Russian — Chinese Student Mathematical Olympiad

Birobidzhan, Russia, September 26, 2017

1. (2 points) For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n+1}}{(2n+1)!}.$$

Find $\sin A$ for $A = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$.

Solution. A direct computation shows that $\sin A = \begin{pmatrix} \sin x & y\cos x \\ 0 & \sin x \end{pmatrix}$.

2. (3 points) You add two random 20-digit base-2 numbers $a = (a_{19}, \dots, a_0)_2$ and $b = (b_{19}, \dots, b_0)_2$ (leading zeroes are allowed). What is the probability to have a carry of 1 from the last column (with a_{19} and b_{19})?

Solution. Let p_n be a probability to have a carry of 1 from the nth column. Then $p_0 = 1/4$ and

$$p_{n+1} = \frac{1}{4}\cdot 0[0+0] + \frac{1}{4}\cdot p_n[0+1] + \frac{1}{4}\cdot p_n[1+0] + \frac{1}{4}[1+1] = \frac{1}{4} + \frac{p_n}{2}.$$

The solution of this recurrence is $p_n = \frac{1}{2} - \frac{1}{2^{n+2}}$. In particular $p_{19} = \frac{1}{2} - \frac{1}{2^{21}}$.

Second solution. We have a carry of 1 from the last column iff $a + b \geq 2^{20}$. The number of such couples is

$$N = \sum_{a=0}^{2^{20}-1} a = 2^{40}\left(\frac{1}{2} - \frac{1}{2^{21}}\right).$$

So the probability to have a carry of 1 is $p_{19} = N/2^{40} = \frac{1}{2} - \frac{1}{2^{21}}$.

3. (3 points) Let $f(x, y)$ be a polynomial with real coefficients. Can $f(x, y)$ satisfy following two conditions:

(1) $\inf_{(x,y)\in\mathbb{R}^2} f(x, y) = 0$,

(2) $\forall (x, y) \in \mathbb{R}^2$ the value of $f(x, y)$ is strictly positive, i.e. $f(x, y) > 0$?

(Give a proof or counterexample.)

Solution. $f(x, y) = (1 - xy)^2 + x^2$.

4. (4 points) Find all pairs (p, q) of positive integers such that $p^{2017} + q$ is divisible by pq.

Solution. It is clear that $p \mid q$. Hence $q = q_{2017}p$. Substituting this gives $q_{2017}p^2 \mid p^{2017} + q^{2017}p$, so $q_{2017}p \mid |p^{2016} + q^{2017}$. It means that $p \mid q_{2017}$ and

$q_{2017} = pq_{2016}$. Continuing down we would have $q = q_1 p^{2017}$ and $pq_1 \mid 1 + q_1$. From this condition follows that either $p = 1$ or $p = 2$. In both cases $q_1 = 1$. **Answer:** $(1, 1)$ and $(2, 2^{2017})$.

5. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) - 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-x}$ for $x \geq 0$. For n a nonnegative integer, define $\mu_n = \int_0^\infty x^n f(x)dx$ (the nth moment of f).

(a) **(3 points)** Express μ_n in terms of μ_0.

(b) **(2 points)** Find the limit $\lim\limits_{n\to\infty} \frac{\mu_n}{\mu_0} \cdot \frac{3^n}{n!}$.

Solution. By the definition

$$\mu_n = \int_0^\infty (2x)^n f(2x)d(2x) = \int_0^\infty (2x)^n \frac{-3f(x) - f'(x)}{3} dx =$$
$$= -2^n \mu_n - \frac{2^n}{3} \int_0^\infty x^n f'(x)dx = -2^n \mu_n + \frac{2^n}{3} \int_0^\infty n x^{n-1} f(x)dx = -2^n \mu_n + \frac{2^n n}{3} \mu_{n-1}.$$

Hence

$$\mu_n = \frac{n}{3} \cdot \frac{1}{1 + 2^{-n}} \mu_{n-1}, \qquad \mu_n = \frac{n!}{3^n} \prod_{k=1}^{n} \frac{1}{1 + 2^{-k}} \mu_0.$$

From the last formula follows that

$$\frac{\mu_n}{\mu_0} \cdot \frac{3^n}{n!} = \prod_{k=1}^{n} \frac{1}{1 + 2^{-k}}.$$

This product is well-defined because it converges absolutely.

6. (4 points) Formal power series $f(z) = c_0 + c_1 z + c_2 z^2 + \ldots$ satisfies the functional equation

$$f(z)^{-t} \ln f(z) = z \qquad (z \neq 0).$$

Find the coefficients c_0, c_1, c_2.

Solution. From the expansion $f(z)^{-t} \ln f(z) = c_0^{-t} \ln c_0 + O(z)$ follows that $c_0 = 1$. In this case $f(z)^{-t} \ln f(z) = c_1 z + O(z^2)$, so $c_1 = 1$. The coefficient c_2 can be calculated in the same way. From the formula $f(z)^{-t} \ln f(z) = z + \left(c_2 - t - \frac{1}{2}\right) z^2 + O(z^3)$ follows that $c_2 = t + \frac{1}{2}$. The general formula is $c_k = \frac{(tk+1)^{k-1}}{k!}$. The series $f(z)$ is known as *generalized exponential series.*