Russian — Chinese
Student Mathematical Olympiad
Birobidzhan, Russia, September 22, 2015

1. (3 points) Let $\alpha \geq 1$ and $x \geq 0$ be real numbers. Prove the inequality

$$1 + x^\alpha \leq (1+x)^\alpha.$$

Solution. Let $f(x) = (1+x)^\alpha - x^\alpha - 1$. Then $f'(x) = \alpha((1+x)^{\alpha-1} - x^{\alpha-1}) \geq 0$. Hence $\min_{x\geq 0} f(x) = f(0) = 0$.

2. (a) **(1 point)** You roll two usual six-sided dice. What is a probability to get a double?

(b) **(1 point)** You roll two dice with 6 and 20 sides. (The die with n sides has numbers from 1 up to n.) What is a probability to get a double?

(c) **(5 points)** You have a collection of dice with 6 and 20 sides (10 dice of each type). You take a random pair of dice from your collection and roll them. What is a probability to get a double?

Solution. (a) 1/6. (b) 1/20. It is the probability to get a fixed number on the second die.

(c) We have a pair of 6-sided (or 20-sided) dice with the probability $\frac{10}{20} \cdot \frac{9}{19} = \frac{9}{38}$ and a pair of different dice with the probability $2 \cdot \frac{10}{20} \cdot \frac{10}{19} = \frac{10}{19}$. The probability to get a double is

$$\frac{9}{38} \cdot \frac{1}{6} + \frac{9}{38} \cdot \frac{1}{20} + \frac{10}{19} \cdot \frac{1}{20} = \frac{1}{19 \cdot 20}\left(15 + \frac{9}{2} + 10\right) = \frac{59}{760}.$$

3. (4 points) Let $P(x)$, $Q(x)$ be polynomials with real coefficients and

$$P(x) + P'(x) + P''(x) + P^{(3)}(x) + \ldots = Q(x).$$

Express $P(x)$ in terms of $Q(x)$.

Solution. From the formula

$$P(x) + P'(x) + P''(x) + P^{(3)}(x) + \ldots = Q(x)$$

follows that

$$P'(x) + P''(x) + P^{(3)}(x) + \ldots = Q'(x).$$

Subtracting one equation from another we get $P(x) = Q(x) - Q'(x)$.

4. How many real roots may have the function

$$f(x) = \int_x^\infty e^{-t} P_n(t)\, dt \qquad (x \in \mathbb{R})$$

where $P_n(t)$ is a polynomial of degree $n \geq 1$ with real coefficients?

(a) **(1 point)** $n = 1$;

(b) **(2 points)** $n = 2$;

(c) **(7 points)** n is arbitrary.

Solution. Integration by parts gives

$$f(x) = -\int_x^\infty P_n(t)\, de^{-t} = P_n(x)e^{-x} + \int_x^\infty P_n'(t)e^{-t}\, dt =$$

$$= P_n(x)e^{-x} + P_n'(x)e^{-x} + \int_x^\infty P_n''(t)e^{-t}\, dt = \cdots =$$

$$= e^{-x}\Big(P_n(x) + P_n'(x) + P_n''(x) + \cdots + P_n^{(n)}(x)\Big).$$

As $e^{-x} \neq 0$, the roots of the function $f(x)$ coincide with the roots of the polynomial $Q_n(x) = P_n(x) + P_n'(x) + P_n''(x) + \cdots + P_n^{(n)}(x)$. Polynomial $Q_n(x)$ can be arbitrary (see problem 1). Hence $f(x)$ can have from 0 up to n real roots if n is even and from 1 up to n roots if n is odd.

Answer. (a) 1; (b) 0, 1, 2; (c) from 0 up to n real roots for even n and from 1 up to n roots for odd n.

5. Find all the (complex) roots $x_1, x_2, \ldots, x_{n-1}$ of the equation

$$(x+1)^n = (x-1)^n$$

(a) **(1 point)** for $n = 3$;

(b) **(1 point)** for $n = 4$;

(c) **(6 points)** for arbitrary n (try to find simplest formula for the roots).

(d) **(5 points)** Calculate the sum $x_1^2 + \cdots + x_{n-1}^2$.

Solution. (a) For $n = 3$ we have the equation $3x^2 + 1 = 0$. Hence $x_{1,2} = \pm i/\sqrt{3}$.

(b) For $n = 4$ the equation has the form $x^3 + x = 0$. Hence $x = 0, \pm i$.

(c) For arbitrary n $x = 1$ is not a root of the given equation. Therefore this equation is equivalent to $\left(\frac{x+1}{x-1}\right)^n = 1$. De Moivre's formula gives

$$\frac{x_k + 1}{x_k - 1} = z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} \qquad (1 \leq k \leq n-1)$$

($k = 0$ must be excluded because $\frac{x+1}{x-1} \neq 1$). Hence

$$x_k = \frac{z_k + 1}{z_k - 1} = \frac{\cos\frac{2\pi k}{n} + 1 + i\sin\frac{2\pi k}{n}}{\cos\frac{2\pi k}{n} - 1 + i\sin\frac{2\pi k}{n}} =$$

$$= \frac{2\cos\frac{\pi k}{n}\left(\cos\frac{\pi k}{n} + i\sin\frac{\pi k}{n}\right)}{2\sin\frac{\pi k}{n}\left(-\sin\frac{\pi k}{n} + i\cos\frac{\pi k}{n}\right)} = -i\cot\frac{\pi k}{n}.$$

(d) If $t_1, \dots, t_n$ are the roots of the equation $t^n - \sigma_1 t^{n-1} + \sigma_2 t^{n-2} + \dots \pm \sigma_n = = 0$ then by Viète's formula

$$t_1^2 + t_2^2 + \dots + t_n^2 = \sigma_1^2 - 2\sigma_2.$$

For the given equation

$$\binom{n}{1} x^{n-1} + \binom{n}{3} x^{n-3} + \dots = 0$$

we have $\sigma_1 = 0, \sigma_2 = \binom{n}{3} / \binom{n}{1} = \frac{(n-1)(n-2)}{6}$ and

$$x_1^2 + \dots + x_{n-1}^2 = -2\sigma_2 = -\frac{(n-1)(n-2)}{3}.$$

Answer. (a) $\pm i/\sqrt{3}$; (b) $0, \pm i$; (c) $-i \cos \frac{\pi k}{n}$ $(1 \leq k \leq n-1)$; (d) $-\frac{(n-1)(n-2)}{3}$.

6. (8 points) Find the limit $\lim\limits_{n\to\infty} \frac{C_n}{A_n}$ where numbers A_n and C_n are defined by the formula

$$\begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}^n = \begin{pmatrix} A_n & B_n \\ B_n & C_n \end{pmatrix}.$$

Solution. We have by induction

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

where F_n are Fibonacci numbers: $F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$ $(n \geq 1)$. Hence

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^6 = \begin{pmatrix} F_5 & F_6 \\ F_6 & F_7 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$$

and

$$\begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}^n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{6n} = \begin{pmatrix} F_{6n-1} & F_{6n} \\ F_{6n} & F_{6n+1} \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ B_n & C_n \end{pmatrix}.$$

Now from Binet's formula $F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$ where $\varphi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$ follows that

$$\lim_{n\to\infty} \frac{C_n}{A_n} = \varphi^2 = \frac{3+\sqrt{5}}{2}.$$

Answer. $\frac{3+\sqrt{5}}{2}$.