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The Distribution of the Rational Points on the Unit Circle[1]

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Abstract—In the paper, the explicit form of distribution function for the lengths of arcs connecting neighbouring rational points on the unit circle whose denominators do not exceed given value, is given.

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The aim of this paper is the investigation of the distribution of the rational points lying on the unit circle. Let us introduce the necessary notations.

Suppose that $Q \geq 2$ and let (x_r, y_r), $r = 1, 2, \ldots, N$, be all the points of the unit circle whose coordinates are positive irreducible fractions with the denominators not exceeding Q and ordered in accordance with the increment of the value $\varphi_r = \arctan\left(\dfrac{y_r}{x_r}\right)$. Further, let $\theta_r = \varphi_r - \varphi_{r-1}$, where $2 \leq r \leq N$. Finally, suppose that $t > 0$ is any fixed positive value and denote by $\mu(Q; t)$ the number of pairs of neighbouring points (x_{r-1}, y_{r-1}), (x_r, y_r) that satisfy the inequality $\theta_r \leq \dfrac{t}{Q}$. The main result is the following

Theorem. *For any fixed $t > 0$ and $Q \to +\infty$ one has*

$$\mu(Q;t) = \frac{Q}{\pi}\int_0^t h(v)dv + O(T),$$
$$\textit{where} \quad T = Q^{5/6}(\ln Q)^{4/3},$$

the implied constant in O-symbol depends on t and the continuous function h is defined as follows:

$$h(v) \equiv 0 \quad \textit{for} \quad 0 \leq v \leq \sqrt{2};$$

$$h(v) \equiv v^{-2}(2\ln v - \ln 2) \quad \textit{for} \quad \sqrt{2} \leq v \leq 2;$$

$$h(v) \equiv v^{-2}(4\ln v - 3\ln 2) \quad \textit{for} \quad 2 \leq v \leq 4;$$

$$h(v) \equiv v^{-2}(2\ln v - 4\ln(1 + \sqrt{1 - 4v^{-1}}) + \ln 2)$$
$$\textit{for} \quad 4 \leq v \leq 2\sqrt{2} + 2;$$

$$h(v) \equiv v^{-2}(2\ln v - \ln 2) \quad \textit{for} \quad 2\sqrt{2} + 2 \leq v \leq 8;$$

$$h(v) \equiv v^{-2}(11\ln 2 - 2\ln v - 8\ln(1 + \sqrt{1 - 8v^{-1}}))$$
$$\textit{for} \quad 8 \leq v \leq 3\sqrt{2} + 4;$$

$$h(v) \equiv v^{-2}(4\ln 2 - 4\ln(1 + \sqrt{1 - 8v^{-1}}))$$
$$\textit{for} \quad v \geq 3\sqrt{2} + 4.$$

Remark. One can prove that the number $N = N(Q)$ of points (x_r, y_r) satisfies the relation $N(Q) = \dfrac{Q}{\pi} + O(\sqrt{Q}\ln Q)$ as $Q \to +\infty$. Hence, the formula for $\mu(Q; t)$ can be expressed in the form

$$\mu(Q;t) = N(Q)\int_0^t h(v)dv + O(T).$$

In what follows, we give the sketch of the proof and the necessary auxiliary assertions.

The coordinates (x, y) of all rational points of the unit circle with the conditions $x > 0$, $y > 0$ have the form

$$x = \frac{b^2 - a^2}{a^2 + b^2}, \quad y = \frac{2ab}{a^2 + b^2}, \tag{1}$$

where $1 \leq a \leq b - 1$ are any positive integers such that $(a, b) = 1$. If a, b have different parity then the fractions (1) are irreducible; if a, b are both odd numbers, then the greatest common divisor of the numerator and the denominator of any such fraction from (1) is equal to 2.

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Suppose that $Q \geq 2$ and consider the following series of the irreducible fractions a/b ordered in the increasing order and satisfying the conditions $a^2 + b^2 \leq Q$, $1 \leq a \leq b - 1$. Inserting the fractions $\frac{0}{1} = 0$, $\frac{1}{1} = 1$, we denote the resulting series as $\mathcal{F}_Q$.

The series $\mathcal{F}_Q$ is the analogue of the classical Farey series $F_Q = \left\{\frac{a}{b}, 1 \leq a \leq b \leq Q, (a, b) = 1\right\}$ and has similar properties. In particular, the series $\mathcal{F}_Q$ can be constructed form $\mathcal{F}_{Q-1}$ by inserting all the possible mediants $\frac{a+c}{b+d}$ with the condition $(a + c)^2 + (b + d)^2 \leq Q$ between neighbouring fractions $\frac{c}{d} < \frac{a}{b}$. Moreover, the neighbouring fractions $\frac{c}{d} < \frac{a}{b}$ of the series $\mathcal{F}_Q$ satisfy the equality $ad - bc = 1$.

Therefore, all the rational points (x, y) of the unit circle such that x , $y > 0$ whose denominators do not exceed Q are given by (1) when a and b run through the natural numbers satisfying to one of the following conditions:

(a) $(a, b) = 1$, $1 \leq a \leq b - 1$, $a^2 + b^2 \leq Q$;

(b) $(a, b) = 1$, $1 \leq a \leq b - 1$, $a, b \equiv 1 \pmod 2$, $Q < a^2 + b^2 \leq 2Q$, that is, the conditions

(a) $\frac{a}{b} \in \mathcal{F}_Q$;

(b) $\frac{a}{b} \in \mathcal{F}_{2Q} \backslash \mathcal{F}_Q$, $a, b \equiv 1 \pmod 2$.

Denote by Φ_Q the series of irreducible fractions $\frac{a}{b}$, ordered in the increasing order and satisfying to one of the conditions (a), (b). If $\frac{c}{d} < \frac{a}{b}$ are neighbouring fractions in Φ_Q, then one can show that the quantity $\delta = ad - bc$ has only two values, namely 1 and 2. By (1), we have

$$x = \frac{2}{1 + \left(\frac{a}{b}\right)^2} - 1, \quad y = \frac{2\left(\frac{a}{b}\right)}{1 + \left(\frac{a}{b}\right)^2},$$

hence, x and y are strictly monotonic functions of the fraction $\frac{a}{b}$. Therefore, there is one-to-one correspondence between neighbouring fractions $\frac{c}{d} < \frac{a}{b}$ of Φ_Q and neighbouring points (x_r, y_r), $1 \leq r \leq N$. Thus, the initial problem is reduced to the determining of the corresponding distribution function for the series Φ_Q.

For the below, the set of pairs $\frac{c}{d} < \frac{a}{b}$ of neighbouring fractions of the series Φ_Q split into 3 families A, B, C according to the number of the fractions in the pair (2, 1 or 0) lying in the series $\mathcal{F}_Q$. Next, it is convenient to split some of these families to the classes and subclasses. Namely, the family A consists of the classes A_1, A_2, A_3 that correspond the following conditions:

(1) $\frac{a+b}{b+d} \notin \mathcal{F}_{2Q}$;

(2) $\frac{a+c}{b+d} \in \mathcal{F}_{2Q}$, but the numerators and denominators of all mediants of the form $\frac{pa+qc}{pb+qd}$, $(p, q) = 1$, $p, q \geq 1$, inserting between $\frac{c}{d}$ and $\frac{a}{b}$, have different parity;

(3) $\frac{a+c}{b+d} \in \mathcal{F}_{2Q}$, and some of mediants of the above form inserting between $\frac{c}{d}$ and $\frac{a}{b}$ have odd numerators and denominators, but all such mediants do not belong to $\mathcal{F}_{2Q}$.

Further, the family B consists of the classes B_1 and B_2 that satisfy the conditions $ad - bc = 1$ and, consequently, $ad - bc = 2$. Finally, the family C consists of the single class denoting by the same letter.

The classes A_2 and A_3 split into subclasses $A_{2,1}$, $A_{2,2}$ and $A_{3,1}$, $A_{3,2}$, that satisfy the conditions

$$\begin{aligned} c < a, \quad d < b, \quad c, d \equiv 1 \pmod 2, \\ a \leq c, \quad b < d, \quad a, b \equiv 1 \pmod 2, \end{aligned}$$

and, consequently

$$\begin{aligned} c < a, \quad d < b, \quad a, b \equiv 1 \pmod 2, \\ a \leq c, \quad b < d, \quad c, d \equiv 1 \pmod 2. \end{aligned}$$

Similarly, the classes B_1 and B_2 split into subclasses $B_{1,1}$, $B_{1,2}$ and $B_{2,1}$, $B_{2,2}$, satisfying the conditions

$$\frac{a}{b} \in \mathcal{F}_{2Q} \backslash \mathcal{F}_Q \text{ (for } B_{1,1} \text{ and } B_{2,1}),$$

$$\frac{a}{b} \in \mathcal{F}_Q \text{ (for } B_{1,2} \text{ and } B_{2,2}).$$

The following assertions allow one to find the numerator and the denominator of neighbouring fraction to given fraction from Φ_Q.

Lemma 1. *The following assertions hold true.*

(1) *If* $\frac{a}{b} \in \mathcal{F}_Q$, *then* $\frac{c}{d}$ *is the neighbouring fraction to* $\frac{a}{b}$ *in* Φ_Q *and the pair* $\left(\frac{c}{d}; \frac{a}{b}\right)$ *belongs to the class X if, and only if the following conditions are satisfied*:

$$ad \equiv \delta \pmod b, \quad c = \frac{ad - \delta}{b},$$

$$D_1 < d \leq D_2, \quad D_r = bf_r + \frac{a\delta}{R},$$

where

$$f_1(a,b) = \sqrt{\frac{2Q}{R} - \frac{1}{R^2}} - 1,$$
$$f_2(a,b) = \sqrt{\frac{Q}{R} - \frac{1}{R^2}},$$

$\delta = 1$ *for* $X = A_1$;

$$f_1(a,b) = \max\left(1, \sqrt{\frac{Q}{R} - \frac{1}{R^2}} - 1\right),$$
$$f_2(a,b) = \min\left(\sqrt{\frac{Q}{R} - \frac{1}{R^2}}, \sqrt{\frac{2Q}{R} - \frac{1}{R^2}} - 1\right),$$

$\delta = 1$ *for* $X = A_{2,\,2}$, *and in this case we necessary have* a, $b \equiv 1$ mod2

$$f_1(a,b) = \max\left(\sqrt{\frac{Q}{R} - \frac{1}{R^2}} - 1, \sqrt{\frac{Q}{2R} - \frac{1}{R^2}} - \frac{1}{2}\right),$$
$$f_2(a,b) = \min\left(1, \sqrt{\frac{2Q}{R} - \frac{1}{R^2}} - 1\right),$$

$\delta = 1$ *for* $X = A_{3,\,1}$, *and in this case we necessary have* a, $b \equiv 1$ (mod2);

$$f_1(a,b) = \max\left(\sqrt{\frac{Q}{R} - \frac{1}{R^2}} - 1, \sqrt{\frac{2Q}{R} - \frac{1}{R^2}} - 2\right),$$
$$f_2(a,b) = \sqrt{\frac{2Q}{R} - \frac{1}{R^2}},$$

$\delta = 1$ *for* $X = B_{1,\,2}$, *and in this case* $d \equiv \gamma$ (mod2b);

$$f_1(a, b) = 2\sqrt{\frac{Q}{R} - \frac{1}{R^2}} - 1,$$
$$f_2(a,b) = \min\left(\sqrt{\frac{2Q}{R} - \frac{4}{R^2}}, 2\sqrt{\frac{2Q}{R} - \frac{1}{R^2}} - 1\right),$$

$\delta = 2$ *for* $X = B_{2,\,2}$, *and in this case* $d \equiv \gamma$ (mod2b); (*here* $R = a^2 + b^2$, *and* $\gamma = \gamma(a, b)$ *denotes some numbers such that* $(\gamma, 2b) = 1$, *different in different relations, in general*).

(2) *If* $\frac{c}{d} \in \mathcal{F}_Q$, *then* $\frac{a}{b}$ *is the neighbouring fraction to* $\frac{c}{d}$ *in* Φ_Q *and the pair* $\left(\frac{c}{d}; \frac{a}{b}\right)$ *belongs to the class Y if, and only if the following conditions are satisfied*:

$$bc \equiv -\delta (\operatorname{mod} d), \quad a = \frac{bc + \delta}{d},$$
$$B_1 < b \le B_2, \quad B_r = dg_r + \frac{\delta c}{R},$$

where the expressions for the functions $g_r = g_r(c, d)$ *are obtained from the expressions for* $f_r(a, b)$ *by the replacement a to c, b to d, that corresponds to the class* $X = A_{2,\,2}$ *for* $Y = A_{2,\,1}$, *to the class* $X = A_{3,\,1}$ *for* $Y = A_{3,\,2}$, *to the class* $X = B_{1,\,2}$ *for* $Y = B_{1,\,1}$, *and* $Y = B_{2,\,1}$, *to the class for* $X = B_{2,\,2}$. *In the case* $X = A_{2,\,2}, A_{3,\,1}$, *the conditions* a, $b \equiv 1$ (mod2) *are replaced by* c, $d \equiv 1$ (mod2) *for* $Y = A_{2,\,1}, A_{3,\,2}$, *and the condition* $d \equiv \gamma$ (mod $2d$), *by the condition* $b \equiv \gamma$ (mod $2d$), *where* $\gamma = \gamma(c, d)$ *are some integers coprime to* $2d$ *and different in different relations, in general*.

(3) *If* $\frac{a}{b} \in \mathcal{F}_{2Q} \backslash \mathcal{F}_Q$, *then* $\frac{c}{d}$ *is the neighbouring fraction to* $\frac{a}{b}$ *in* Φ_Q *and the pair* $\left(\frac{c}{d}; \frac{a}{b}\right)$ *belongs to the class C if, and only if the following conditions are satisfied*: a, $b \equiv 1$ (mod 2), $ad \equiv 2$ (mod b), *where* $c = \frac{ad - 2}{b}$, $D_1 < d \le D_2$,

$$D_1(a,b) = 2b\sqrt{\frac{Q}{4R} - \frac{1}{R^2}} + \frac{2a}{R},$$
$$D_2(a,b) = 2b\sqrt{\frac{Q}{2R} - \frac{1}{R^2}} + \frac{2a}{R},$$

$R = a^2 + b^2$, *and* $d \equiv \gamma$ (mod2b), *where* $(\gamma, 2b) = 1$.

Let φ', φ'' be the angles of inclination of the lines connecting the origin with the points (1) of the unit circle corresponding to the neighbouring fractions from Φ_Q. Then

$$\tan\varphi' = \frac{2cd}{d^2 - c^2}, \quad \tan\varphi'' = \frac{2ab}{b^2 - a^2},$$

and hence, setting $\theta = \varphi'' - \varphi'$ we get $\tan\left(\frac{\theta}{2}\right) = \frac{\delta}{ac + bd}$. Thus, the investigation of the distribution of the arc lengths θ reduces to the derivation of the asymptotic for the number of pairs $\left(\frac{c}{d}; \frac{a}{b}\right)$ such that $\frac{\delta}{ac + bd} < \frac{t}{Q}$, where $t > 0$ is any given quantity.

Denote $\lambda = \frac{\delta}{t}$, $x = \lambda Q$; then the above condition is written as follows:

$$ac + bd > x. \tag{2}$$

Let $W_r(Q, \lambda)$ be the number of pairs $\left(\frac{c}{d}; \frac{a}{b}\right)$, satisfying to (2) and belonging to the class X_r, where $X_1 = A_1$, $X_2 = A_{2,\,1}$, $X_3 = A_{2,\,2}$, $X_4 = A_{3,\,1}$, $X_5 = A_{3,\,2}$, $X_6 = B_{1,\,1}$, $X_7 = B_{1,\,2}$, $X_8 = B_{2,\,1}$, $X_9 = B_{2,\,2}$, $X_{10} = C$.

In the cases $r = 1, 3, 4, 7, 9$, the condition (2) has the form $d > \frac{bx + a\delta}{R}$, $R = a^2 + b^2$ and in the cases $r = 2, 5, 6, 8$ has the form $b > \frac{dx - c\delta}{R}$, $R = c^2 + d^2$. Consequently, the accessory conditions to the given class (sub-class) together with (2) are obtained from the formulas of Lemma 1 by inserting into the signs of maximum in the expressions for f_1, g_1 the quantity $\frac{x}{R}$ (or by the replacement f_1, g_1 by the maximum of two quantities in the cases of classes (sub-classes) A_1, $B_{2,\,2}$, $B_{2,\,1}$, and C).

Transforming $W_r(Q, \lambda)$ by Lemma 2 and Theorem 1 from [1] we arrive at the following assertion.

Lemma 2. *For any fixed* $\lambda > 0$ *and* $Q \to +\infty$ *one has*

$$W_r(Q,\lambda) = \kappa_r j_r N(Q) + O(T),$$

where $\kappa_1 = \frac{3}{2}$, $\kappa_r = \frac{1}{2}$ *for* $2 \le r \le 7$, $\kappa_r = \frac{1}{4}$ *for* $8 \le r \le 10$, *the implied constants in O-symbols depend on* λ,

$$j_r(\lambda) = \int_0^{\xi} [h_1(u) < h_2(u)] \cdot (h_2(u) - h_1(u))du,$$

$\xi = 1$ *for* $1 \le r \le 9$, $\xi = \sqrt{2}$ *for* $r = 10$,

$$h_1(u) = \max\left(\sqrt{2} - u, \frac{\lambda}{u}\right), \quad h_2(u) \equiv 1 \quad \text{for} \quad r = 1;$$

$$h_1(u) = \max\left(u, 1 - u, \frac{\lambda}{u}\right),$$
$$h_2(u) \equiv \min(1, \sqrt{2} - u) \quad \text{for} \quad r = 2, 3;$$

$$h_1(u) = \max\left(1 - u, \frac{1}{\sqrt{2}} - \frac{u}{2}, \frac{\lambda}{u}\right),$$
$$h_2(u) \equiv \min(u, \sqrt{2} - u) \quad \text{for} \quad r = 4, 5;$$

$$h_1(u) = \max\left(1, \sqrt{2} - 2u, \frac{\lambda}{u}\right),$$
$$h_2(u) \equiv \sqrt{2} \quad \text{for} \quad r = 6, 7;$$

$$h_1(u) = \max\left(2 - u, \frac{\lambda}{u}\right),$$
$$h_2(u) \equiv \min(\sqrt{2}, 2\sqrt{2} - u) \quad \text{for} \quad r = 8, 9;$$

$$h_1(u) = \max\left(1, \frac{\lambda}{u}\right),$$
$$h_2(u) \equiv \sqrt{2} \quad \text{for} \quad r = 10,$$

and the value [A] *is equal to one if the condition A is true and equal to zero in the opposite case.*

Denoting by $W(Q, \lambda)$ the number of neighbouring pairs from the series Φ_Q such that $\tan\left(\frac{\theta}{2}\right) \le \frac{1}{\lambda Q}$, by Lemma 2 we get:

$$W(Q,\lambda) = j(\lambda)N(Q) + O(T),$$

where $j(\lambda) = \frac{3}{2}j_1(\lambda) + j_2(\lambda) + j_4(\lambda) + j_6(\lambda) + \frac{1}{8}j_8(2\lambda) + \frac{1}{4}j_{10}(\lambda) = \int_{\lambda}^{+\infty} f(u)du$, $f(u) = -j'(\lambda)$. Since $\frac{\theta}{2} \le \tan\left(\frac{\theta}{2}\right) \le \frac{\theta}{2} + \theta^3$ for $0 \le \theta \le \frac{\pi}{2}$, then the number $V(Q, \lambda)$ of neighbouring pairs from Φ_Q with the condition $\theta \le \frac{2}{\lambda Q}$ satisfies the inequalities

$$W(Q,\lambda) \le V(Q,\lambda) \le W(Q,\nu),$$
$$\nu = \lambda + O(\lambda^{-1}Q^{-2}).$$

Since the function $j'(\lambda)$ is continuous, setting $t = \frac{2}{\lambda}$, we get:

$$\mu(Q;t) = V(Q,\lambda) = j(\lambda)N(Q) + O(T),$$

where

$$j(\lambda) = \int_{2/t}^{+\infty} f(u)du = \int_0^t h(\nu)d\nu, \quad h(\nu) = 2\nu^{-2}f(2\nu^{-1}).$$

The initial assertion follows now from the explicit expressions for the function f, that can be obtained by direct calculation of the quantities $j_r(\lambda)$ in Lemma 2.

Remark. It is interesting to compare the formulas of the theorem with the results of [2] and, in particular, with the formulas of the Corollary 0.4 of Theorem 0.3 for the distribution function of the angle between the neighbouring segments connecting the origin with the primitive points lying inside the disk of unboundedly increasing radius.

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