The ring $\mathcal{S}_3 \otimes \mathbb{Q}$ is multiplicatively generated by $h_3(e_1)$ and $h_3(e_3)$. Moreover, $\mathcal{S}_3 \otimes \mathbb{Q} \cong \mathbb{Q}[c_1, c_3]$ (see the proof of Theorem* 12.7*).*

Theorem 12.4. *The ring $\mathcal{S}_3$ does not contain elements of finite order.*

To prove the theorem, we need the following assertions.

Lemma 12.5. *The ring $\mathcal{S}_3$ does not contain elements of order $p \neq 3$.*

Proof. Consider the ring homomorphism $\mathbb{Z}[c_1, c_3] \to \mathcal{S}_3$, where c_1 and c_3 are formal variables that are mapped to the corresponding coefficients of the series $C(u)$ over $\mathcal{S}_3$. By Lemma 9.8, we have the classifying ring homomorphism $\mathcal{S}_3 \to \mathbb{Z}[\mu_1, \mu_3, \mu_6]/\{\mu_3^2 = -3\mu_6\}$, under which $c_1 \mapsto -\mu_1$ and $c_3 \mapsto -\mu_3$. Theorem 12.2 implies that the ring homomorphism $\mathbb{Z}_p[c_1, c_3] \to \mathcal{S}_3 \otimes \mathbb{Z}_p$ is an epimorphism for prime $p \neq 3$. Since it can be continued to an isomorphism $\mathbb{Z}_p[c_1, c_3] \to \mathbb{Z}_p[\mu_1, \mu_3]$, $\mathbb{Z}_p[c_1, c_3] \to \mathcal{S}_3 \otimes \mathbb{Z}_p$ is an isomorphism.

Thus, in the ring $\mathcal{S}_3$ there is no p-torsion for $p \neq 3$. □

Lemma 12.6. *In the ring $\mathcal{S}_3$ the multiplicative generators c_{3^r} are related as follows:*

$$\xi(r)c_{3^r}^3 = 3\eta(r)c_{3^{r+1}} + P_r(c_1, c_3, c_{3^2}, \dots, c_{3^r}), \qquad r \geq 1,$$

where $\xi(r)$ and $\eta(r)$ are integers coprime to 3 *and the degree of the polynomial P_r with respect to c_{3^r} is less than* 3.

Proof. By equating the coefficient of vw in $\mathscr{F}_3(u, \mathscr{F}_3(v, w)) - \mathscr{F}_3(\mathscr{F}_3(u, v), w)$ to zero, we get the equation

$$2\big(uC'(u)\big(c_1u - C(u)^2\big) - 2c_1uC(u) + C(u)^3 - 1\big) = 0. \tag{12.12}$$

To verify the assertion of the lemma, it suffices to compare the coefficient of $u^{3^{r+1}}$ in the equality obtained with zero. □

Proof of Theorem 12.4. Let us check that there are no elements of order 3 in the ring $\mathcal{S}_3$. On the one hand, the isomorphism $\mathcal{S}_3 \otimes \mathbb{Q} \cong \mathbb{Q}[c_1, c_3]$ implies that the group $\mathcal{S}_3^{-2m} \otimes \mathbb{Q}$, which consists of elements of degree $-2m$ of the ring $\mathcal{S}_3 \otimes \mathbb{Q}$, is generated by the monomials $c_1^{m_1}c_3^{m_3}$, where

$$m_1 + 3m_3 = m, \tag{12.13}$$

and thus its rank is equal to the number of solutions of equation (12.13) in nonnegative integers. Hence, in the group $\mathcal{S}_3^{-2m} \otimes \mathbb{F}_3$ the number of generators is not less than this number. On the other hand, by Lemma 12.6 all elements of the group $\mathcal{S}_3^{-2m} \otimes \mathbb{F}_3$ can be expressed in terms of monomials of the form $c_1^{j_0}c_3^{j_1}c_{3^2}^{j_2}c_{3^3}^{j_3}\dots$, where $j_0 \geq 0$, $0 \leq j_1, j_2, j_3, \dots \leq 2$, and $j_0 + 3j_1 + 3^2j_2 + 3^3j_3 + \dots = m$. The number of such monomials coincides with the number of solutions of equation (12.13). Therefore, the ring $\mathcal{S}_3$ cannot have 3-torsion. □

Theorem 12.7. *The ring homomorphism $h_3^{\mathrm{V}}\colon \mathcal{S}_3 \to \mathcal{R}_3$ classifying the formal group* (9.7) *as a group of the form* (12.2) *is an isomorphism.*

Proof. Since $\mathcal{S}_3$ is a coefficient ring, h_3^{V} is an epimorphism. Let us show that $h_3^{\mathrm{V}}\colon \mathcal{S}_3 \to \mathbb{Z}[\mu_1, \mu_3, \mu_6]/\{\mu_3^2 = -3\mu_6\}$ is a monomorphism. From Theorem 12.2 we find that the ring $\mathcal{S}_3 \otimes \mathbb{Q}$ is multiplicatively generated by the elements c_1 and c_3. From the expression for $A(u)$ we get $h_3^{\mathrm{V}}(c_1) = -\mu_1$ and $h_2^{\mathrm{V}}(c_3) = -\mu_3$; therefore, $\mathcal{S}_3 \otimes \mathbb{Q} \to \mathbb{Q}[\mu_1, \mu_3]$ is an isomorphism. Because of the absence of torsion in $\mathcal{S}_3$ we find that $\mathcal{S}_3 \to \mathbb{Z}[\mu_1, \mu_3, \mu_6]/\{\mu_3^2 = -3\mu_6\}$ is a monomorphism, which proves the theorem. $\square$

Let us choose generators e_k in $\mathcal{R}_{\mathrm{U}}$ by Lemma 3.1 for $\beta_k = c_k$.

Corollary 12.8. *The value $h_3(e_k)$ is equal to the coefficient of u^k in the series $A(u) = 1 - \mu_1 u - \mu_3 s(u)$, where $s(u) = u^3 + \dots$ is determined by equation* (12.1).

Corollary 12.9. *The series $C(u)$ satisfies the cubic equation*

$$C(u)^3 - 3c_1 u C(u) - \big(1 + (c_1^3 + 3c_3)u^3\big) = 0. \tag{12.14}$$

Proof. By substituting $C(u) = 1 - \mu_1 u - \mu_3 s(u)$, $c_1 = -\mu_1$, $c_3 = -\mu_3$, and $3\mu_6 = -\mu_3^2$ into (12.14), we check that it follows from (12.1). $\square$

Corollary 12.10. *The function $\omega(u)$ defined by the equality*

$$\omega(u) = \frac{1}{g'(u)} = f'(g(u)) = \frac{\partial}{\partial v} F(u,v)\Big|_{v=0} \in R[[u]]$$

satisfies the cubic equation

$$\omega(u)^3 - 3c_1 u\, \omega(u)^2 = 1 + 2(3c_3 - c_1^3)u^3 + (3c_3 + c_1^3)^2 u^6. \tag{12.15}$$

Proof. From (9.2), in the case of $A(u) = C(u)$ and $B(u) = C(u)^2$, we obtain

$$\omega(u) = C(u)^2 - c_1 u. \tag{12.16}$$

With this substitution, equation (12.15) follows from (12.14). $\square$

Corollary 12.11.

$$C(u)(\omega(u) - 2c_1 u) = 1 + (c_1^3 + 3c_3)u^3.$$

13. EXPONENTIAL OF THE FORMAL GROUP DETERMINING THE ELLIPTIC GENUS OF LEVEL 3

From Lemma 9.8 and Theorems 12.7 and 5.3 we get

Theorem 13.1. *The formal group* (9.7) *is determined by its exponential*

$$f(x) = -2\frac{\wp(x) + \frac{\mu_1^2}{4}}{\wp'(x) - \mu_1\wp(x) - \mu_3 - \frac{\mu_1^3}{4}}, \tag{13.1}$$

where $\wp(x)$ and $\wp'(x)$ are the Weierstrass functions with the parameters

$$g_2 = \frac{1}{4}\mu_1\big(3\mu_1^3 + 8\mu_3\big), \qquad g_3 = \frac{1}{24}\big(3\mu_1^6 + 12\mu_1^3\mu_3 + 8\mu_3^2\big).$$

Proof. The ring $\mathcal{R}_3$ has no torsion; therefore, the exponential determines the formal group. Taking into account that $\omega(u) = f'(g(u))$ and $c_1 = -\mu_1$, $c_3 = -\mu_3$, from (12.15) we get a differential equation for $f(x)$:

$$f'(x)^3 + 3\mu_1 f(x)\, f'(x)^2 = 1 + 2\big(\mu_1^3 - 3\mu_3\big) f(x)^3 + \big(3\mu_3 + \mu_1^3\big)^2 f(x)^6 \tag{13.2}$$

with solution (13.1). $\square$

The solution of (13.2) is the elliptic function of level 3, which can therefore be expressed in the form (13.1). For more details see [4].

14. BUCHSTABER FORMAL GROUP UNDER THE CONDITION $B(u) = A(u)^2$

In this section we present results concerning the series $C(u)$ that are based on the form of the formal group (12.2). Some results of this section are also corollaries to the results of Sections 12 and 13 (in particular, to Theorem 12.7).

Theorem 14.1. *For the universal formal group of the form* (12.2), *the series* (12.3) *has the following properties*:

1. *The series $C(u)$ satisfies the cubic equation* (12.14).
2. $C(u) \in \mathbb{Z}[1/3][c_1, c_3]$.
3. $C(u) \in \mathbb{Z}[c_1, c_3, \widetilde{c}_6]/\{c_3^2 = 3\widetilde{c}_6\}$.
4. *The series $C(u)$ can be represented as*

$$C(u) = C_0(u^3) + c_1 u C_1(u^3),$$

where $C_0(0) = 1$ and $C_1(0) = 1$. In addition, the functions

$$Y_0(t) = C_0(t)^3 = 1 + c_3 t + 3c_1^3 c_3 t^2 + \ldots \in \mathbb{Z}[c_1, c_3][[t]],$$
$$Y_1(t) = t c_1^3 C_1(t)^3 = c_1^3 t - 3c_1^3 c_3 t^2 - \ldots \in \mathbb{Z}[c_1, c_3][[t]]$$

are solutions of the quadratic equation

$$Y(t)^2 - \bigl(1 + (c_1^3 + 3c_3)t\bigr)Y(t) + tc_1^3 = 0; \tag{14.1}$$

in particular, $C_0(t)C_1(t) = 1$.

Proof. 1. Due to the absence of 2- and 3-torsion, the substitution $q(u) = C(u)^3 - 3c_1 u C(u)$ reduces formula (12.12) to the equation

$$q(u) - 1 = \frac{u}{3} q'(u)$$

with general solution $q(u) = 1 + bu^3$. Hence

$$C(u)^3 - 3c_1 u C(u) = 1 + bu^3.$$

Comparing the coefficients of u^3 on the two sides of the equation obtained, we find that $b = c_1^3 + 3c_3$.

2. Equating the coefficient of u^n in equation (12.14) to zero, we obtain a recurrence from which one can find c_n:

$$3c_n + \sum_{\substack{0 \le i \le j \le k < n \\ i+j+k=n}} T(i,j,k) c_i c_j c_k - 3c_1 c_{n-1} = 0, \qquad n \ge 4, \tag{14.2}$$

where

$$T(i,j,k) = \begin{cases} 1 & \text{if } i = j = k, \\ 3 & \text{if } i = j < k \text{ or } i < j = k, \\ 6 & \text{if } i < j < k. \end{cases}$$

Expressing successively the coefficients c_n in terms of c_1 and c_3, we find that $C(u) \in \mathbb{Z}[1/3][c_1, c_3]$.

3. By induction we verify that for all $n \ge 4$ the coefficient c_n can be represented in the form $c_n = c_3 P_{n-3}(c_1, c_3, \widetilde{c}_6)$, where P is a polynomial with integer coefficients. To find c_n, we will apply equality (14.2). If $n \ne 3m$, then we obtain the desired formula for c_n, as all the coefficients $T(i,j,k)$ are divisible by 3 and by the induction hypothesis at least one of the factors in the product $c_i c_j c_k$

has the form $c_3P(c_1, c_3, \widetilde{c}_6)$. If $n = 3m$, then among the coefficients $T(i, j, k)$ exactly one is not divisible by 3, namely, $T(m, m, m) = 1$. Since $m > 1$, the corresponding term has the form

$$c_m^3 = c_3^3 P_{m-3}(c_1, c_3, \widetilde{c}_6)^3 = 3c_3\widetilde{c}_6 P_{m-3}(c_1, c_3, \widetilde{c}_6).$$

Hence, even in this case, the desired representation for the coefficient c_n holds.

4. Let us substitute the series $C(u)$ represented as

$$C(u) = X_0(u^3) + uX_1(u^3) + u^2X_2(u^3)$$

into (12.14). Equating the two series containing u to powers of the form $3n + 1$ and $3n + 2$ to zero, we obtain the respective equations

$$X_0(u^3)^2X_1(u^3) + u^3X_1(u^3)^2X_2(u^3) + u^3X_2(u^3)^2X_0(u^3) = c_1X_0(u^3), \tag{14.3}$$

$$X_0(u^3)^2X_2(u^3) + X_1(u^3)^2X_0(u^3) + u^3X_2(u^3)^2X_1(u^3) = c_1X_1(u^3). \tag{14.4}$$

Combining them with the coefficients $X_1(u^3)$ and $-X_0(u^3)$, we arrive at the equation

$$X_2(u^3)\big(u^3X_1(u^3)^3 - X_0(u^3)^3\big) = 0,$$

which implies that $X_2(u) = 0$. Then equations (14.3) and (14.4) take the form

$$X_0(u^3)X_1(u^3) = c_1. \tag{14.5}$$

Let us make a single series from all the terms on the left-hand side of (12.14) that contain u to powers of the form $3n$. Equating this series to zero, we additionally obtain the equation

$$X_0(u^3)^3 + u^3X_1(u^3)^3 = u^3(c_1^3 + 3c_3) + 1. \tag{14.6}$$

It follows from (14.5) and (14.6) that the series $Y_0(t) = X_0(t)^3$ and $Y_1(t) = tX_1(t)^3$ satisfy the quadratic equation (14.1). These are series of the form $Y(t) = y_0 + y_1t + y_2t^2 + \dots$ that are characterized by their initial coefficients

$$Y_0(t) = 1 + 3c_3t + \dots, \qquad Y_1(t) = c_1^3t + \dots.$$

Equating the coefficient of t^n $(n \ge 2)$ on the left-hand side of equation (14.1) to zero, we find that the coefficients of $Y_0(t)$ and $Y_1(t)$ satisfy the recurrence relations

$$\pm y_n = (c_1^3 + 3c_3)y_{n-1} - \sum_{k=1}^{n-1} y_k y_{n-k}.$$

Therefore, both series $Y_0(t)$ and $Y_1(t)$, being solutions of equation (14.1), have coefficients in $\mathbb{Z}[c_1, c_3]$.

For $c_1 = 0$ equation (14.1) takes the form

$$Y(t)\big(Y(t) - 1 - 3c_3t\big) = 0.$$

Therefore, all the coefficients of the series $Y_1(t)$ are divisible by c_1^3, and thus the series $C_1(t) = (Y_1(t)/(tc_1^3))^{1/3}$ is well defined. Setting $C_0(t) = Y_0(t)^{1/3}$, we get the required representation for the function $C(u)$:

$$C(u) = X_0(u^3) + uX_1(u^3) = Y_0(u^3)^{1/3} + Y_1(u^3)^{1/3} = C_0(u^3) + c_1uC_1(u^3). \qquad \square$$

Corollary 14.2. $c_2 = c_5 = c_8 = \dots = c_{3k+2} = \dots = 0.$

15. COEFFICIENT RING OF THE BUCHSTABER FORMAL GROUP UNDER THE CONDITION $F(u,u)=0$

Theorem 15.1. *Let $F(u,v)$ be a formal group of the form (7.1) over a ring R without zero divisors and $F(u,u)=0$. Then the ring R is an algebra over the field $\mathbb{F}_2$ and*

$$A(u)=B(u), \qquad a_{2k+1}=0, \quad a_{4k+4}=0, \quad k\geq 0.$$

Proof. The condition $F(u,u)=0$ implies that $2=0$. By l'Hôpital's rule we have the formula

$$F(u,u)=\frac{u(2A(u)-uA'(u))}{B(u)-uB'(u)}; \tag{15.1}$$

thus $a_{2k+1}=0$.

Let us introduce the notation $N(u,v,w)$ for the expression $F(F(u,v),w)-F(u,F(v,w))$ and d_2 for the operator $\mathbb{F}_2[[u]]\to\mathbb{F}_2[[u]]$ resulting from the reduction modulo 2 of the operator $\frac{1}{2}\frac{d^2}{du^2}:\mathbb{Z}[[u]]\to\mathbb{Z}[[u]]$.

In this notation, modulo 2 the coefficient of vw in $N(u,v,w)$ is equal to $B'(u)(B(u)+a_1u)$ and, since $B(0)=1$, we have $B'(u)=0$. At the same time the coefficient of v^2w in $N(u,v,w)$ is equal to

$$\frac{B(u)}{u^2}\big(A(u)+u^2(a_2+b_2)+B(u)^2+u^2B(u)d_2(B(u))\big).$$

Since the coefficient a_2 of $A(u)$ does not affect the form of (7.1), one can set $a_2=b_2$. We obtain the equation

$$A(u)=B(u)\big(B(u)+u^2d_2(B(u))\big).$$

Equating the coefficient of w in $N(u,v,w)$ to zero, we arrive at the formula

$$B(F(u,v))=\frac{B(u)B(v)F(u,v)}{vB(u)-uB(v)}. \tag{15.2}$$

The equation

$$\big(B(u)+u^2d_2(B(u))\big)^2=1 \tag{15.3}$$

is derived from (15.2) by substituting $v=u$, as by l'Hôpital's rule

$$\frac{F(u,v)}{vB(u)-uB(v)}\bigg|_{v=u}=\left(\frac{B(u)+u^2d_2(B(u))}{B(u)}\right)^2.$$

From the expansion

$$\big(B(u)+u^2d_2(B(x))\big)^2=1+\sum_{k=1}^{\infty}b_{4k}^2u^{4k}$$

and the absence of zero divisors in the ring R, we obtain the relations $b_{4k}=0$, $k\geq 1$, and $B(u)+u^2d_2(B(u))=1$. Thus

$$A(u)=B(u). \qquad \square$$

Corollary 15.2. *The coefficient ring $\mathcal{R}_{B,2}$ of the universal Buchstaber formal group with the condition $F(u,u)=0$ is isomorphic to the ring $\mathbb{F}_2[a_2,a_6,\dots,a_{2^k-2},\dots]$. Its exponential has the form $f(x)=x+\sum_{k\geq 2}a_{2^k-2}x^{2^k-1}$, and $A(u)=u/g(u)$, where $g(u)$ is the logarithm of this group.*

Proof. Consider the series

$$\psi(x) = x + \sum_{k=2}^{\infty} a_{2^k-2} x^{2^k-1}$$

over the ring $\mathbb{F}_2[a_{2^k-2},\ k \ge 2]$. Since for $n = 2^k - 1$ all binomial coefficients $\binom{n}{j}$, $1 \le j \le n-1$, are odd numbers, we have

$$\psi(x+y) = x + y + \sum_{k=2}^{\infty} a_{2^k-2} \frac{x^{2^k} - y^{2^k}}{x-y} = \frac{x\psi(x) - y\psi(y)}{x-y}. \tag{15.4}$$

Let $\overline{\psi}(u)$ be a series inverse to $\psi(x)$ and $A(u) = u/\overline{\psi}(u)$. Setting $x = \overline{\psi}(u)$ and $y = \overline{\psi}(v)$ in equation (15.4), we get a linearization of the formal group

$$\psi(\overline{\psi}(u) + \overline{\psi}(v)) = \frac{u\overline{\psi}(u) - v\overline{\psi}(v)}{\overline{\psi}(u) - \overline{\psi}(v)} = \frac{u^2 A(v) - v^2 A(u)}{uA(v) - vA(u)}$$

over the ring $\mathbb{F}_2[a_{2^k-2},\ k \ge 2]$. Its logarithm $\overline{\psi}(u)$ is equal to $u/A(u)$.

By Theorems 15.1 and 8.1, the classifying homomorphism $\mathcal{R}_{B,2} \to \mathbb{F}_2[a_2, a_6, \dots, a_{2^k-2}, \dots]$ is an isomorphism. $\square$

The specialization of the Buchstaber formal group with the condition $F(u,u) = 0$ is the formal group (9.9),

$$\mathcal{F}_4(u,v) = \frac{u^2(1 + \mu_6 s(v)^2) - v^2(1 + \mu_6 s(u)^2)}{u(1 + \mu_2 v^2 + \mu_6 s(v)^2) - v(1 + \mu_2 u^2 + \mu_6 s(u)^2)}.$$

It is defined over the ring $\mathbb{F}_2[\mu_2, \mu_4, \mu_6]$. Here $s(u) = u^3 + \dots$ is determined by the equation

$$s(u) = u^3 + \mu_2 u^2 s(u) + \mu_4 u s(u)^2 + \mu_6 s(u)^3. \tag{15.5}$$

We have $\mathcal{F}_4(u,u) = 0$.

The specialization of the Buchstaber formal group (7.1) is determined by the relations $A(u) = B(u) = 1 + \mu_2 u^2 + \mu_6 s(u)^2$.

The initial coefficients of the series expansion of the formal group show that over the ring $\mathbb{F}_2[\mu_2, \mu_4, \mu_6]$ the formal group is not generating; namely, μ_4 does not belong to its coefficient ring $\mathcal{R}_4$.

Theorem 15.3. *The addition law in the Tate formal group with the additional condition $F(u,u) = 0$ is defined over the ring $\mathbb{F}_2[\mu_2, \mu_4, \mu_6]$ and has the form* (9.9). *The logarithm of the formal group $\mathcal{F}_4(u,v)$ is given by the function*

$$g(u) = \frac{u}{1 + \mu_2 u^2 + \mu_6 s(u)^2} = \frac{s(u)}{u^2 + \mu_4 s(u)^2}.$$

Proof. The initial coefficients of the series expansion of $\mathcal{F}_{\mathrm{T}}(u,u)$ (see (5.6)) lead to the relations $2 = 0$, $\mu_1 = 0$, and $\mu_3 = 0$. According to Lemma 9.9, these relations determine the form (9.9).

For (9.9) we have $A'(u) = 0$ and $B'(u) = 0$; therefore, from (15.1) we obtain $\mathcal{F}_4(u,u) = 0$.

For the formal group $\mathcal{F}_4(u,v)$, from (2.3) and (9.2) we have

$$g'(u) = \frac{1}{1 + \mu_2 u^2 + \mu_6 s(u)^2}.$$

Over the ring $\mathbb{F}_2[\mu_2, \mu_4, \mu_6]$ the series $g(u)$ is odd due to the grading; therefore,

$$g(u) = \frac{u}{1 + \mu_2 u^2 + \mu_6 s(u)^2}.$$

In view of the relations for $s(u)$ from Lemma 9.9, this proves the last statement of the theorem. □

Corollary 15.4 (to Theorem 15.3 and Corollary 4.2)**.** *The Tate formal group $F(u, v)$ with the additional condition $F(u, u) = 0$ determines a Hirzebruch genus $L_f \colon \Omega_{\mathrm{U}} \to \mathbb{F}_2[\mu_2, \mu_4, \mu_6]$ such that*

$$\sum_{n\geq 0} L(\mathbb{C}\mathrm{P}^n)u^n = \frac{1}{1 + \mu_2 u^2 + \mu_6 s(u)^2} = \frac{s'(u)}{u^2 + \mu_4 s(u)^2},$$

where the series $s(u)$ is determined by formula (15.5).

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